

Numerically stable homotopy methods without an extra dimension

EUGENE L. ALLGOWER^{1,3,4,5} and KURT GEORG^{1,2,4}

¹Department of Mathematics, Colorado State University
Ft. Collins, Colorado 80523

²Institut für Angewandte Mathematik, Universität Bonn
53 Bonn, West Germany

³Partially supported by the Office of Naval Research
under grant # URI - ONR N 00014 - 86K 0687

⁴Partially supported by the National Science Foundation
under grant # DMS - 8805682

⁵Partially supported by the Alexander von Humboldt-Stiftung

October 1988

Abstract. We give new versions of the global Newton method and the Kellogg & Li & Yorke method for calculating zero points and fixed points of nonlinear maps, which are numerically stable, but do not require an extra homotopy dimension. In addition, regularity results are established so that predictor-corrector continuation methods will lead to solutions if appropriate boundary conditions are satisfied.

1. Introduction

The global Newton method for calculating zeros of maps and the Kellogg & Li & Yorke method for calculating fixed points of maps have fallen somewhat out of favor vis a vis the global homotopy and linear homotopy methods respectively, because of observed numerical instabilities. The homotopy remedy for overcoming the numerical instability is usually achieved at the mild cost of increasing the dimension of the system by one parameter. Our aim here is to show that by an appropriate reparametrization one may overcome this and some other difficulties without increasing the dimension of the system. In the course of the discussion some striking analogies between the zero point or fixed point problems and aspects of the above-mentioned solution methods will become apparent. Before coming to these aspects, we will give brief reviews of the main ideas of predictor-corrector continuation methods, the global Newton method and the Kellogg & Li & Yorke method.

2. A brief review of predictor-corrector continuation methods

Underlying all of the zero point and fixed point methods we discuss below is the fact that an implicitly defined curve must be numerically traced. Consequently, we briefly review some ideas for doing this task. For readers wishing to learn more of the details we suggest [3], [14] or [18], where also many further references may be found.

Let us make the following

(2.1) HYPOTHESES.

- (1) $H : \mathbf{R}^{N+1} \rightarrow \mathbf{R}^N$ is a C^∞ -map;
- (2) 0 is a regular value of H .

Under (2.1), it is well known (see e.g. [16]) that $H^{-1}(0)$ consists of finitely many smooth curves each of which is homeomorphic either to \mathbf{R} or to the unit circle S^1 . The task now is to numerically trace a curve $c \subset H^{-1}(0)$. For simplicity we can assume that c is parametrized according to arclength s . The tangent $\dot{c}(s)$ to $c(s)$ is given by $\tau(H'(c(s)))$ where H' denotes the Jacobian of H and τ denotes a normalized vector in the kernel of H' :

(2.2) DEFINITION. Let A be an $(N + 1) \times N$ -matrix which has full rank. The **tangent vector** $\tau(A) \in \mathbf{R}^{N+1}$ is uniquely defined by the following three conditions:

- (1) $A \tau(A) = 0$;
- (2) $\|\tau(A)\|_2 = 1$;
- (3) $\det \begin{pmatrix} A \\ \tau(A)^* \end{pmatrix} > 0$.

For this special case the **Moore-Penrose inverse** A^+ is defined by the equation

$$(4) \quad \begin{pmatrix} A \\ \tau(A)^* \end{pmatrix}^{-1} = (A^+, \tau(A)).$$

Here $*$ denotes transposition.

Differentiating the equation $H(c(s)) = 0$ with respect to arclength s , a curve $c(s)$ in $H^{-1}(0)$ can be viewed as the solution of the following autonomous initial value problem

$$(2.3) \quad \begin{aligned} \dot{x} &= \tau(H'(x)); \\ x(0) &\in H^{-1}(0). \end{aligned}$$

The point $x(0)$ is a given starting point, and we assume positive orientation. One could employ an initial value solver on (2.3) to trace $c(s)$. However, it is preferable to use a predictor-corrector continuation method, because contrary to the situation with the usual initial value solvers, it is possible to precisely correct back “orthogonally” to the curve $c(s)$ via the Newton method:

$$(2.4) \quad x_{i+1} = x_i - H'(x_i)^+ H(x_i).$$

This method is locally quadratically convergent, see e.g. [5]. The method consists of the following basic steps:

(2.5) Generic Euler-Newton Method. *comment:*

```

input
  begin
     $x \in \mathbf{R}^{N+1}$  such that  $H(x) = 0$ ;           initial point
     $h > 0$ ;                                       initial steplength
  end;
repeat
  approximate  $A \approx H'(x)$ ;                       approximate Jacobian
   $x := x + h\tau(A)$ ;                               predictor step
  repeat
    approximate  $A \approx H'(x)$ ;                       approximate Jacobian
     $x := x - A^+H(x)$ ;                               corrector loop
  until convergence;
  choose a new steplength  $h > 0$ ;                 steplength adaptation
until traversing is stopped.

