Exploiting Symmetry in Solving Linear Equations

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Introduction

Recent efforts have shown the efficacy of applying group theoretical methods to the numerical treatment of certain partial differential equations via finite differences and finite elements, see [2, 3, 4, 5, 7, 8, 9, 12]. The articles, e.g., [4, 7], have demonstrated that the use of discretizations of partial differential equations which are suitably adapted to respect symmetry properties yield highly useful decompositions which can reduce computational effort, improve the numerical conditioning of problems, and significantly facilitate the study of bifurcation behavior at singularities. The results in the present paper extend those in [1, 4] via a systematic introduction of group representation theory.

For simplicity we deal only with the problem of solving linear equations (usually coming from discretization methods) in the presence of a group action. We find a decomposition of the given linear problem into block form which is most efficient in the sense that a general problem cannot be further decomposed. We do this in a direct way without first determining bases for the corresponding subspaces (as the method of symmetry adapted bases, see [12]). Our method can be expressed in terms of tensor products, which is described in the last section of this article; the tensor approach is more fully explored in [10].

1 Groups Acting as Permutations of Indices

Let G be a finite group with identity I.

We assume that we have a set of N indices $\mathcal{N} = \{1, \ldots, N\}$ which G permutes. We will denote this action as follows: if n is an index and g is a group element, then g sends n to gn. The action satisfies the following basic group action axioms:

$$In = n \text{ for every } n \text{ in } \mathcal{N}, \tag{1.1}$$

and

$$(gh)n = g(hn) \text{ for every } g \text{ and } h \text{ in } G \text{ and } n \text{ in } \mathcal{N}. \tag{1.2}$$

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Our basic assumption is that G acts on $\mathcal{N}$ without any fixed points: i.e.,

$$gn = n \quad \text{for some } n \text{ only if} \quad g = I. \tag{1.3}$$

The *orbit* of an index n is the set $\{gn : g \in G\}$ of indices to which n is moved by the group elements in G. Using (1.3), we see that there are exactly $\#G$ elements in each orbit. Any two orbits are either identical or disjoint, so that there are exactly $M = N/\#G$ orbits.

A *selection* of indices is a subset $\mathcal{M}$ of $\mathcal{N}$ which consists of one index in each orbit of G.

Thus, if $\mathcal{M}$ is a selection of indices, then the set $\mathcal{N}$ of all indices is equal to the set $\{gm : g \in G, m \in \mathcal{M}\}$. Moreover, with the assumption (1.3), there is no duplication in this presentation of the full set $\mathcal{N}$. There are $M = N/\#G$ elements in a selection (one element for each orbit).

2 The Induced Action on Column Vectors

Suppose now that G acts as in Section 1 on a set of indices $\mathcal{N}$. Let $V = \mathbb{C}^{\mathcal{N}}$ be the space of complex column vectors indexed by $\mathcal{N}$. If n is an index in $\mathcal{N}$ and v is a vector in V, we will denote the n^{th} coordinate of v by $v[n]$.

Since G acts on the indices $\mathcal{N}$, we obtain an action of G on the vector space $V = \mathbb{C}^{\mathcal{N}}$ by setting $(gv)[n] = v[g^{-1}n]$. This gives a representation of G on V, which is referred to in the representation theory literature as a *permutation representation*. The axioms (1.1) and (1.2) extend to the action of G on V as:

$$Iv = v \text{ for every } v \text{ in } V, \tag{2.1}$$

and

$$(gh)v = g(hv) \text{ for every } g \text{ and } h \text{ in } G \text{ and } v \text{ in } V. \tag{2.2}$$

Note that if e_n is the standard basis vector of V with 1 in the n^{th} position and 0 in all other positions, then the definition of the action of G on V implies that

$$g\, e_n = e_{gn}. \tag{2.3}$$

Fix an irreducible matrix representation $\rho : G \to GL(d, \mathbb{C})$; here $d = \dim \rho$ is the *dimension*, or *degree*, of the representation. We denote the ij^{th} entry of the matrix of the representation ρ on the group element g by $A^{\rho}_{ij}(g)$.

