Mathematical induction and the natural numbers

For this work we assume $\mathcal{F}$ is an ordered field, that is to say it satisfies all the axioms listed in Ross except completeness.

**Definition 1** A subset $S \subseteq \mathcal{F}$ is called inductive if for all $x \in S$ we must have $x + 1 \in S$ as well.

**Definition 2** Set $\mathcal{I} = \{ S \subseteq \mathcal{F} | 1 \in S \text{ and } S \text{ is inductive} \}$. Now set $S_0 = \bigcap_{S \in \mathcal{I}} S$, the intersection of all elements in $\mathcal{I}$.

**Problem 1** Show that $1 \in S_0$ and that if $x \in S_0$ then $x + 1 \in S_0$ and hence that $S_0 \in \mathcal{I}$.

**Definition 3** We call $S_0$ the “natural numbers” in $\mathcal{F}$.

The set $S_0$ is the smallest set for which “mathematical induction” works. To explain this, suppose $P$ is some formula or property that a value $x \in \mathcal{F}$ might have. That is to say, $P(x)$ is either a true or a false statement. Now suppose you show $P(1)$ is true and further that whenever $P(x)$ is true, you also must have that $P(x + 1)$ is true. You probably recall that these are the two hypotheses of mathematical induction. In our situation what this means is that the set $S = \{ x | P(x) \text{ is true} \}$ must belong to $S_0$. Since $S_0$ is the intersection of all such sets this implies $P$ is true for all elements of $S_0$. Thus by defining the “natural numbers” as the intersection $S_0$ we find we can use mathematical induction to prove properties hold on the set $S_0$.

**Problem 2** Show that if $a$ and $b$ belong to $S_0$ so does $a + b$.

*hint:* Fix $a \in S_0$ and set $S_1 = \{ b \in S_0 | a + b \in S_0 \}$. Now show $S_0 \subseteq S_1$ by mathematical induction.

In $\mathcal{F}$ we can define open and closed intervals $(a, b) = \{ x \in \mathcal{F} | a < x < b \}$ and $[a, b] = \{ x \in \mathcal{F} | a \leq x \leq b \}$.

**Problem 3** Show that if $b \in S_0$ then $(b, b + 1) \cap S_0 = \emptyset$.

*hint:* First show that $S_0 \cap (-\infty, 1) = \emptyset$. Now show that for all $b \in S_0$ we have $S_0 \cap (b, b + 1) = \emptyset$ by induction on $b$.

**Problem 4** Show that $S_0$ is complete in that for any Cauchy sequence $\{s_i\}$ of terms from $S_0$, there is a $I$ and for $i \geq I$, $s_i = s_I$, i.e. the sequence is constant once $i$ is large enough.

**Problem 5** Now assume also that $\mathcal{F}$ satisfies the least upper bound property in that any bounded subset has a least upper bound and show that $S_0$ cannot be bounded above.

*hint:* Show that if $S_0$ is bounded above, then the supremum is in $S_0$ and this is impossible.

**Problem 6** Show that if $\mathcal{F}$ satisfies the least upper bound property then for all $x \in \mathcal{F}$ there is a unique $b \in S_0$ with $x \in [b, b + 1)$.