```

Numerous variations of the Euler-Newton method are possible e.g.

- modifications of Newton's method can be used i.e. the method of obtaining the approximation $A \approx H'(x)$ may vary;
- various stepsize selection strategies can be incorporated, see e.g. [7], [10–11], [19];
- special points on the curve e.g. where a homotopy parameter attains the value 1, can be approximated by various techniques, see [4, chap. 9] or [10].

3. Remarks on the numerical linear algebra

Let us briefly describe how the numerical linear algebra involved in algorithm (2.5) can be performed. Given an $N \times (N + 1)$ -matrix A which has maximal rank, let us briefly explain how the tangent vector $\tau(A)$ is calculated and how the system

$$(3.1) \quad \left. \begin{array}{l} Aw = b \\ \tau(A)^*w = 0 \end{array} \right\} \iff w = A^+b$$

can be solved for w whenever a vector $b \in \mathbf{R}^N$ is given. For our first case, assume that a QR factorization of A^* is available:

$$A^* = Q \begin{pmatrix} R \\ 0^* \end{pmatrix},$$

where Q is an $(N + 1) \times (N + 1)$ orthogonal matrix and R is a nonsingular $N \times N$ upper triangular matrix. Then the last column of Q , viz. $z := Qe_{N+1}$ satisfies (2.2)(1)–(2), and hence $\tau(A) = \pm z$, according as $\det Q \det R$ is positive or negative, since

$$(3.2) \quad \det \begin{pmatrix} A \\ z^* \end{pmatrix} = \det(A^*, z) = \det Q \det \begin{pmatrix} R & 0 \\ 0^* & 1 \end{pmatrix} = \det Q \det R.$$

Now, $\det R$ is the product of the diagonal elements of R , and its sign is obvious. Also $\text{sign det } Q$ is usually easily obtained. For example, if Givens rotations are used, it is equal to unity. If Householder reflections are used, each reflection changes the sign, and so $\text{sign det } Q = (-1)^p$ where p is the number of reflections which are involved in the factorization of A^* by Householder's method. In any event, the question of determining $\tau(A)$ is easily resolved. Furthermore, the numerical solution of (3.1) is accomplished by a forward solving of $R^*u = b$ and a matrix multiplication

$$w := Q \begin{pmatrix} u \\ 0 \end{pmatrix} .$$

Let us now discuss how in general any linear solver can be incorporated into the continuation methods which we have outlined above, see also [2]. In our situation, a linear solver might be generically described as follows: Given an $N \times (N+1)$ -matrix A and some vector $e \in \mathbf{R}^{N+1}$ which is not yet specified, we have a method for obtaining the solution $x \in \mathbf{R}^{N+1}$ for the linear system

$$(3.3) \quad \begin{aligned} Ax &= y, \\ e^*x &= 0, \end{aligned}$$

whenever $y \in \mathbf{R}^N$ is given. Among such methods might be linear conjugate gradient methods, direct factorization methods etc. The choice of the vector e in (3.3) may be regarded as representing a local parametrization, which usually is changed in the process of numerically traversing a solution curve. Of primary importance in the choice of e is its influence upon the condition of the coefficient matrix in (3.3) viz. we should require that

$$(3.4) \quad \text{cond} \begin{pmatrix} A \\ e^* \end{pmatrix} \approx \sqrt{\text{cond}(AA^*)}$$

are approximately of the same order. Intuitively speaking, the vector e should be as parallel as possible to $\ker A$. A very typical choice for e is the i^{th} co-ordinate unit vector, where the co-ordinate i must be carefully chosen. This leads to deleting the corresponding column and co-ordinate in (3.3). Let us show how the tangent vector $\tau(A)$ and the numerical calculation $w := A^+b$ can be cheaply obtained. For convenience, let us denote by

$$(3.5) \quad x = By$$

the solution operator of (3.3). We emphasize that the $(N+1) \times N$ -matrix B is not explicitly given, but instead we have some way of calculating the **result** $x := By$.

