

A GENERALIZED FOURIER TRANSFORM FOR BOUNDARY ELEMENT METHODS WITH SYMMETRIES*

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Abstract. We study the solution of linear systems that typically arise in discretizations of boundary value problems on a domain with geometrical symmetries. If the discretization is done in an appropriate way, then such a system commutes with a group of permutation matrices. Recently, algorithms have been developed that exploit this special structure, however these methods are limited to the case that the permutations have no fixed points. Here a new symmetry exploiting algorithm, based on the Fourier transform on the symmetry group, is introduced which is capable of handling fixed points. The techniques developed can also be used to achieve further reductions when the right hand side of the proposed system has symmetries. The approach is illustrated by the boundary element method on an equilateral triangle and on a 3-cube. The reduction technique can also be applied to other solution methods, e.g., finite element methods.

Key words. Fourier transform, symmetry, representation theory, discretization methods

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1. Introduction. Consider solving a boundary value problem defined on a domain with geometrical symmetries. In recent years, several methods have been proposed to exploit this structure for numerical solutions, see, e.g., [14] for symmetry exploiting methods in general, [1, 5, 16] for boundary element methods, [9, 10] for finite element methods, [2, 3, 11, 12, 18] for bifurcation problems, and [19] for structural problems. In this article we address linear systems and eigenvalue problems that arise when the discretization of a boundary value problem is generated in a symmetry preserving way. This usually results in a large $n \times n$ -matrix A which commutes with a permutation representation of the underlying symmetry group Γ , that is,

$$(1) \quad A\Pi_\gamma = \Pi_\gamma A \quad \text{for } \gamma \in \Gamma,$$

where Γ is a subgroup of permutations of the index set $\mathcal{N} := \{1, 2, \dots, n\}$ and Π_γ is a permutation matrix, i.e., $\Pi_\gamma x_i = x_{\gamma^{-1}i}$ for a column vector $x = \{x_i\}_{i=1, \dots, n}$. A matrix with the above property is called Γ -*equivariant*. Definition (1) is equivalent to

$$(2) \quad A_{i,j} = A_{\gamma i, \gamma j} \quad \text{for } \gamma \in \Gamma.$$

An example showing how to obtain symmetry preserving discretizations is given in [5] and at the end of this article.

We will demonstrate how the Generalized Fourier Transformation (GFT) on a finite group Γ can be used to block diagonalize the equivariant matrix A . This yields a significant reduction of the numerical work necessary to solve the linear system $Ax =$

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b or the eigenvalue problem $Ax = \lambda x$. Originally, the application of the GFT for equivariant systems was described in [21], and it was shown there how the method reduces to the discrete Fourier transform when the matrix A is circulant, i.e., equivariant with respect to the cyclic group \mathbb{Z}_n .

However, the discussion there is limited to the case when the group acts without fixed points on \mathcal{N} , i.e., if $\gamma(j) = j$ for some $\gamma \in \Gamma$ and $j \in \mathcal{N}$ always implies $\gamma = e$ (= the unit element of Γ). Unfortunately, this is a severe limitation since fixed points occur quite naturally in finite elements or boundary element methods when elements of higher order are used. The primary objective of this article is to present a modification of the GFT-algorithm to handle the case that Γ is not fixed point free.

The GFT is numerically equivalent to the symmetry exploiting methods described by Georg and Miranda [15, 16], although their method is based on a different approach. An adaptation of this approach for the case that fixed points occur can be found in [4]. This adaptation requires the choice of a basis which depends on the discretization scheme. The advantage of the method proposed here is its independence of the particular application.

Like all other symmetry reduction methods, the GFT depends on the irreducible representations (*irreps*) R of the finite group Γ . These are linear transformations $R(\gamma) : \mathbb{C}^d \rightarrow \mathbb{C}^d$, $\gamma \in \Gamma$ with the additional properties:

- (3) $R(\alpha\gamma) = R(\alpha)R(\gamma)$ for $\alpha, \gamma \in \Gamma$,
- (4) $R(e) = I_d$ (= the identity on \mathbb{C}^d),
- (5) $R(\gamma)V \subset V$ for a subspace $V \subset \mathbb{C}^d$ implies $V = \{0\}$ or $V = \mathbb{C}^d$.