Since V is a representation of G, we have the standard projector and transfer operators P^{ρ}_{ij} defined on V, where i and j run up to the dimension of ρ. (See [11, section 2.7].) These are defined by the formula

$$P^{\rho}_{ij}(v) = \frac{\dim \rho}{\#G} \sum_{g \in G} A^{\rho}_{ji}(g^{-1}) gv.$$

Note the reversal of indices and the use of the inverse element in the sum. These operators satisfy the formulas

$$gP^{\rho}_{ij} = \sum_{k} A^{\rho}_{ki}(g) P^{\rho}_{kj} \tag{2.4}$$

and, if ρ and τ are two irreducible representations of G which are either identical or non-conjugate, then

$$P_{ij}^{\rho} \circ P_{kl}^{\tau} = P_{il}^{\tau}\, \delta_{jk}\, \delta_{\rho\tau}. \tag{2.5}$$

Note that this implies that the operators P_{ii}^{ρ} are orthogonal projectors on V. The image of P_{ii}^{ρ} will be denoted by V_i^{ρ}.

Not only are the P_{ii}^{ρ} mutually orthogonal projectors, but they form a complete set of projectors in the following sense. Define *a full set of irreducible matrix representations* of G to be a set of non-conjugate irreducible matrix representations $\{\rho_\alpha\}$ such that every irreducible representation of G is conjugate to exactly one ρ_α. Since G is finite, general results from representation theory say that such a full set exists, and is finite; the number of representations in a full set equals the number of conjugacy classes of G. Moreover, one has the useful and standard formula

$$\sum_{\alpha} (\dim \rho_\alpha)^2 = \#G \tag{2.6}$$

which is proved in [6, section 31]. The completeness of the set of projectors is stated as follows.

Proposition 2.7 *Let G be a finite group acting on V. Then V splits into a direct sum of the subspaces V_i^{ρ}, as ρ ranges over a full set of irreducible matrix representations of G and i ranges from 1 to $\dim\rho$:*

$$V = \bigoplus_{\rho} \bigoplus_{i=1}^{\dim\rho} V_i^{\rho}.$$

In fact,

$$\sum_{\rho} \sum_{i=1}^{\dim\rho} P_{ii}^{\rho} \text{ is the identity on } V.$$

We will denote $P_{ij}^{\rho}(v)$ by v_{ij}^{ρ}. Explicitly,

$$v_{ij}^{\rho}[m] = \frac{\dim\rho}{\#G} \sum_{g\in G} A_{ji}^{\rho}(g^{-1}) v[g^{-1}m]. \tag{2.8}$$

By Proposition 2.7,

$$v = \sum_{\rho} \sum_{i=1}^{\dim\rho} v_{ii}^{\rho}. \tag{2.9}$$

As a consequence of (2.4) and the definitions, we have the following Lemma, whose proof is a simple computation which we leave to the reader.

Lemma 2.10 *Fix an irreducible matrix representation ρ, an index m, a group element g, and a vector v in V. Then for each j between 1 and $\dim\rho$,*

$$v_{jj}^{\rho}[gm] = \sum_{i=1}^{\dim\rho} A_{ij}^{\rho}(g^{-1}) v_{ij}^{\rho}[m].$$

We interpret this property as a condition on the coordinates of vectors in the subspaces V_j^ρ. In particular, if one knows $v_{ij}^\rho[m]$ for all $i = 1, \ldots, \dim\rho$ and all m in a selection $\mathcal{M}$, then one knows v_{jj}^ρ. Note that it then requires only $M \dim\rho = N \dim\rho/\#G$ numbers to determine the vector v_{jj}^ρ (not N); this is in fact the dimension of V_j^ρ.

Proposition 2.11 *The dimension of the subspace V_j^ρ is $M \dim\rho = N \dim\rho/\#G$.*

Proof: The above lemma shows that the dimension of V_j^ρ is at most $N \dim\rho/\#G$. However,

$$\begin{aligned} N &= \dim V = \dim\left(\bigoplus_\rho \bigoplus_{j=1}^{\dim\rho} V_j^\rho\right) = \sum_\rho \sum_{j=1}^{\dim\rho} \dim V_j^\rho \\ &\leq \sum_\rho \sum_{j=1}^{\dim\rho} [N \dim\rho/\#G] = \frac{N}{\#G} \sum_\rho (\dim\rho)^2 = N \end{aligned}$$

where the last equality uses (2.6). Therefore the inequalities must all be equalities, proving the proposition. □

3 Equivariant Linear Systems

Keeping the notation of the previous section, let $\mathcal{L} : V \to V$ be a linear transformation. We say $\mathcal{L}$ is *G-equivariant* (or simply *equivariant*) if

$$\mathcal{L}(gv) = g\mathcal{L}(v) \tag{3.1}$$

for all vectors $v \in V$ and all group elements $g \in G$.