The tangent vector $\tau(A)$ is determined as follows. By its definition B satisfies

$$(3.6) \quad \begin{aligned} AB &= \text{Id} ; \\ e^*B &= 0^* . \end{aligned}$$

If we set

$$(3.7) \quad z := e - BAe ,$$

then it can be seen that

$$(3.8) \quad \tau(A) = \pm \frac{z}{\|z\|}.$$

We note that the cost of calculating $\tau(A)$ requires essentially one calculation of Ae (which is cost free in case $e = e_i$) and one solving of (3.3) i.e. $x := B(Ae)$.

In most applications, the choice of sign in (3.7) will be clear from the context e.g. we take the tangent which has a small angle with a previously obtained tangent along the curve. Occasionally, it may be desirable to explicitly obtain this sign by calculating

$$(3.9) \quad \text{sign det} \begin{pmatrix} A \\ z^* \end{pmatrix} = \text{sign det} \begin{pmatrix} A \\ e^* \end{pmatrix},$$

which may be cheaply calculated for some linear solvers of (3.3).

Using the tangent vector $\tau(A)$ which we already obtained in the previous step, it is readily checked that

$$A^+ = [\text{Id} - \tau(A)\tau(A)^*] B.$$

Hence, once $\tau(A)$ has been obtained, the cost of calculating $w := A^+y$ amounts to one solving of (3.3) i.e. $x := By$, and then calculating $w = x - [\tau(A)^*x]\tau(A)$, which is essentially the cost of one scalar product.

4. The global Newton method

Let us begin by stating the problems we want to study and the underlying assumptions. These assumptions could be considerably relaxed in our discussions below. It is not our aim to make the most general hypotheses. Since we want only to convey the essential ideas and have our discussions simple, we make the following

(4.1) HYPOTHESES.

- (1) $f : \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a C^∞ -map;
- (2) $\Omega \subset \mathbf{R}^N$ is an open bounded set having a smooth connected boundary $\partial\Omega$;
- (3) 0 is a regular value of f .

The global Newton method calculates a zero point of f in Ω . This method has been promulgated by Branin [6] and has found frequent use in scientific applications. Smale [20] has studied this method from a theoretical standpoint and given an existence theorem which we will state below. The method consists of the following steps.

(4.2) GLOBAL NEWTON METHOD.

- (1) Choose a starting point $p \in \partial\Omega$;
- (2) Follow the curve defined by the initial value problem

$$(4.3) \quad \begin{aligned} \dot{x} &= -\lambda(x)f'(x)^{-1}f(x) \\ x(0) &= p \end{aligned}$$

into Ω until a zero point of f is found (**success**) or until the curve exits from Ω without having found a zero point (**failure**).

A number of choices for $\lambda(x)$ are possible. A standard choice is $\lambda(x) = \det f'(x)$. In order to guarantee success of the global Newton method a boundary condition needs to be satisfied. Smale [20] has shown that success is assured under the following

(4.4) BOUNDARY CONDITION. For all $p \in \partial\Omega$

- (1) $f(p) \neq 0$;
- (2) $f'(p)$ is nonsingular;
- (3) the Newton direction $-f'(p)^{-1}f(p)$ is not tangent to $\partial\Omega$ at p .

We note that the factor $\lambda(x)$ in (4.3) allows the overcoming of simple singular points of f in the flow, but nevertheless the evaluation of the right hand side remains numerically unstable. Keller [13] observed that the solution of (4.3) can be obtained in a numerically stable way using his global homotopy method. Independently, Garcia & Gould [8–9] obtained similar results. The global homotopy method consists of the following steps.

(4.5) GLOBAL HOMOTOPY METHOD.

- (1) Choose a starting point $p \in \partial\Omega$;
- (2) Follow the curve defined by the equation

$$(4.6) \quad f(x) - tf(p) = 0$$

into the cylinder $\Omega \times \mathbf{R}$ starting from $(p, 1)$ until the level $t = 0$ is reached (**success**) or until the curve exits from $\Omega \times \mathbf{R}$ without reaching the level $t = 0$ (**failure**).