The number d is called the *degree* of the irrep R . We will also use the notation d_R to emphasize the dependence on the irrep. A particularly simple irrep of degree one is the *unit irrep* which assigns the number 1 to each group element.

Two irreps R, \tilde{R} are said to be equivalent if they differ only by a similarity transformation, i.e., there exists a nonsingular matrix T such that for all $\gamma \in \Gamma$ we have $TR_\gamma = \tilde{R}_\gamma T$. An irrep is usually considered significant only up to equivalence.

For the following discussion we need some facts from group representation theory, for more details we refer the reader to any standard text, e.g., [20].

1. The matrices $R(\gamma)$ of the irrep R can be chosen to be unitary, i.e., R has an equivalent unitary irrep. We assume therefore in the following that all irreps are unitary.
2. The number of non-equivalent irreps is finite. We assume \mathcal{R} to be a maximal collection of non-equivalent irreps. Then the following relation holds between their degrees and the group order g :

$$(6) \quad \sum_{R \in \mathcal{R}} d_R^2 = g.$$

3. The following orthogonality relation between two irreps $R, S \in \mathcal{R}$ holds:

$$(7) \quad \sum_{\gamma \in \Gamma} R_{i,j}(\gamma) \overline{S_{p,q}(\gamma)} = \frac{g}{d_R} \delta_{R,S} \delta_{i,p} \delta_{j,q}.$$

In particular, this orthogonality implies

$$(8) \quad \sum_{R \in \mathcal{R}} \frac{d_R}{g} \operatorname{tr} \{R(\gamma)R(\alpha^{-1})\} = \delta_{\gamma,\alpha} \quad \forall \alpha, \gamma \in \Gamma.$$

4. The irreps of many important groups are known and listed in the literature.

The last two equations turn out to be the key to constructing the Fourier transform on the group Γ . The irreps of the dihedral group D_3 are listed in Table 1.

Now let us outline this article. After a brief review of the GFT method without fixed points in Section 2 we develop the new theory for group actions with fixed points in Sections 3, 4 and 5. Section 6 describes the modified GFT-Algorithm in matrix notation. In Section 7 the previous techniques are applied to obtain additional savings when the right hand side has symmetries. Section 8,9 contain two illustrations of our symmetry reduction method with fixed points. We conclude the article with a discussion of the complexity of the method in Section 10.

2. The method without fixed points. If the group acts without fixed points, then the size of the orbits $\{\gamma i : \gamma \in \Gamma\}$ is the group order g for all indices. We call a subset \mathcal{S} of \mathcal{N} which contains exactly one index from each orbit a *selection* of indices.

Each index in \mathcal{N} can be written as the unique product of a group element and an index in the selection \mathcal{S} , i.e., all sets $\{\gamma \mathcal{S}\}_{\gamma \in \Gamma}$ are disjoint and contain the same number of elements. Therefore the group order divides the dimension of A , i.e., $n = mg$. Here m is the cardinality of the selection \mathcal{S} . The indices in \mathcal{N} can be renumbered such that each vector $x \in \mathbb{C}^n$ splits into the components

$$(9) \quad x = (x(\gamma))_{\gamma \in \Gamma}$$

where the components of the block $x(\gamma) \in \mathbb{C}^m$ are given by $x(\gamma)_i = x_{\gamma i}$ for $i \in \mathcal{S}$.

The blocks of the matrix A can be indexed in the same way:

$$(10) \quad A = (A(\alpha, \gamma))_{\alpha, \gamma \in \Gamma}$$

where $A(\alpha, \gamma) \in \mathbb{C}^{m \times m}$ and $A(\alpha, \gamma)_{i,j} = A_{\alpha i, \gamma j}$ for $i, j \in \mathcal{S}$. In this notation, the equivariance of A reads

$$(11) \quad A(\alpha, \gamma) = A(\gamma^{-1}\alpha, e) =: a(\gamma^{-1}\alpha),$$

this demonstrates the block structure of A . Thus the whole matrix is determined by its leading $n \times m$ block column $a = (a(\gamma_1), \dots, a(\gamma_g))^T$. The matrix vector product is a convolution in the following sense:

$$(12) \quad (Ax)(\gamma) = \sum_{\nu \in \Gamma} A(\gamma, \nu) x(\nu) = \sum_{\nu \in \Gamma} a(\nu^{-1}\gamma) x(\nu) =: (a * x)(\gamma).$$

