A New Exclusion Test [1]

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Abstract

Exclusion tests are a well-known tool for obtaining *all* solutions of a nonlinear system of equations, see, e.g., the work of R. Krawczyk or R. E. Moore. Recently, K. Yamamura and collaborators have developed Linear Programming exclusion tests that turned out to be highly successful for nonlinear problems in electrical networking. The author has developed higher order exclusion tests based on Taylor expansions. In the present paper it is shown that the ideas behind both approaches can be combined: We present a new class of higher order Linear Programming exclusion tests, investigate their computational complexity, and illustrate their performance on several examples.

Key words: numerical solution of nonlinear systems of equations (65H10), exclusion tests, computational complexity (65Y20)

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1 Introduction

Exclusion tests are a well-known tool for obtaining *all* solutions of a nonlinear system of equations. This idea goes back at least to [1,2], see also [3,4]. Such algorithms seem to be very useful for finding all solutions of low-dimensional, but highly nonlinear systems which have many solutions. Such systems occur, e.g., in mechanical engineering.

A simple version of an exclusion algorithm can be described in the following way: Let $\Lambda \subset \mathbb{R}^n$ be an (initial) interval and $F : \Lambda \to \mathbb{R}^n$ a map. Given an interval $\sigma \subset \Lambda$, we consider necessary conditions for F to have a zero in

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σ. We call such conditions *exclusion tests*. The exclusion algorithm consists of applying an exclusion test on an interval. If the test is *negative*, i.e., the necessary condition does not hold, then we are sure that the interval contains no zero, and we exclude it. Otherwise, we bisect the interval and apply the test again. This leads to a simple recursive algorithm:

Algorithm 1 (Exclusion Algorithm)

```
$\Gamma \leftarrow \{ \Lambda \}$ (initial interval)
for $\ell = 1 :$ maximal_level
    for $\mathtt{a} = 1 : n$
        let $\tilde{\Gamma}$ be obtained by bisecting each $\sigma \in \Gamma$ along the axis $\mathtt{a}$
        for $\sigma \in \tilde{\Gamma}$
            if the exclusion test for $\sigma$ is negative
                drop $\sigma$ from $\tilde{\Gamma}$ ($\sigma$ is excluded)
        $\Gamma \leftarrow \tilde{\Gamma}$
    $\Gamma_\ell \leftarrow \Gamma$ (for later reference in this paper)
```

Note the preceding definition of the list Γ_ℓ of intervals which were not discarded after the initial interval Λ has been bisected ℓ times along each axis. We will occasionally refer to this list in the sequel.

For clarity of exposition and notation, the list of intervals is processed breadth-first rather than depth-first. However, the (mathematically equivalent) depth-first choice (i.e., a standard recursion, which uses less memory in the presence of garbage collection) was actually implemented by the author for the numerical examples. Note that this different choice has no influence on the final form of the list Γ_ℓ.

Also note that the algorithm may stop with an empty list Γ_ℓ indicating that F has no zero in Λ.

Let us introduce some notation. Components of column vectors $x \in \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{n\times n}$ are denoted by $x[i]$ and $A[i,j]$. In $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ we use the component-wise '$\leq$' as a partial ordering, and '$|\cdot|$' is the component-wise absolute value. An interval in $\mathbb{R}^n$ is a set

$$\sigma = \{ x \in \mathbb{R}^n \ : \ |m_\sigma - x| \leq r_\sigma \} =: [m_\sigma - r_\sigma \ , \ m_\sigma + r_\sigma] \tag{1}$$

where we used the definition $[a,b] := \{ x \in \mathbb{R}^n \ : \ a \leq x \leq b \}$. We call m_σ the midpoint and r_σ the radius of σ. We require that $r_\sigma[i] > 0$ for $i = 1 \ldots n$.

The simplest exclusion test is the well-known *Lipschitz Test*: Let L be a Lipschitz constant for F on σ, i.e., $\|F(x) - F(y)\|_\infty \leq L\|x - y\|_\infty$ for $x, y \in \sigma$.

Then

$$\|F(m_\sigma)\|_\infty \leq L\,\|r_\sigma\|_\infty \tag{2}$$

if σ contains a zero of F. See also [5].