The set $\{t \mid f(x) = tf(p), x \in \bar{\Omega}\}$ is bounded for $f(p) \neq 0$. Hence, if 0 is a regular value of the map $(x, t) \mapsto f(x) - tf(p)$, then the curve of (4.5) must exit $\Omega \times \mathbf{R}$. As Keller has observed, the relationship between the global Newton method (4.2) and his global homotopy method (4.5) is easily seen by differentiating (4.6) according to arclength to obtain

$$\dot{x} = \frac{\dot{t}}{t} f'(x)^{-1} f(x).$$

Comparing this with (4.3) we see that the x -component of the solution curve for (4.6) and the solution curve for (4.3) are the same — only their parametrizations are different. However, the global homotopy method (4.5) handles singular points of f in a more natural way. It also enabled Keller to show success for method (4.5) i.e. to give a simple proof of Smale’s theorem that the solution curve of (4.2) hits a zero point of f before leaving the domain Ω if the boundary condition (4.4) is satisfied.

Let us define the global homotopy $\tilde{H} : \mathbf{R}^N \times \mathbf{R} \times \partial\Omega \longrightarrow \mathbf{R}^N$ by

$$(4.7) \quad \tilde{H}(x, t, p) := f(x) - tf(p).$$

In order to guarantee that the solution curves given by (4.3) and (4.6) are well defined and smooth, we have to assume that the starting point $p \in \partial\Omega$ is chosen in such a way that the restricted map

$$\tilde{H}(\cdot, \cdot, p) : \mathbf{R}^N \times \mathbf{R} \rightarrow \mathbf{R}^N$$

has 0 as a regular value. However, the proof that this is true for almost all $p \in \partial\Omega$ becomes technically complicated, because p only varies over the surface $\partial\Omega$. Percell [17] gave such a proof.

We now want to give a numerically stable version of the global Newton method. Using the boundary condition (4.4)(1) we obtain from (4.6)

$$(4.8) \quad t = \frac{f(p)^* f(x)}{\|f(p)\|_2^2}.$$

Thus the map (4.7) induces a new map $H : \mathbf{R}^N \times \partial\Omega \rightarrow \mathbf{R}^N$ defined by

$$(4.9) \quad \begin{aligned} H(x, p) &:= f(x) - t(x, p)f(p), \\ \text{where } t(x, p) &:= \frac{f(p)^* f(x)}{\|f(p)\|_2^2}. \end{aligned}$$

Hence for $p \in \partial\Omega$ and $x \in \mathbf{R}^N$, the value $H(x, p)$ is obtained by orthogonally projecting $f(x)$ onto $\{f(p)\}^\perp$. To show that the homotopy method defined by (4.9) is numerically stable and that for almost all $p \in \partial\Omega$ the solution curve $\mathcal{C}_p \subset H^{-1}(0)$ with starting point $x = p$ consists of regular points of H , we use the following partial derivatives which are routine to check. Note that t_x and t_p are row vectors.

(4.10) EQUATIONS.

$$(1) \quad t_x(x, p) = \frac{f(p)^* f'(x)}{\|f(p)\|_2^2};$$

$$(2) \quad H_x(x, p) = f'(x) - f(p)t_x(x, p) = \left(Id - \frac{f(p)f(p)^*}{\|f(p)\|_2^2} \right) f'(x);$$

$$(3) \quad t_p(x, p) = \frac{1}{\|f(p)\|_2^2} (f(x) - 2t(x, p)f(p))^* f'(p);$$

$$(4) \quad t_p(x, p) = -\frac{t(x, p)}{\|f(p)\|_2^2} f(p)^* f'(p) \quad \text{for } H(x, p) = 0;$$

$$(5) \quad H_p(x, p) = -t(x, p)f'(p) - f(p)t_p(x, p);$$

$$(6) \quad H_p(x, p) = -t(x, p) \left(Id - \frac{f(p)f(p)^*}{\|f(p)\|_2^2} \right) f'(p) \quad \text{for } H(x, p) = 0.$$

From (4.8)-(4.10) and since $\|f(p)\|_2 \neq 0$ we see that the evaluations of $H(x, p)$ and $H_x(x, p)$, which would be required for a predictor-corrector tracing of $H(x, p) = 0$, are numerically stable.

(4.11) PROPOSITION. *Let $x \in \mathbf{R}^N$, $p \in \partial\Omega$ be such that $H(x, p) = 0$. Let T_p denote the tangent space of $\partial\Omega$ at p . Then for the Jacobian $H'(x, p) = (H_x(x, p), H_p(x, p))$ we have*

$$H'(x, p)(\mathbf{R}^N \times T_p) = \{f(p)\}^\perp.$$

PROOF: We have two cases to consider.