The Fourier transform \hat{x} of $x \in \mathbb{C}^n$ on the finite group Γ is defined in block notation by the following equation:

$$(13) \quad \hat{x}(R)_i = \sqrt{\frac{d_R}{g}} \sum_{\alpha \in \Gamma} R(\alpha^{-1}) x(\alpha)_i \quad \forall i \in \mathcal{S}, R \in \mathcal{R}.$$

Note that $\hat{x}(R)_i$ is a $d_R \times d_R$ -matrix. Thus \hat{x} has $\sum_{R \in \mathcal{R}} m d_R^2 = m g = n$ entries, see Formula (6); i.e., the transform again lies in an n -dimensional vector space. We view $\hat{x}(R)$ as a block column with block entries $\hat{x}(R)_i$ for $i \in \mathcal{S}$, and we denote by $\hat{x}^p(R)$ its p -th column; hence $\hat{x}(R) \in \mathbb{C}^{d_{Rm} \times d_R}$ and $\hat{x}^p(R) \in \mathbb{C}^{d_{Rm}}$. Finally, the vector $\hat{x} \in \mathbb{C}^n$ consists of all columns of all $\hat{x}(R)$.

Since the transformation (13) is linear, it can be written in matrix form, i.e., $\hat{x} = Fx$. The transformation matrix F is unitary by the orthogonality relation (7), and hence the inverse is easily obtained by multiplying with the Hermitian transpose of F , i.e., $x = F^H \hat{x}$. Using Equation (8) it is straightforward to verify that the inversion is equivalent to the following formula, which is more convenient to work with:

$$(14) \quad x(\gamma)_i = \sum_{R \in \mathcal{R}} \sqrt{\frac{d_R}{g}} \operatorname{tr} \{ R(\gamma) \hat{x}(R)_i \} \quad \forall i \in \mathcal{S}, \gamma \in \Gamma.$$

In an analogous fashion, we define the Fourier transform of the $n \times m$ block column a :

$$(15) \quad \hat{a}(R)_{i,j} = \sqrt{\frac{d_R}{g}} \sum_{\gamma \in \Gamma} R(\gamma^{-1}) a(\gamma)_{i,j} \quad \forall i, j \in \mathcal{S}, R \in \mathcal{R}.$$

Note that each entry $\hat{a}(R)_{i,j}$ is again a $d_R \times d_R$ -matrix, and hence $\hat{a}(R) \in \mathbb{C}^{dm \times dm}$.

As usual, the Fourier transformation turns the convolution (12) to a component-wise multiplication, this is demonstrated in the following calculation:

$$\begin{aligned} (a * x)(R)_i &= \sqrt{\frac{d}{g}} \sum_{\alpha \in \Gamma} R(\alpha^{-1}) \left[\sum_{\nu \in \Gamma} a(\nu^{-1} \alpha) x(\nu) \right]_i \\ &= \sqrt{\frac{d}{g}} \sum_{j \in \mathcal{S}} \sum_{\nu \in \Gamma} \sum_{\alpha \in \Gamma} R(\alpha^{-1}) a(\nu^{-1} \alpha)_{i,j} x(\nu)_j \\ &= \sqrt{\frac{d}{g}} \sum_{j \in \mathcal{S}} \sum_{\nu \in \Gamma} \sum_{\beta \in \Gamma} R(\beta^{-1}) R(\nu^{-1}) a(\beta)_{i,j} x(\nu)_j \quad \text{where } \beta = \nu^{-1} \alpha \\ &= \sqrt{\frac{d}{g}} \sum_{j \in \mathcal{S}} \sum_{\beta \in \Gamma} R(\beta^{-1}) a(\beta)_{i,j} \sum_{\nu \in \Gamma} R(\nu^{-1}) x(\nu)_j \\ &= \sqrt{\frac{g}{d}} \sum_{j \in \mathcal{S}} \hat{a}(R)_{i,j} \hat{x}(R)_j \end{aligned}$$

and hence,

$$(16) \quad (a * x)(R)_i = \sqrt{\frac{g}{d}} \sum_{j \in \mathcal{S}} \hat{a}(R)_{i,j} \hat{x}(R)_j \quad \forall R \in \mathcal{R}, i \in \mathcal{S}.$$

The above calculation demonstrates how the matrix-vector product is simplified by the Fourier transform: The sum in (16) is only taken over indices in the selection \mathcal{S} and not over all indices, and moreover the R -th component of the matrix-vector product depends only on the R -th components of the vectors \hat{a} and \hat{x} . Thus an equivariant matrix is block diagonal in the Fourier transformed space.