Many exclusion tests have been developed, in particular those based on tools in interval analysis, see, e.g., [1–4]. Recently, K. Yamamura and collaborators [6–8] have developed Linear Programming exclusion tests that turned out to be highly successful for nonlinear problems in electrical networking which are typically only mildly nonlinear. The author and collaborators [9,10] have developed higher order exclusion tests based on Taylor expansions which aim at low-dimensional, but highly nonlinear problems.

The present paper combines these last two classes of tests into a new class of tests, and investigates their complexity. The new tests aim at low-dimensional, but highly nonlinear problems. The main contribution of the paper is contained in Section 3 where we define and analyze the new class of tests.

2 Taylor Tests

In this section we review the exclusion tests given in [9]. First we introduce some notation.

Definition 1 *We denote by $\mathbb{Z}_+$ the set of nonnegative integers. For a multi-index $\alpha \in {\mathbb{Z}_+}^n$ we consider the following definitions:*

- *The length of α is defined by $\|\alpha\|_1 = \sum_i \alpha[i]$.*
- *The factorial of α is defined by $\alpha! := \prod_i \alpha[i]!$.*
- *If $x \in \mathbb{R}^n$, then we define $x^\alpha := \prod_i x[i]^{\alpha[i]}$*
- *We define the partial derivatives $\partial^\alpha = (\alpha!)^{-1} \prod_i \partial_i^{\alpha[i]}$.*
- *Furthermore, we introduce the probability measures $\omega_k(dt) = k(1-t)^{k-1}\,dt$ on the interval $[0,1]$.*

These definitions are introduced to facilitate the use of Taylor's formula with integral remainder:

$$F(m+x) = \sum_{0 \leq \|\alpha\|_1 < q} \partial^\alpha F(m) x^\alpha + \sum_{\|\beta\|_1 = q} \int_0^1 \partial^\beta F(m+tx)\,\omega_q(dt)\,x^\beta\ . \tag{3}$$

We introduce the concept of dominant functions which was developed in [9].

Definition 2 *Let* $F, G : \mathbb{R}^n \to \mathbb{R}^n$. *We say that* F *is dominated by* G *with order* q *and write* $F \prec_q G$ *if and only if:*

- *For all multi-indices* $\alpha \in {\mathbb{Z}_+}^n$ *with* $\|\alpha\|_1 \leq q$ *we have that the derivatives* $\partial^\alpha F$ *and* $\partial^\alpha G$ *are integrable over all intervals* $\sigma \subset \mathbb{R}^n$.
- *For all* $x, y \in \mathbb{R}^n$ *with* $|x| \leq |y|$, *and for all* $\alpha \in {\mathbb{Z}_+}^n$ *with* $\|\alpha\|_1 \leq q$ *we have*

$$|\partial^\alpha F(x)| \leq \partial^\alpha G(|x|) \leq \partial^\alpha G(|y|). \tag{4}$$

Definition 3 *We say that* F *is dominated by* G *with order* ∞ *and write* $F \prec_\infty G$ *if and only if* $F \prec_q G$ *holds for all* $q = 1, 2, \ldots$

For simplicity we assume that our maps $F, G, \ldots$ are defined on all of $\mathbb{R}^n$. However, it is simple to change the statements into more localized statements.

Example 4 *The following estimates will be used in our numerical examples:*

$$\sin(t) \prec_1 t \qquad \cos(t) \prec_1 1 + t \tag{5}$$

$$\sin(t) \prec_2 t + t^2/2 \qquad \cos(t) \prec_2 1 + t + t^2/2 \tag{6}$$

$$\sin(t) \prec_3 t + t^3/6 \qquad \cos(t) \prec_3 1 + t + t^3/6 \tag{7}$$

Example 5 *Let* $F(x) = \sum_{\|\alpha\|_1 \leq k} C_\alpha x^\alpha$, $C_\alpha \in \mathbb{R}^n$, *be a polynomial system of degree* k. *Define* $G(x) = \sum_{\|\alpha\|_1 \leq k} |C_\alpha| x^\alpha$. *Then* $F \prec_\infty G$.