- (1) $t(x, p) = 0$. In this case $H(x, p) = 0$ implies $f(x) = 0$. Thus x is a regular point of f by (4.1)(3). Then $H_x(x, p)(\mathbf{R}^N) = \{f(p)\}^\perp$ by (4.10)(2).
- (2) $t(x, p) \neq 0$. We prove that $H_p(x, p)(T_p) = \{f(p)\}^\perp$. As a consequence of (4.10)(6) we need only to show the following:

$$(4.12) \quad u \in \mathbf{R}^N, \quad u \perp f(p), \quad u \perp f'(p)(T_p) \quad \implies \quad u = 0.$$

Since $f'(p)$ is nonsingular by (4.4)(2), we have $\text{span}\{f(p), f'(p)(T_p)\} = \mathbf{R}^N$ by (4.4)(3). This proves our claim (4.12).

Hence the assertion of the proposition follows. ■

Now we are prepared to show the following regularity result.

(4.13) THEOREM. *For almost all $p \in \partial\Omega$, the homotopy map $H(\cdot, p) : \mathbf{R}^N \rightarrow \{f(p)\}^\perp$ defined by (4.8)–(4.9) has 0 as a regular value.*

PROOF: Let us choose a fixed $q \in \partial\Omega$ and consider the orthogonal projection $P_q : \mathbf{R}^N \rightarrow \{f(q)\}^\perp$. By the previous proposition there exists a q -neighborhood $U_q \subset \partial\Omega$ such that $P_q H'(x, p)(\mathbf{R}^N \times T_p) = \{f(q)\}^\perp$ for all $p \in U_q$ and $x \in \mathbf{R}^N$ such that $H(x, p) = 0$. By a generalized version of Sard's theorem (see e.g. [1] or [12]), we conclude that $P_q H(\cdot, p) : \mathbf{R}^N \rightarrow \{f(q)\}^\perp$ has 0 as a regular value for almost all $p \in U_q$. Since $H(\mathbf{R}^N, p) = \{f(p)\}^\perp$ and P_q is only an orthogonal projection onto $\{f(q)\}^\perp$, it follows that also $H(\cdot, p) : \mathbf{R}^N \rightarrow \{f(p)\}^\perp$ has 0 as a regular value for almost all $p \in U_q$. Since $\partial\Omega \subset \mathbf{R}^N$ is compact, we can cover $\partial\Omega$ with finitely many U_q 's and the theorem follows. ■

We conclude this section with some brief remarks about implementing a predictor-corrector method for our version of the global Newton method defined by the homotopy (4.9) which does not have an extra dimension. Given a starting point $p \in \partial\Omega$, we fix $N - 1$ basis vectors u_1, u_2, \dots, u_{N-1} of $\{f(p)\}^\perp$, and define $\hat{H} : \mathbf{R}^N \rightarrow \mathbf{R}^{N-1}$ by

$$\hat{H}(x) := \begin{pmatrix} u_1^* H(x, p) \\ \vdots \\ u_{N-1}^* H(x, p) \end{pmatrix}.$$

Then the equation $\hat{H}(x) = 0$ is to be used in the general Euler-Newton method outlined in sections 2–3.

5. The Kellogg & Li & Yorke method

The method of Kellogg & Li & Yorke [15] calculates a fixed point of $f : \bar{\Omega} \rightarrow \bar{\Omega}$ where $\bar{\Omega}$ is a compact convex set in \mathbf{R}^N and f is a smooth map. Analogously to our sketch of the global Newton method we again simplify our discussion by making the following somewhat stronger

(5.1) HYPOTHESES.

- (1) $f : \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a C^∞ -map;
- (2) $\Omega = \{x \in \mathbf{R}^N \mid \|x\|_2 < 1\}$ is the open unit ball;
- (3) 0 is a regular value of $Id - f$;
- (4) $\overline{f(\mathbf{R}^N)} \subset \Omega$.

Let $C := \{x \in \mathbf{R}^N \mid f(x) = x\}$ be the fixed point set of f . Define a map $G : \overline{\Omega} \setminus C \rightarrow \partial\Omega$ by $G(x) = f(x) + \mu(x)(x - f(x))$ where $\mu(x) > 0$ is uniquely defined via the condition $G(x) \in \partial\Omega$. The method consists of the following steps.

(5.2) KELLOGG & LI & YORKE METHOD.