Since V is a space of column vectors, $\mathcal{L}$ has a matrix $\mathbf{L}$, whose mn^{th} entry will be denoted by $L[m, n]$. Hence $\mathcal{L}(v) = \mathbf{L}v$, or, equivalently, $\mathcal{L}(e_n)[m] = L[m, n]$, where e_n is the n^{th} standard basis vector for V. The equivariance of $\mathcal{L}$ is expressed in terms of its matrix $\mathbf{L}$ as follows.

Lemma 3.2 *With the above notation, $\mathcal{L}$ is equivariant if and only if $L[m, gn] = L[g^{-1}m, n]$ for every $g \in G$ and every $m, n \in \mathcal{N}$.*

Proof: Since both sides of (3.1) are linear in v, the operator $\mathcal{L}$ is equivariant if and only if $\mathcal{L}(ge_n) = g\mathcal{L}(e_n)$ for every index n. But $\mathcal{L}(ge_n)[m] = \mathcal{L}(e_{gn})[m] = L[m, gn]$ and $(g\mathcal{L}(e_n))[m] = \mathcal{L}(e_n)[g^{-1}m] = L[g^{-1}m, n]$, proving the lemma. □

Note that equivariance can also be expressed by the formula

$$L[gm, gn] = L[m, n] \tag{3.3}$$

for every $g \in G$ and every $m, n \in \mathcal{N}$.

The following lemma gives the dimension of the space of equivariant linear operators on V.

Lemma 3.4 *Let V have dimension N and G have order $\#G$. Then the dimension of the space of G-equivariant linear operators $\mathcal{L}$ on V is $N^2/\#G$.*

Proof: The formula $L[m, gn] = L[g^{-1}m, n]$ for the matrix $\mathbf{L}$ shows that $\mathbf{L}$ is determined by its entries in the columns coming from a selection. Moreover, the columns in a selection can be arbitrary. Hence the dimension of the space of equivariant operators on V is

$$(\text{\# of selected columns })(\text{\# of entries in each column})$$

which is $(N/\#G)(N) = N^2/\#G$. □

Since the equivariance of $\mathcal{L}$ means that $\mathcal{L}$ commutes with the action of each group element g, it also implies that $\mathcal{L}$ commutes with all of the operators P^ρ_{ij}, since they are just linear combinations of group elements:

$$\mathcal{L}P^\rho_{ij} = P^\rho_{ij}\mathcal{L}. \tag{3.5}$$

In particular $\mathcal{L}$ commutes with the projectors P^ρ_{ii}, and so $\mathcal{L}$ respects the decomposition of Proposition 2.7 of V:

Lemma 3.6 *If $\mathcal{L}$ is equivariant, then $\mathcal{L}$ maps V^ρ_i into itself for each ρ and each i.*

We will denote the restriction of $\mathcal{L}$ to V^ρ_i by $\mathcal{L}^\rho_i$:

$$\mathcal{L}^\rho_i = \mathcal{L} \mid V^\rho_i. \tag{3.7}$$

In particular, $\mathcal{L}$ is invertible if and only if $\mathcal{L}^\rho_i$ is invertible for each ρ and each i.

4 Reducing to the Sub-Problems

In this section we fix a finite group G acting on an index set $\mathcal{N}$ without fixed points as above. We also fix a selection $\mathcal{M}$ of indices, and a full set of irreducible matrix representations $\{\rho_\alpha\}$.