The following theorems have been shown in [9]:

Theorem 6

- $F_1 \prec_q G_1$ *and* $F_2 \prec_q G_2$ *implies* $c_1 F_1 + c_2 F_2 \prec_q |c_1| G_1 + |c_2| G_2$ *for* $c_1, c_2 \in \mathbb{R}$.
- $F_1 \prec_q G_1$ *and* $F_2 \prec_q G_2$ *implies* $F_1 \cdot F_2 \prec_q G_1 \cdot G_2$.
- $F_1 \prec_q G_1$ *and* $F_2 \prec_q G_2$ *implies* $F_1 \circ F_2 \prec_q G_1 \circ G_2$.

Here "$\cdot$" indicates multiplication of functions and "$\circ$" indicates composition of functions.

Note that these rules allow us to calculate more complex estimates from simple ones.

Example 7

$$\begin{aligned} \sin(t) &\prec_2 t + t^2/2 \\ \sin(\sin(t)) &\prec_2 \left(t + 1/2\, t^2\right) + 1/2 \left(t + 1/2\, t^2\right)^2 \\ &= t + t^2 + 1/2\, t^3 + 1/8\, t^4 \end{aligned}$$

Theorem 8 *If $F \prec_q G : \mathbb{R}^n \to \mathbb{R}^n$ for some $q > 0$ and $\sigma \subset \mathbb{R}^n$ is an interval, then*

$$\begin{aligned} |F(m_\sigma)| \leq & \, G(|m_\sigma + r_\sigma|) - G(|m_\sigma|) \\ & - \sum_{0 < \|\alpha\|_1 < q} \left(\partial^\alpha G(|m_\sigma|) - |\partial^\alpha F(m_\sigma)|\right) r_\sigma^\alpha \end{aligned} \tag{8}$$

is an exclusion test for F on σ.

Definition 9 (Taylor Test) *We will call the preceding test (8) a* Taylor test of order q.

We note in passing that the Taylor test of order one is very similar to the Lipschitz test (2). The test in [11] can also be interpreted as a Taylor test of order one, except that the authors impose the condition $F \prec_\infty G$ which is much stronger than our condition $F \prec_1 G$.

The following technical definition is needed in the sequel:

Definition 10 *We say that a zero point ξ of F has uniform order p if and only if*

1. $\partial^\alpha F(\xi) = 0$ *for* $\|\alpha\|_1 < p$.
2. *There exists an $\varepsilon > 0$ such that*

$$\varepsilon \, \|m - \xi\|_\infty^{\,p} \leq \|F(m)\|_\infty \quad \text{for} \quad \|m - \xi\|_\infty \leq \varepsilon.$$

The following is a well-known result from analysis:

Example 11 *If ξ is a regular zero point, i.e., $F(\xi) = 0$ and $F'(\xi)$ is invertible, then ξ is a zero point of F of uniform order 1.*

Our main complexity result for Taylor tests is contained in the following

Theorem 12 (Complexity Result) *Assume $F \prec_q G$, $q > 0$, and let the exclusion algorithm run with the Taylor test of order q. Let each zero point of F be of some uniform order which is at most q. Then there exists a constant $A > 0$ such that the following holds: if $\sigma \in \Gamma_k$ with $k > A$, then there exists a zero point $\xi \in \Lambda$ of F such that $\|m_\sigma - \xi\|_\infty \leq A \|r_\sigma\|_\infty$.*

In other words: If an interval on a sufficiently fine bisection level is not discarded, then its distance to the nearest zero point does not exceed a certain (uniform) number of steps, where a step has the size of its own diameter.

Corollary 13 *Under the assumptions of Theorem 12, the list $\#\Gamma_\ell$ of not discarded intervals on bisection level ℓ is bounded as $\ell \to \infty$.*

3 LP Tests

Our aim now is to tighten the previously analyzed Taylor tests even more with the help of Linear Programs. The idea of this section has been influenced by tests designed in [6–8] for handling electrical networks.

In [6–8] the authors analyze mildly nonlinear systems, where nearly all terms are linear and only some terms are nonlinear. These nonlinear terms are then estimated on a given interval, say σ, which leads to inequalities that must be true on σ if σ contains a solution. This leads to an exclusion test that is formulated as the first phase of a linear program, i.e., a feasibility check.

Here we pick up this idea, but localize it: we expand the nonlinear map about the midpoint of a given interval using the Taylor formula, and then we exploit the ideas of the previous section to estimate the nonlinear part of the Taylor formula. This leads again to inequalities that can be checked with the first phase of a linear program. Note that the resulting new exclusion tests can now also be applied to highly nonlinear systems.