In matrix notation, we see that the transform $\widehat{A} := FAF^H$ has the block diagonal form

$$(17) \quad \widehat{A} = \text{diag} \left\{ \widehat{A}(R) \right\}_{R \in \mathcal{R}},$$

where each block $\widehat{A}(R) \in \mathbb{C}^{d_{Rm} \times d_{Rm}}$ is repeated d_R -times because the block column $\widehat{x}(R)$ has width d_R . One of the diagonal blocks of \widehat{A} has the following form:

$$(18) \quad \widehat{A}(R) = \sqrt{\frac{g}{d_R}} \begin{pmatrix} \widehat{a}(R)_{1,1} & \cdots & \widehat{a}(R)_{1,m} \\ \vdots & & \vdots \\ \widehat{a}(R)_{m,1} & \cdots & \widehat{a}(R)_{m,m} \end{pmatrix}.$$

The above diagonalization of equivariant matrices is of central importance for the efficient solution of linear systems and eigenvalue problems. Instead of solving the proposed problem directly, computational effort can be saved by solving the problem in the transformed space. More formally, we have:

$$(19) \quad \begin{aligned} Ax = b &\Leftrightarrow F(Ax) = Fb \\ &\Leftrightarrow \widehat{a} * x = \widehat{b} \\ &\Leftrightarrow \widehat{A} \widehat{x} = \widehat{b} \\ &\text{where } x = F^H \widehat{x} \end{aligned}$$

3. Extended Group Actions. How does the previous discussion change when there are fixed points in the group action? In this case the isotropy subgroup

$$\Gamma_i := \{\alpha \in \Gamma : \alpha i = i\}$$

is non-trivial for some indices $i \in \mathcal{S}$. The size of the orbit containing the index i is given by the index of the isotropy subgroup

$$[\Gamma : \Gamma_i] := \#\{\gamma \Gamma_i : \gamma \in \Gamma\}$$

and is no longer constant for all indices as in the previous case. In general, the size of the selection does not divide the dimension of the matrix, but the dimension is the sum of the sizes of the orbits, i.e.,

$$(20) \quad n = \sum_{i \in \mathcal{S}} [\Gamma : \Gamma_i].$$

Thus there is no re-numbering of the indices so that the equivariant matrix has the block structure (10) as in the fixed point free case. Therefore the convolution formula (12) does not hold and our previous discussion cannot be applied when there are fixed points in the group action.

To overcome this difficulty we will extend the underlying vector space so that all fixed points of the group action are removed, and then the results of the previous section are applicable.

We define the extended index set to be the Cartesian product $\Gamma \times \mathcal{S}$ and the action of the group by

$$(21) \quad \gamma \cdot (\alpha, i) = (\gamma\alpha, i) \quad \forall \alpha, \gamma \in \Gamma, i \in \mathcal{S}.$$

Obviously, the new group action has no fixed points. The extended vector space consists of all tuples indexed by $\Gamma \times \mathcal{S}$ and is isomorphic to \mathbb{C}^{gm} . Moreover a vector can be decomposed into blocks as in equation (9).

Define the inclusion $\mathcal{I} : \mathbb{C}^n \rightarrow \mathbb{C}^{gm}$ component wise by

$$(22) \quad [\mathcal{I}(x)(\alpha)]_i := \frac{x_{\alpha i}}{\sqrt{\#\Gamma_i}} \quad \forall \alpha \in \Gamma, i \in \mathcal{S}.$$

Since $\Gamma_{\alpha i} = \alpha\Gamma_i\alpha^{-1}$ holds, all isotropy subgroups of an orbit are conjugate, and therefore they have the same cardinality, i.e.,

$$\#\Gamma_{\alpha i} = \#\Gamma_i \quad \forall \alpha \in \Gamma, i \in \mathcal{S}.$$

Definition (22) implies that the image of the inclusion \mathcal{I} is the n -dimensional linear subspace

$$(23) \quad V := \{x \in \mathbb{C}^{gm} : x(\alpha)_i = x(e)_i \quad \forall \alpha \in \Gamma_i, i \in \mathcal{S}\},$$

and \mathcal{I} is an isomorphism from \mathbb{C}^n onto V .