Let $\mathcal{L}$ be an invertible G-equivariant linear operator $\mathcal{L} : V \to V$, with matrix $L[m, n]$ as described in the previous section. Let an arbitrary vector v in V be given. Suppose that we wish to solve the equation

$$\mathcal{L}(u) = v \tag{4.1}$$

or equivalently

$$\mathbf{L}u = v \tag{4.2}$$

where u is an unknown vector in V. Applying the projector P^ρ_{kk} to both sides of (4.1), and using the equivariance of $\mathcal{L}$, gives the sub-problem

$$\mathcal{L}^\rho_k(u^\rho_{kk}) = v^\rho_{kk} \tag{4.3}$$

which is also an invertible linear system (on the subspace V^ρ_k). If we can solve these subproblems, for each ρ and each $k = 1, \ldots, \dim\rho$, then we can reconstruct the global solution u to (4.1) as

$$u = \sum_{\rho} \sum_{k=1}^{\dim \rho} u^{\rho}_{kk}. \tag{4.4}$$

Our job is to write down matrices for the subproblems in an efficient manner. For this we make use of the remarks following Lemma 2.10: to determine all the coordinates of the vector u^{ρ}_{kk}, it is enough to know only the transfer vectors u^{ρ}_{ik} in the coordinates from the selection $\mathcal{M}$. Moreover, the sub-problem (4.3) has as input the vector v^{ρ}_{kk}, and by Lemma 2.10, it too is determined by the selected coordinates of the transfer vectors v^{ρ}_{jk}. Therefore the heart of the sub-problem can be expressed as follows:

For a fixed ρ and k, given the selected coordinates of the vectors v^{ρ}_{jk}, determine the selected coordinates of the vectors u^{ρ}_{ik}.

For a fixed ρ and k, the scalar variables in the above set-up are therefore naturally doubly-indexed, by pairs (i, m) where $i = 1, \ldots, \dim \rho$ and $m \in \mathcal{M}$. For example, the pair (i, m) corresponds to the m^{th} coordinate in the vector u^{ρ}_{ik}.

Therefore the matrix of the sub-problem (4.3) is a matrix

$$\mathbf{L}^{\rho}_{k} = L^{\rho}_{k}[(i, m), (j, n)]$$

such that

$$\sum_{(j,n)} L^{\rho}_{k}[(i, m), (j, n)] u^{\rho}_{jk}[n] = v^{\rho}_{ik}[m]. \tag{4.5}$$

The main theorem, which gives a formula for the matrix $\mathbf{L}^{\rho}_{k}$, can now be stated.

Theorem 4.6 *Fix ρ and k between 1 and $\dim \rho$. Then*

$$L^{\rho}_{k}[(i, m), (j, n)] = \sum_{g \in G} L[g^{-1}m, n] A^{\rho}_{ji}(g^{-1}).$$

Proof: Fix the pair (i, m). Then

$$\begin{aligned}
v^{\rho}_{ik}[m] &= (P^{\rho}_{ik}(v^{\rho}_{kk}))[m] \\
&= \frac{\dim \rho}{\#G} \sum_{h \in G} A^{\rho}_{ki}(h^{-1})(h v^{\rho}_{kk})[m] \\
&= \frac{\dim \rho}{\#G} \sum_{h \in G} A^{\rho}_{ki}(h^{-1}) v^{\rho}_{kk}[h^{-1}m] \\
&= \frac{\dim \rho}{\#G} \sum_{h \in G} A^{\rho}_{ki}(h^{-1}) \sum_{p \in \mathcal{N}} L[h^{-1}m, p] u^{\rho}_{kk}[p] \\
&\quad \text{(setting p = gn)} \\
&= \frac{\dim \rho}{\#G} \sum_{h \in G} A^{\rho}_{ki}(h^{-1}) \sum_{g \in G} \sum_{n \in \mathcal{M}} L[h^{-1}m, gn] u^{\rho}_{kk}[gn] \\
&\quad \text{(using the equivariance of L and Lemma 2.10)} \\
&= \frac{\dim \rho}{\#G} \sum_{h \in G} A^{\rho}_{ki}(h^{-1}) \sum_{g \in G} \sum_{n \in \mathcal{M}} L[g^{-1}h^{-1}m, n] \sum_{j=1}^{\dim \rho} A^{\rho}_{jk}(g^{-1}) u^{\rho}_{jk}[n]
\end{aligned}$$

$$= \sum_{(j,n)} \left[\frac{\dim \rho}{\#G} \sum_{h \in G} \sum_{g \in G} L[g^{-1}h^{-1}m, n] A^{\rho}_{ki}(h^{-1}) A^{\rho}_{jk}(g^{-1}) \right] u^{\rho}_{jk}[n].$$