In the following we use the notation e for $[1 \ \dots \ 1]^T \in \mathbb{R}^n$, $\mathbf{I}$ for the identity matrix in $\mathbb{R}^{n\times n}$, and $\mathbb{R}_+$ for the non-negative real numbers.

Theorem 14 *Let $F \prec_q G : \mathbb{R}^n \to \mathbb{R}^n$ with $q > 1$, and let $\sigma \subset \mathbb{R}^n$ be an interval. Then a necessary condition for F to have a zero in σ is that the following linear program (in standard form) has optimal value zero:*

$$\begin{aligned} &\textit{Maximize} \quad -e^T t \quad \textit{subject to} \\ &\qquad EF'(m)\tilde{x} + E\tilde{y} + t = p \\ &\qquad \tilde{x} + u = 2r_\sigma \\ &\qquad \tilde{y} + v = 2d_\sigma \\ &\qquad \tilde{x}, \tilde{y}, u, v, t \in \mathbb{R}^n_+ \end{aligned} \tag{9}$$

where $E \in \mathbb{R}^{n\times n}$ is a diagonal matrix with ± 1 on its diagonal and such that

$$p := E(-F(m_\sigma) + F'(m_\sigma)r_\sigma + d_\sigma) \geq 0 \tag{10}$$

and where

$$\begin{aligned} d_\sigma := &\, G(|m_\sigma| + r_\sigma) - G(|m_\sigma|) - G'(|m_\sigma|)\, r_\sigma \\ &- \sum_{1<\|\alpha\|_1<q} (\partial^\alpha G(|m_\sigma|) - |\partial^\alpha F(m_\sigma)|)\, r_\sigma^\alpha \geq 0. \end{aligned} \tag{11}$$

Definition 15 (LP Test) *We call the preceding test (i.e., the test whether*

(9) has an optimal value zero) an LP test of order q.

Note that the linear program (9) has the feasible basis $t = p$, $u = 2r_\sigma$, and $v = 2d_\sigma$, which can be used to start the simplex method.

PROOF. We prove Theorem 14. Let $m_\sigma + x \in \sigma$ be a zero of F. Then, using (3) and (11), we obtain

$$F(m_\sigma + x) = 0 = F(m_\sigma) + F'(m_\sigma)x + y \tag{12}$$

where

$$y = \sum_{1<\|\alpha\|_1<q} \partial^\alpha F(m_\sigma)\, x^\alpha + \int_0^1 \sum_{\|\alpha\|_1=q} \partial^\alpha F(m_\sigma + tx)\, \omega_q(dt)\, x^\alpha \tag{13}$$

and hence

$$\begin{aligned} |y| &\leq \sum_{1<\|\alpha\|_1<q} |\partial^\alpha F(m_\sigma)|\; r_\sigma^\alpha + \int_0^1 \sum_{\|\alpha\|_1=q} \partial^\alpha G(|m_\sigma| + tr_\sigma)\, \omega_q(dt)\, r_\sigma^\alpha \\ &= \sum_{1<\|\alpha\|_1<q} |\partial^\alpha F(m_\sigma)|\; r_\sigma^\alpha + G(|m_\sigma| + r_\sigma) - \sum_{0\leq\|\alpha\|_1<q} \partial^\alpha G(|m_\sigma|)\, r_\sigma^\alpha \\ &= d_\sigma\ . \end{aligned} \tag{14}$$

Hence the existence of a zero in σ implies that the set

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2n} \ : \ \begin{matrix} F(m_\sigma) + F'(m_\sigma)x + y = 0 \\ |x| \leq r_\sigma\,,\ |y| \leq d_\sigma \end{matrix} \right\} \tag{15}$$

is not empty. Introducing the substitution $\tilde{x} = x + r_\sigma$ and $\tilde{y} = y + d_\sigma$, the slack variables $u, v \in \mathbb{R}_+^n$, and the definition (10), we conclude that the set (15) is not empty if and only if the set

$$\left\{ \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ u \\ v \end{bmatrix} \in \mathbb{R}_+^{4n} \ : \ \begin{matrix} E\,F'(m_\sigma)\tilde{x} + E\,\tilde{y} = E\,(-F(m_\sigma)F'(m_\sigma)r_\sigma + r_\sigma) \\ \tilde{x} + u = 2r_\sigma\,,\ \tilde{y} + v = 2d_\sigma \end{matrix} \right\} \tag{16}$$

is not empty (i.e., has a feasible basis). Now we introduce our final slack variable $t \in \mathbb{R}_+^n$ and the objective function $z = -e^T t$ to find such a feasible basis of (16). Hence, we have the equivalence of the following three statements:

- The linear program (9) has the optimal value 0.
- The set (15) is not empty.
- The set (16) is not empty. □

Theorem 16 *The LP test of order q implies the Taylor test of order q.*

This means that the LP test is stricter and discards more intervals than the Taylor test.