In the following we want to find the matrix representation of A as a map on V . It turns out that this can be easily done by copying rows and columns and multiplying with appropriate constants. More precisely, the matrix $A^{\mathcal{I}} \in \mathbb{C}^{mg \times mg}$, defined by

$$(24) \quad A^{\mathcal{I}}_{(\alpha,i),(\gamma,j)} := \frac{1}{\sqrt{(\#\Gamma_i)(\#\Gamma_j)}} A_{\alpha i, \gamma j} \quad \forall i, j \in \mathcal{S}, \alpha, \gamma \in \Gamma,$$

has the desired property of rendering the following diagram commutative:

$$(25) \quad \begin{array}{ccccc} \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n & & \\ \downarrow \mathcal{I} & & \downarrow \mathcal{I} & & \\ V & \xrightarrow{A^{\mathcal{I}}} & V & & \\ \downarrow & \nearrow & \downarrow & & \\ \mathbb{C}^{gm} & \xrightarrow{A^{\mathcal{I}}} & \mathbb{C}^{gm} & & \end{array}$$

To show this, let us first verify that $A^{\mathcal{I}}\mathcal{I} = \mathcal{I}A$. Let $x^{\mathcal{I}} = \mathcal{I}(x) \in V$. Then:

$$\begin{aligned} [(A^{\mathcal{I}}x^{\mathcal{I}})(\alpha)]_i &= \sum_{j \in \mathcal{S}} \sum_{\gamma \in \Gamma} A^{\mathcal{I}}_{(\alpha,i),(\gamma,j)} x^{\mathcal{I}}(\gamma)_j \\ &= \sum_{j \in \mathcal{S}} \sum_{\gamma \in \Gamma} \frac{A_{\alpha i, \gamma j}}{\sqrt{(\#\Gamma_i)(\#\Gamma_j)}} \frac{x_{\gamma j}}{\sqrt{\#\Gamma_j}} \quad \text{by (22) and (24)} \\ &= \frac{1}{\sqrt{\#\Gamma_i}} \sum_{j \in \mathcal{S}} \sum_{\gamma \in \Gamma} \frac{1}{\#\Gamma_j} A_{\alpha i, \gamma j} x_{\gamma j} \\ &= \frac{1}{\sqrt{\#\Gamma_i}} \sum_{j \in \mathcal{N}} A_{\alpha i, j} x_j \\ &= (\mathcal{I}Ax)(\alpha)_i. \end{aligned}$$

Furthermore, we show that $A^{\mathcal{I}}\mathbb{C}^{gm} \subset V$. Let $\alpha \in \Gamma_i$ and $x \in \mathbb{C}^{gm}$. Then:

$$\begin{aligned} [(A^{\mathcal{I}}x)(\alpha)]_i &= \sum_{j \in \mathcal{S}} \sum_{\gamma \in \Gamma} A_{(\alpha,i),(\gamma,j)}^{\mathcal{I}} x^{\mathcal{I}}(\gamma)_j \\ &= \sum_{j \in \mathcal{S}} \sum_{\gamma \in \Gamma} \frac{A_{\alpha i, \gamma j}}{\sqrt{(\#\Gamma_i)(\#\Gamma_j)}} \frac{x_{\gamma j}}{\sqrt{\#\Gamma_j}} \\ &= \sum_{j \in \mathcal{S}} \sum_{\gamma \in \Gamma} \frac{A_{e i, \gamma j}}{\sqrt{(\#\Gamma_i)(\#\Gamma_j)}} \frac{x_{\gamma j}}{\sqrt{\#\Gamma_j}} \\ &= [(A^{\mathcal{I}}x)(e)]_i, \end{aligned}$$

and the assertion follows from (23).