Thus, by (4.5), we have

$$\begin{aligned}
L^{\rho}_k[(i,m),(j,n)] &= \frac{\dim \rho}{\#G} \sum_{h \in G} \sum_{g \in G} L[g^{-1}h^{-1}m, n] A^{\rho}_{ki}(h^{-1}) A^{\rho}_{jk}(g^{-1}) \\
&\quad \text{(setting a = hg)} \\
&= \frac{\dim \rho}{\#G} \sum_{a \in G} \sum_{h \in G} L[a^{-1}m, n] A^{\rho}_{ki}(h^{-1}) A^{\rho}_{jk}(a^{-1}h) \\
&= \dim \rho \sum_{a \in G} L[a^{-1}m, n][\frac{1}{\#G} \sum_{h \in G} A^{\rho}_{jk}(a^{-1}h) A^{\rho}_{ki}(h^{-1})] \\
&\quad \text{(using the orthogonality relations for the matrices } A^{\rho}) \\
&= \dim \rho \sum_{a \in G} L[a^{-1}m, n][\frac{1}{\dim \rho} A^{\rho}_{ji}(a^{-1})] \\
&= \sum_{a \in G} L[a^{-1}m, n] A^{\rho}_{ji}(a^{-1})
\end{aligned}$$

which, after replacing a by g, is the required formula. The orthogonality relations used to obtain the second-to-last equality above can be found in [6, section 31]. □

We have now the following remarkable observation:

Corollary 4.7 *The matrix L^{ρ}_k is independent of k.*

Thus there is actually only one sub-matrix for each irreducible matrix representation. We will therefore drop the subscript and define

$$L^{\rho}[(i,m),(j,n)] = \sum_{g \in G} L[g^{-1}m, n] A^{\rho}_{ji}(g^{-1}). \tag{4.8}$$

Thus the subproblems can be written in the form

$$\sum_{(j,n)} L^{\rho}[(i,m),(j,n)] u^{\rho}_{jk}[n] = v^{\rho}_{ik}[m]. \tag{4.9}$$

One final remark which should be made is that the matrices $\mathbf{L}^{\rho}$ for the subproblems are formed by using the entries of the full matrix $L[gm, n]$, for $g \in G$ and $m, n \in \mathcal{M}$. Therefore it is not necessary to know the entire matrix $\mathbf{L}$; the reductions only require those columns of $\mathbf{L}$ indexed by the selection $\mathcal{M}$.

5 Putting it All Together

In this section we will give an overview of the entire process for solving the linear system (4.1), where $\mathcal{L}$ is a G-equivariant operator on the column space V, on which G acts through an action on the indices $\mathcal{N}$ for the coordinates of V without fixed points.

Given: a vector v of V.
To Solve: $\mathcal{L}(u) = v$ for the unknown vector u.

1: Choose a selection $\mathcal{M}$ of indices.

2: Find a full set of irreducible complex matrix representations for G.

3: Determine the entries $L[m,n]$ of the matrix for $\mathcal{L}$, in the selected columns.

4: For each irreducible complex matrix representation ρ in the full set, perform steps 4a - d:

4a: Form the submatrices $\mathbf{L}^\rho$ using (4.8):

$$L^\rho[(i,m),(j,n)] = \sum_{g\in G} L[g^{-1}m,n]A^\rho_{ji}(g^{-1}).$$

4b: Using the projectors P^ρ_{ik} (formula (2.8)), determine the selected coordinates of the vectors v^ρ_{ik}:

$$v^\rho_{ik}[m] = \frac{\dim\rho}{\#G}\sum_{g\in G} A^\rho_{ki}(g^{-1})v[g^{-1}m].$$

4c: Solve the subproblems (4.9) for the selected coordinates of the vectors u^ρ_{jk}.

4d: Reconstruct all the coordinates of the vector u^ρ_{kk} using Lemma 2.10:

$$u^\rho_{kk}[gm] = \sum_{j=1}^{\dim\rho} A^\rho_{jk}(g^{-1})u^\rho_{jk}[m].$$

5: Reconstruct the solution $u = \sum_\rho \sum_{k=1}^{\dim\rho} u^\rho_{kk}$.

If one has many instances of the linear system to solve with different right-hand-side vectors v, one of course only performs steps 1–4a once; then steps 4b–5 are done for each separate problem.