- (1) Choose a starting point $p \in \partial\Omega$;
- (2) trace the curve defined by

$$(5.3) \quad \begin{aligned} G(x) &:= f(x) + \mu(x)(x - f(x)) = p \\ \text{where } \mu(x) &:= \frac{1 - p^* f(x)}{p^*(x - f(x))} \end{aligned}$$

inward into Ω starting from $x = p$, $\mu(p) = 1$ until a fixed point of f is approached (**success**) or until the curve exits from Ω without having found a fixed point (**failure**).

The classical “boundary condition” in this case comes from the Brouwer fixed point theorem i.e. in our case we assume the somewhat stronger condition (5.1)(4). By Sard’s theorem, cf. [16] almost all $p \in \partial\Omega$ are regular values of G , and for such starting points p , Kellogg & Li & Yorke showed the success of method (5.2) under the hypothesis of the Brouwer fixed point theorem.

By examining (5.3) we can see that for x near C the determination of $\mu(x)$ can become numerically unstable and yet this is precisely what one hopes to do viz. approach C via the curve defined by $G(x) = p$. In order to overcome this difficulty we replace $\mu(x, p)$ by $(1 - t(x, p))^{-1}$ in the equation (5.3) to obtain a zero point problem for the following homotopy $H : \mathbf{R}^N \times \partial\Omega \rightarrow \mathbf{R}^N$ defined by

$$(5.4) \quad \begin{aligned} H(x, p) &:= x - p - t(x, p)(f(x) - p), \\ \text{where } t(x, p) &:= \frac{p^*(x - p)}{p^*(f(x) - p)}. \end{aligned}$$

Clearly, for fixed $p \in \partial\Omega$, the range of the restricted map $H(\cdot, p)$ is contained in $\{p\}^\perp$. Much of the following discussion is now similar to that which was made in section 4 for the global Newton method. Hence we may abbreviate some of the remarks. Let us list the partial derivatives to be used below. They are routine to check:

(5.5) EQUATIONS.

- (1) $t_x(x, p) = \frac{p^*}{p^*(f(x) - p)} (Id - t(x, p)f'(x))$;
- (2) $t_p(x, p) = \frac{1}{p^*(f(x) - p)} [(x - 2p)^* - t(x, p)(f(x) - 2p)^*]$;
- (3) $t_p(x, p) = \frac{t(x, p) - 1}{p^*(f(x) - p)} p^*$ for $H(x, p) = 0$;

$$(4) \quad H_x(x, p) = \left(Id - \frac{(f(x) - p)p^*}{p^*(f(x) - p)} \right) (Id - t(x, p)f'(x));$$

$$(5) \quad H_p(x, p) = (t(x, p) - 1)Id - (f(x) - p)t_p(x, p);$$

$$(6) \quad H_p(x, p) = (t(x, p) - 1) \left(Id - \frac{(f(x) - p)p^*}{p^*(f(x) - p)} \right) \quad \text{for } H(x, p) = 0.$$

Since

$$\inf_{x \in \mathbf{R}^N} |p^*(f(x) - p)| > 0$$

by our assumption (5.1)(4), we see from the above equations (5.5) that the evaluations of $H(x, p)$ and $H_x(x, p)$, which would be required for an Euler-Newton tracing of $H(x, p) = 0$ are numerically stable.

(5.6) PROPOSITION. *Let $x \in \mathbf{R}^N$, $p \in \partial\Omega$ be such that $H(x, p) = 0$. Let T_p denote the tangent space of $\partial\Omega$ at p i.e. $T_p = \{p\}^\perp$. Then for the Jacobian $H'(x, p) = (H_x(x, p), H_p(x, p))$ we have*

$$H'(x, p)(\mathbf{R}^N \times T_p) = \{p\}^\perp.$$

PROOF: We have two cases to consider.

- (1) $t(x, p) = 1$. In this case $H(x, p) = 0$ implies $f(x) = x$. Thus by (5.1)(3) x is a regular point of $Id - f$. Then $H_x(x, p)(\mathbf{R}^N) = \{p\}^\perp$ by (5.5)(4).
- (2) $t(x, p) \neq 1$. As an immediate consequence of (5.5)(6) we obtain $H_p(x, p)(T_p) = \{p\}^\perp$.