PROOF. Let (9) have the optimal value zero. According to the last statement of the preceding proof, this implies that the set (15) is not empty. Let $[x^T\,,\, y^T]^T$ be a point in this set. Then, using the conditions in (15) and the definition in (11), we have

$$\begin{aligned}|F(m_\sigma)| &\leq |F'(m_\sigma)\, x| + |y| \\ &\leq |F'(m_\sigma)|\, r_\sigma + d_\sigma \\ &= G(|m_\sigma + r_\sigma|) - G(|m_\sigma|) - \sum_{0<\|\alpha\|_1<q} \partial^\alpha \left(G(|m_\sigma|) - |F(m_\sigma)|\right) r_\sigma^\alpha\end{aligned}$$

which implies the Taylor test (8). □

Corollary 17 *The complexity results in Theorem 12 and Corollary 13 hold if we replace the Taylor test with the (stricter) LP test of the same order (for $q > 1$).*

4 End Strategy

A typical feature of the exclusion algorithm is that each zero point causes the generation of several intervals, and therefore in a final step we have to sort out which intervals represent the same zero point. For the purpose of this discussion we assume that the exclusion algorithm started with the initial interval Λ and stopped with the generation of the list Γ_k. We further assume that a constant $C \geq 2$ has been chosen.

Definition 18 *We call two intervals $\sigma, \tau \in \Gamma_k$* close *if and only if $|r_\sigma - r_\tau| \leq C2^{-k} r_\Lambda$. This introduces a graph with nodes Γ_k. Two intervals $\{\,\sigma, \tau\,\}$ form an edge if and only if they are close. This leads to the notion of connected components of Γ_k.*

Note that a choice $C < 2$ makes no sense. A simple consequence of our complexity results is the following

Corollary 19 *Let the assumptions of Theorem 12 hold. We consider the exclusion algorithm either under Taylor tests of order* $q \geq 1$ *or under LP tests of order* $q > 1$*. Then there is a* $C \geq 2$ *such that for sufficiently large* k *each zero point is represented by exactly one connected component in* Γ_k *.*

In this case we can say that the algorithm has isolated all zeros. It is not difficult to write a program that generates such connected components.

Furthermore, as a last step, our codes (optionally) replace each connected component by the smallest interval σ containing it (i.e., by its convex hull), and m_σ is then viewed as an approximation of the zero with error $\pm r_\sigma$.

5 Polynomial Systems

Although exclusion tests are not only applicable to polynomial systems $F(x) = 0$, such special systems are of great interest in many areas of applications. Here the method of choice seems to be a state of the art homotopy continuation method, see, e.g., [12] for a survey of such methods. However, these methods calculate all complex roots, not only real ones, and their numerical effort cannot be reduced by restricting the attention only to real roots contained in a given interval Λ. Exclusion algorithms with efficient tests may have an advantage in this case.

As indicated in Example 5, $F \prec_\infty G$ is easily obtained, and we can set $q = \infty$ in the summation occuring in (8) or (9). This leads to the following tests:

Definition 20 (Polynomial Taylor Test of Order ∞) *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial system and* G *the dominating function described in Example 5. Then for a given interval* $\sigma \subset \mathbb{R}^n$ *we call*

$$\begin{aligned} |F(m_\sigma| \leq & \, G(|m_\sigma + r_\sigma|) - G(|m_\sigma|) - \sum_{0<\|\alpha\|_1} \partial^\alpha \left(G(|m_\sigma|) - |F(m_\sigma)|\right) r_\sigma^\alpha \\ = & \sum_{0<\|\alpha\|_1} \partial^\alpha |F(m_\sigma)| \, r_\sigma^\alpha \end{aligned} \qquad (17)$$

a polynomial Taylor Test of Order ∞*.*

The first form of the right hand side of the preceding inequality may be useful in discarding intervals without a full generation of partial derivatives.