6 The Reduction in Complexity

Let us discuss the computational effort of the method under the assumption that the matrix is full and the linear equation solver is one of the standard direct solvers such as Gaussian elimination. The essential overhead cost of our method occurs in step 4a which amounts to N^2 flops. For large problems, the greatest part of the cost of the method is in Step 4c as outlined above: solving the sub-problems. The dimension of the subproblem for the irreducible matrix representation ρ is the dimension of V^ρ_k for any k, and this is $N\dim\rho/\#G$ by (2.11). For a direct solver as mentioned above, this gives a computational effort of

$$C\sum_\rho \left(\frac{N\dim\rho}{\#G}\right)^3 \quad \text{flops} \tag{6.1}$$

for some constant C. Without using our method, we are faced with a full size matrix of order N, which using the same direct solver gives an effort of CN^3 flops. Neglecting the lower order overhead expense and taking the quotient, we obtain the reduction factor

$$\frac{1}{(\#G)^3}\sum_{\rho}(\dim \rho)^3. \tag{6.2}$$

7 The Sharpness of the Method

What we have essentially done in the reduction of the full linear system (4.1) to the sub-systems (4.3) is block-diagonalize the matrix $\mathbf{L}$, without in fact finding an explicit basis for the subspaces V_k^ρ. If R is the number of irreducible matrix representations of G, (which is the number of conjugacy classes of G), we have found that $\mathbf{L}$ block-diagonalizes to R sets of blocks (one for each ρ), and the set of blocks corresponding to the irreducible matrix representation ρ is the matrix $\mathbf{L}^\rho$ repeated $\dim \rho$ times.

The reader may wonder whether any further reduction can be made. The answer is no, as a simple dimension count of available parameters shows. Each matrix $\mathbf{L}^\rho$ is of size $N \dim \rho/\#G$ (by Proposition 2.11), and so there are

$$\sum_{\rho}(N \dim \rho/\#G)^2 = N^2/\#G$$

total parameters (note that we have used (2.6)). Now this is precisely the dimension of the space of equivariant operators on V, by Lemma 3.4. Hence no further reduction in the problem is possible.

8 The Tensor Formulation

The reader will have noticed that the sub-matrices $\mathbf{L}^\rho$ defined in (4.8) have doubly-indexed entries. This suggests that these matrices can be conveniently expressed as tensor products. In this section we will develop this aspect of the construction. This approach is taken *ab initio* in [10].

Fix an irreducible representation ρ, and let $d = \dim \rho$ be its dimension. For any selected index m, let $\mathbf{v}^\rho[m]$ denote the d-by-d matrix whose ij^{th} entry is $v_{ij}^\rho[m]$:

$$\mathbf{v}^\rho[m] = \begin{pmatrix} v_{11}^\rho[m] & \dots & v_{1d}^\rho[m] \\ \vdots & \ddots & \vdots \\ v_{d1}^\rho[m] & \dots & v_{dd}^\rho[m] \end{pmatrix} \tag{8.1}$$

Using the formula (2.8), we see that the definition of $\mathbf{v}^\rho[m]$ can be expressed as a linear combination of transposes of the matrices $\mathbf{A}^\rho$ which define the representation ρ:

$$\mathbf{v}^\rho[m] = \frac{\dim \rho}{\#G}\sum_{g\in G} v[g^{-1}m]\mathbf{A}^\rho(g^{-1})^\top. \tag{8.2}$$

Using (8.2), we can express the matrices $\mathbf{v}^\rho[m]$ in terms of the original vector v. Inverting this procedure is also straightforward. Fix a (possibly unselected) index n, and write $n = gm$ for a unique group element g and a unique selected index m. Then by Lemma 2.10,

$$\begin{aligned} v_{jj}^\rho[gm] &= \sum_{i=1}^d A_{ij}^\rho(g^{-1}) v_{ij}^\rho[m] \\ &= jj^{th} \text{ entry of } \mathbf{A}^\rho(g^{-1})^\top \mathbf{v}^\rho[m]. \end{aligned}$$

Hence

$$\text{trace}(\mathbf{A}^\rho(g^{-1})^\top \mathbf{v}^\rho[m]) = \sum_{j=1}^d v_{jj}^\rho[gm]$$

and so by (2.9)

$$v[gm] = \sum_\rho \text{trace}(\mathbf{A}^\rho(g^{-1})^\top \mathbf{v}^\rho[m]). \tag{8.3}$$