Hence the assertion of the proposition follows. ■

In a fashion similar to the discussion for theorem (4.13) the above proposition gives us the following

(5.7) THEOREM. *For almost all $p \in \partial\Omega$, the homotopy map $H(\cdot, p) : \mathbf{R}^N \rightarrow \{p\}^\perp$ defined by (5.6)-(5.7) has 0 as a regular value.*

Analogously to the remarks at the end of section 4, we describe how to implement an Euler-Newton method for our version of the Kellogg & Li & Yorke homotopy defined by (5.4). Given a starting point $p \in \partial\Omega$, we fix $N - 1$ basis vectors u_1, u_2, \dots, u_{N-1} of $\{p\}^\perp$, and define $\hat{H} : \mathbf{R}^N \rightarrow \mathbf{R}^{N-1}$ by

$$\hat{H}(x) := \begin{pmatrix} u_1^* H(x, p) \\ \vdots \\ u_{N-1}^* H(x, p) \end{pmatrix}.$$

Then the equation $\hat{H}(x) = 0$ is to be used in the general Euler-Newton method outlined in sections 2–3.

REFERENCES

1. R. Abraham & J. Robbin, "Transversal mappings and flows," W. A. Benjamin, New York, Amsterdam, 1967.
2. E. L. Allgower & C.-S. Chien & K. Georg, *Large sparse continuation problems*, Journal of Computational and Applied Mathematics (to appear).
3. E. L. Allgower & K. Georg, *Predictor-corrector and simplicial methods for approximating fixed points and zero points of nonlinear mappings*, in "Mathematical Programming: The State of the Art," A. Bachem & M. Grötschel & B. Korte editors, Springer Verlag, Berlin, Heidelberg, New York, 1983, pp. 15–56.
4. _____, "Introduction to numerical continuation methods," In preparation.
5. A. Ben-Israel, *A Newton-Raphson method for the solution of systems of equations*, J. Math. Anal. Appl. **15** (1966), 243–252.
6. F. H. Branin, Jr., *Widely convergent methods for finding multiple solutions of simultaneous nonlinear equations*, IBM J. Res. Develop. **16** (1972), 504–522.
7. C. Den Heijer & W. C. Rheinboldt, *On steplength algorithms for a class of continuation methods*, SIAM J. Numer. Anal. **18** (1981), 925–948.
8. C. B. Garcia & F. J. Gould, *A theorem on homotopy paths*, Math. of Op. Res. **3** (1978), 282–289.
9. _____, *Relations between several path following algorithms and local and global Newton methods*, SIAM Review **22** (1980), 263–274.
10. K. Georg, "Zur numerischen Realisierung von Kontinuitätsmethoden mit Prädiktor-Korrektor- oder simplizialen Verfahren," Habilitationsschrift, University of Bonn, 1982.
11. _____, *A note on stepsize control for numerical curve following*, in "Homotopy methods and global convergence," B. C. Eaves & F. J. Gould & H.-O. Peitgen & M. J. Todd editors, Plenum Press, New York, pp. 145–154.
12. M. W. Hirsch, "Differential topology," Springer Verlag, Berlin, Heidelberg, New York, 1976.
13. H. B. Keller, *Global homotopies and Newton methods*, in "Recent advances in numerical analysis," C. De Boor & G. H. Golub editors, Academic Press, New York, London, 1978, pp. 73–94.
14. _____, "Lectures on numerical methods in bifurcation problems," Springer Verlag, Berlin, Heidelberg, New York, 1987.
15. R. B. Kellogg & T. Y. Li & J. A. Yorke, *A constructive proof of the Brouwer fixed point theorem and computational results*, SIAM J. Numer. Anal. **13** (1976), 473–483.
16. J. W. Milnor, "Topology from the differential viewpoint," The University Press of Virginia, Charlottesville, Virginia, 1969.
17. P. Percell, *Note on a global homotopy*, Numer. Func. Anal. and Optim. **2** (1980), 99–106.
18. W. C. Rheinboldt, "Numerical analysis of parametrized nonlinear equations," John Wiley & Sons, New York, 1986.
19. H. Schwetlick & J. Cleve, *Higher order predictors and adaptive steplength control in path following algorithms*, SIAM J. Numer. Anal. **14** (1987), 1382–1393.
20. S. Smale, *A convergent process of price adjustment and global Newton method*, J. of Math. Economics **3** (1976), 1–14.

Keywords. Numerical homotopy methods, global Newton methods, numerical stability, nonlinear systems of equations, zero points, fixed points

1980 *Mathematics subject classifications:* 65H10