Definition 21 (Polynomial LP Test of Order ∞) *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial system and* G *the dominating function described in Example 5. If* *we replace eq. (11) of the LP test with*

$$d_\sigma := \sum_{1<\|\alpha\|_1} \partial^\alpha |F(m_\sigma)|\, r_\sigma^\alpha, \tag{18}$$

then we call the resulting test a polynomial LP test of order ∞.

Exclusion algorithms using Taylor tests of order ∞ have been programmed in JAVA by the author. All derivatives are generated automatically in run-time, hence the implemention can be used as a *black box* algorithm: the only input required are the coefficients of the polynomials and an initial interval.

However, these JAVA codes still need to be adjusted to allow also for LP tests of order ∞. The numerical results for the polynomial examples below were obtained with preliminary MATLAB experiments.

6 Numerical Examples

In our numerical experiments we observed occasionally numerical instabilities due to the fact that the tested inequalities can become so tight that round-off effects can make them wrong.

To give a simple example, a test such as

$$|F(m_\sigma)| \le G(|m_\sigma + r_\sigma|) - G(|m_\sigma|)$$

may be true in exact arithmetic since in fact equality holds. But small round-off effects may result in

$$|F(m_\sigma)| > G(|m_\sigma + r_\sigma|) - G(|m_\sigma|)$$

which will consequently cause the algorithm to discard the interval σ.

This seems to happen more easily for tests on very high bisection levels, or if the underlying equations and their solutions are quite simple. This is not really surprising.

A fool-proof remedy would be to use techniques from interval analysis to guarantee that results are always rounded up or down so as to stay safely on the correct side of an inequality.

There is another simple remedy which works quite well and which we used. Instead of performing a test on an interval σ, we perform it on a slightly enlarged interval $\tilde{\sigma}$ with $m_{\tilde{\sigma}} = m_\sigma$ and $r_{\tilde{\sigma}} = \phi\, r_\sigma$ where a stabilizing factor

$\phi > 1$ is chosen. Note that a larger ϕ generates a larger list Γ_ℓ of not discarded intervals, so one wants to choose ϕ relatively small.

We present a few numerical examples $F(x) = 0$ for which we find all solutions in a given interval $\Lambda \subset \mathbb{R}^n$. We will always indicate how the dominating function G was constructed for the tests, and we will indicate the choice of the stabilizing factor ϕ. We will then present a table where we list the number of non-discarded intervals $\#\Gamma_\ell$ on bisection level ℓ for various tests.

The presented results are preliminary, and have been extracted from various numerical experiments which the author has performed over the last two years. The experiments were usually done in MATLAB using different computers (laptops, PCs, and a UNIX workstation with and without a MATLAB to C compiler). For the special case of Taylor tests for polynomial systems, also a JAVA code was written. Therefore, the only comparison that seems to make sense is to count the number of tests that were performed. We are aware that not all tests come with the same computational expense. Nevertheless, the given numbers contain some valuable information. A more systematic study of various tests is planned in the future.

Note also that for an implementation of a test of order q we only need to exploit derivatives of F and G up to order $q - 1$. These can be generated symbolically (which we did using the `maple` function in MATLAB), or via automatic differentiation, see, e.g., [13]. In the case of polynomials, any order derivative can be implemented quite easily, see [9].

6.1 Example

The following is a simple example taken from [5]:

$$\begin{aligned}
&2\,\sin(x[1]) + 0.8\,\cos(2\,x[1]) + 7\,\sin(x[2]) - x[1] = 0\\
&4\,\sin(2\,x[1]) + 1.4\,\sin(3\,x[2]) + 3.1\,\cos(2\,x[2]) - x[2] = 0
\end{aligned}$$

The equations are separable, i.e., mixed derivatives vanish. This makes the lower order tests perform relatively well. In fact, we observed numerical instabilities. This led us to use a stabilizing factor $\phi = 1.2$ which we chose uniformly for all tests. The dominating functions for orders 1,2,3 were obtained via the estimates (5),(6),(7), respectively, in combination with Theorem 6. The initial interval is $[-10, 10]^2$.