Now let $\mathbf{v}^\rho$ be the Md-by-d matrix formed by stacking the d-by-d matrices $\mathbf{v}^\rho[m]$ for the M selected indices m_α in a column:

$$\mathbf{v}^\rho = \begin{pmatrix} \mathbf{v}^\rho[m_1] \\ \mathbf{v}^\rho[m_2] \\ \vdots \\ \mathbf{v}^\rho[m_M] \end{pmatrix}. \tag{8.4}$$

We want to think of this matrix $\mathbf{v}^\rho$ as a column vector of length M, with d-by-d matrices as entries. The linear system for the subproblem (4.3) can now be written as

$$\mathbf{L}^\rho \, \mathbf{u}^\rho = \mathbf{v}^\rho \tag{8.5}$$

where $\mathbf{L}^\rho$ is an appropriately-indexed version of the matrix $L^\rho[(i,m),(j,n)]$ constructed in Section 4. This matrix $\mathbf{L}^\rho$ is an Md-by-Md matrix, which because of the double-indexing, can be viewed naturally as an M-by-M matrix whose mn^{th} entry is a d-by-d matrix. Indeed, using the definition (4.8) of $L^\rho[(i,m),(j,n)]$, we see that the mn^{th} entry is the matrix $\sum_{g\in G} L[g^{-1}m,n]\mathbf{A}^\rho(g^{-1})^\top$. Thus $\mathbf{L}^\rho$ in this form is naturally *a sum of tensor product matrices.*

Recall that the tensor product of two matrices $\mathbf{X}$ and $\mathbf{Y}$ which are d-by-d and e-by-e respectively is the de-by-de matrix $\mathbf{X} \otimes \mathbf{Y}$ which breaks naturally into a d-by-d matrix with e-by-e matrix entries: the matrix in the rs^{th} block is $X_{rs}\mathbf{Y}$.

To write our matrix $\mathbf{L}^\rho$ with this notation, one more definition is useful. Specifically, for $g \in G$ define the M-by-M matrix $\mathbf{L}_g$ by

$$\mathbf{L}_g[m,n] = L[g^{-1}m,n]. \tag{8.6}$$

Then

$$\mathbf{L}^\rho = \sum_{g\in G} \mathbf{L}_g \otimes \mathbf{A}^\rho(g^{-1})^\top. \tag{8.7}$$

The overview of the method given here can now be written with this notation as follows:

Given: a vector v of V.
To Solve: $\mathcal{L}(u) = v$ for the unknown vector u.

1: Choose a selection $\mathcal{M}$ of indices.

2: Find a full set of irreducible complex matrix representations for G.

3: Determine the entries $L[m,n]$ of the matrix for $\mathcal{L}$, in the selected columns, and find the matrices $\mathbf{L}_g$ for each group element g.

4: For each ρ in the full set of irreducible complex matrix representations, perform steps 4a - b:

4a: Form the column vector (with matrix entries) $\mathbf{v}^\rho$ from the given vector v, using (8.2):

$$\mathbf{v}^\rho[m] = \frac{\dim \rho}{\#G} \sum_{g \in G} v[g^{-1}m] \mathbf{A}^\rho(g^{-1})^\top$$

4b: Solve the subproblems (8.5) for the column vector (with matrix entries) $\mathbf{u}^\rho$:

$$\mathbf{L}^\rho \mathbf{u}^\rho = \left[\sum_{g \in G} \mathbf{L}_g \otimes \mathbf{A}^\rho(g^{-1})^\top \right] \mathbf{u}^\rho = \mathbf{v}^\rho.$$

5: Reconstruct the solution vector u from the $\mathbf{u}^\rho$'s using (8.3):

$$u[gm] = \sum_\rho \operatorname{trace}(\mathbf{A}^\rho(g^{-1})^\top \mathbf{u}^\rho[m]).$$

The reader will notice that the above outline of our method is a reorganization of the outline given in Section 5, expressed with a tensor formulation which emphasizes the block structure underlying the approach. It is our opinion that this latter organization is more suitable for a computer implementation. A small numerical example illustrating this feature has been given in [10].

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