Level	0	1	2	3	4	5	6	7	8	9	10	11	12
Taylor order 1	1	4	16	64	179	285	370	319	326	335	315	278	286
Taylor order 2	1	4	16	60	149	154	114	99	93	86	84	93	86
Taylor order 3	1	4	16	64	162	167	118	99	91	83	83	93	86
LP order 2	1	4	16	60	132	94	64	60	45	50	46	45	41
LP order 3	1	4	16	64	150	111	62	58	42	49	45	45	41

The authors of [5] claim that there are 27 solutions, but we found 29. They are all regular. The wrong count in [5] is probably due to numerical instabilities which the authors did not take into consideration. All our tests found all 29 solutions and nicely separated them.

6.2 Example

The following three-dimensional polynomial system $F(x) = 0$ has been represented as a general economic equilibrium model in [14]. The functions are taken from Verschelde's web page

`http://www.math.uic.edu/~jan/Demo/rose.html`,

see also [15].

Verschelde reports 136 complex solutions (obtained with a homotopy continuation method), however only 14 are real. They are contained in the interval $[-2, 2]^3$ which we take as an initial interval. We used the polynomial dominance (see Example 5) for all runs, with a factor $\phi = 1$.

Level	0	1	2	3	4	5	6	7	8	9	10	11	12
Taylor order 1	1	8	48	256	658	1594	3884	9288	21052	22304	17144	19526	25354
Taylor order 2	1	8	48	240	542	718	652	244	158	96	76	60	72
Taylor order 3	1	8	48	240	506	342	180	98	80	72	60	54	68
LP order 2	1	8	48	228	486	596	500	143	77	50	30	24	28
LP order 3	1	8	48	228	429	238	98	40	28	24	22	20	24

It should be noted that three of the real solutions are singular, so our complexity results are not valid for order 1. In fact, the preceding table shows that $\#\Gamma_\ell$ does not remain bounded for the test of order one. In this example, the order three tests were considerably superior to the order two tests. All tests of order > 1 found all 14 solutions and nicely separated them.

6.3 Example

The following problem, see [16], is an example from mechanical engineering.

$$\begin{aligned}
&C_1\ (x[3] - \alpha\ \sin(x[1])\cos(x[2])) = 0\\
&C_2\ (x[4] - \alpha\ \cos(x[1])\sin(x[2])) = 0\\
&D_1\ (x[3] - \alpha\ \sin(x[1])\cos(x[2])) - x[3] = 0\\
&D_2\ (x[4] - \alpha\ \cos(x[1])\sin(x[2])) - x[4] = 0
\end{aligned}$$

where

$$\begin{aligned}
&C_1 = \tfrac{1-e^{-2\mu_1}}{2\mu_1}, \quad \mu_1 = 0.1\pi\ , \quad \mu_2 = 0.2\pi\ ,\\
&C_2 = \tfrac{1-e^{-2\mu_2}}{2\mu_2}, \quad D_1 = e^{-2\mu_1}, \quad D_2 = e^{-2\mu_2}, \quad \alpha = 5
\end{aligned}$$

We seek solutions in the interval $[-\pi,\pi]^2 \times [-1.5, 1.5]^2$. It turns out that there are 13, and they are all regular. Note, however, that the equations are unchanged under the actions $x \mapsto P_1 x,\ P_2 x$ with

$$P_1 := \begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P_2 := \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Hence, we need only to seek solutions x with $x[1], x[2] \geq 0$. This reduces our search to $[0,\pi]^2 \times [-1.5, 1.5]^2$ which we take as our initial interval. The reduced problem has 5 solutions, the other solutions are obtained via symmetry. We chose $\phi = 1$. The dominating functions for orders 1,2,3 were obtained via the estimates (5),(6),(7), respectively, in combination with Theorem 6.

Level	0	1	2	3	4	5	6	7	8	9	10	11	12
Taylor order 1	1	16	256	2552	4927	5950	6142	6238	6269	6230	6230	6140	6126
Taylor order 2	1	16	204	556	159	60	40	32	32	32	32	32	32
Taylor order 3	1	16	184	75	32	32	32	32	32	32	32	32	32
LP order 2	1	16	204	446	114	52	36	32	32	32	32	32	32
LP order 3	1	16	171	54	32	32	32	32	32	32	32	32	32

In this example, the order three tests were considerably superior to the order two tests. All tests of order > 1 found all 5 solutions and nicely separated them.

6.4 Example

The four-dimensional polynomial system investigated in this example comes from a planar four-bar design problem, see [17]. The equations were taken from Verschelde's web page

`http://www.math.uic.edu/~jan/Demo/fourbar.html`,

see also [15].

Verschelde reports 36 complex solutions, but only three are real. One real solution is $[0\ 0\ 0\ 0]^T$ (singular), and the other two real solutions are regular and close to each other and contained in $[0.1, 2.2]^4$ which we take as our initial interval. We chose $\phi = 1$. We used the polynomial dominance (see Example 5) for all runs, with a factor $\phi = 1$.

Level	0	1	2	3	4	5	6	7	8	9	10	11	12
Taylor order 1	1	16	256	3570	27574	124824	540507	?	?	?	?	?	?
Taylor order 2	1	16	253	1297	3282	5940	5593	2199	820	651	474	347	290
Taylor order 3	1	16	219	765	1571	2423	1504	520	380	340	296	281	260
LP order 2	1	16	251	1175	2751	3691	2693	682	215	124	56	18	17
LP order 3	1	16	210	595	857	610	198	72	44	26	10	9	5

In this example, the order three tests were considerably superior to the order two tests. Note that our complexity results apply also for order one in this case, but it needs a lot of patience (more than the author had) to confirm this numerically. All tests of order > 1 found both solutions and nicely separated them.

7 Concluding Remarks

From the preceding examples (and other examples that the author calculated) we draw the following conclusions:

- The complexity results Theorem 12, Corollary 13, Corollary 17 are confirmed by our numerical experiments.
- The order one test is usually very inefficient, compared to the other tests, and can be recommended only for very simple problems.
- For more difficult problems, the order three tests are usually superior to order two tests, even when taking into account the increased computational effort per step. The same holds for LP tests versus Taylor tests, but there the difference is not always so pronounced.

- LP tests produce typically fewer non-discarded intervals on high bisection levels than Taylor tests. The explanation is that LP tests take a local linearization of the problem into account and thus can reflect the local structure more accurately in accordance with the Newton-Kantorovitch theorems than the Taylor tests.
- Since higher derivatives of polynomials are readily implemented, it could be advantageous to use *all* derivatives (i.e., order ∞ tests) for polynomial systems. Software (in JAVA) using Taylor tests of order ∞ for polynomial systems has been written by the author and will be modified to include LP tests.

Note, however, that exclusion tests are only of interest for nonlinear systems that possess several zeros, and if there is a need to find them all. Note, also, that we typically want to run exclusion algorithms only until each zero is nicely isolated: If a zero of a nonlinear system is regular and we have a sufficiently close approximation, then, in view of the celebrated Newton-Kantorovitch theorems, traditional locally superconvergent (Newton-like or inexact Newton) methods are the best choice for obtaining precise approximations.

There exist also various globalization methods to find a zero without having a good initial guess, see, e.g., the survey [12]. However, these methods typically do not find all zeros. An exception is mentioned at the beginning of Section 5.

8 Note Added in Proof

After the first submission of this manuscript, one referee pointed out that the LP test described in Section 3 could be modified to reduce a non-discarded interval to a smaller interval: once a solution to the LP problem has been found with $t = 0$, the variables t could be removed from the set of variables, and new objective functions

$$\pm e_j^T x = \pm e_j^T (\tilde{x} - r_\sigma) \ , \qquad \begin{array}{l} \text{for } j = 1, \ldots, n \\ e_j := j\text{-th column of the identity matrix} \end{array}$$

could be maximized to find a (possibly much tighter) restriction $x_* \le x \le x^*$ on the zeros $m_\sigma + x$ in σ. This leads to a reduction

$$\tilde{\sigma} := [m_\sigma + x_* \ , \ m_\sigma + x^*] \subset \sigma$$

of the non-discarded interval σ. Of course, this modification increases the computational cost of an exclusion test, but it might nonetheless lead to drastic improvements in the efficiency of exclusion algorithms, in particular for higher-dimensional and only mildly nonlinear systems. Another aspect is that small intervals which contain one regular zero should reduce drastically in view of the Newton-Kantorovitch theorems. These ideas will be pursued and analyzed elsewhere.

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