Note that in Diagram (25) we did not distinguish between $A^{\mathcal{I}}$ and its restriction on V . This is justified by the preceding properties. The commutativity of the diagram has a number of simple consequences:

1. $A^{\mathcal{I}}$ maps \mathbb{C}^{gm} into V , and hence $A^{\mathcal{I}}$ as a map on \mathbb{C}^{gm} is always singular if the action of Γ is not fixed point free.
2. $A^{\mathcal{I}}$ maps the subspace V into V .
3. $\text{rank} A^{\mathcal{I}} = \text{rank} A$.
4. If A is non-singular, then $A^{\mathcal{I}}$ maps V isomorphically onto V .

Thus, in the non-singular case, the linear system $A^{\mathcal{I}}x^{\mathcal{I}} = \mathcal{I}(b)$ has a unique solution $x^{\mathcal{I}} = \mathcal{I}(x)$ in the subspace V , and moreover, the pre-image x under \mathcal{I} solves $Ax = b$.

The matrix $A^{\mathcal{I}}$ is equivariant with respect to the extended group action defined in (21). We show this by verifying equation (2):

$$A_{(\gamma,i),(\gamma,j)}^{\mathcal{I}} = \frac{1}{\sqrt{(\#\Gamma_i)(\#\Gamma_j)}} A_{\gamma i, \gamma j} = \frac{1}{\sqrt{(\#\Gamma_i)(\#\Gamma_j)}} A_{i,j} = A_{(e,i),(e,j)}^{\mathcal{I}}.$$

Since the group acts without fixed points on the extended index set, it is possible to apply the reduction method discussed in the previous section on the linear system $A^{\mathcal{I}}x^{\mathcal{I}} = b^{\mathcal{I}}$, with $b^{\mathcal{I}} = \mathcal{I}(b)$. Then the solution x of the original system is given by $x^{\mathcal{I}} = \mathcal{I}(x)$.

The problem here is that the extended matrix $A^{\mathcal{I}}$ is singular, since the columns (or the rows) corresponding to non-trivial isotropy subgroups are equal. Since the Fourier transform preserves the rank, the matrix \hat{A} is singular as well. However, when the original matrix is non-singular, there is a unique solution in the subspace V for a right hand side in V . This means for the transformed system that both the solution and the right hand side are in the transformed space \hat{V} . Hence we have to describe how V is transformed. This will enable us to identify the linearly dependent rows and columns in the matrix \hat{A} . Before we do that, let us introduce some new notions in the next section.

4. Invariant Subspaces. The main purpose of this section is to describe the Fourier transform of a vector which has some components that are invariant under the action of a subgroup $\Gamma_0 < \Gamma$, like the isotropy subgroup Γ_i . We begin with a definition.

DEFINITION 4.1. *The invariant subspace $M_R(\Gamma_0)$ of the irrep R and the subgroup Γ_0 is defined by*

$$(26) \quad M_R(\Gamma_0) = \{x \in \mathbb{C}^{d_R} : x = R(\alpha)x \quad \forall \alpha \in \Gamma_0\}.$$

It is easy to find the orthogonal projector onto the invariant subspace, it is given by

$$(27) \quad P_{R,\Gamma_0} = \frac{1}{\#\Gamma_0} \sum_{\alpha \in \Gamma_0} R(\alpha).$$

A straightforward calculation shows that $P_{R,\Gamma_0} : \mathbb{C}^{d_R} \rightarrow \mathbb{C}^{d_R}$ is indeed the orthogonal projector onto the subspace $M_R(\Gamma_0)$. An immediate consequence is that the dimension of the range is given by the trace of the projector, i.e.,

$$(28) \quad m_R(\Gamma_0) := \dim M_R(\Gamma_0) = \text{tr} \{P_{R,\Gamma_0}\}.$$

The columns of the matrix P_{R,Γ_0} span the subspace $M_R(\Gamma_0)$. An orthogonal basis of the range $M_R(\Gamma_0)$ can be found with a slight modification of the Gram-Schmidt procedure over the columns of P_{R,Γ_0} . Or, more stably, a singular value decomposition of P_{R,Γ_0} can be performed. Note, however, that in typical applications the dimension d_R will be relatively small.

Now we are in a position to describe the Fourier transform of the subspace V defined in Equation (23). Definition (22) also implies the equivalent characterization

$$(29) \quad V = \{x \in \mathbb{C}^{g^m} : x(\gamma\alpha)_i = x(\gamma)_i \quad \forall \alpha \in \Gamma_i, \gamma \in \Gamma, i \in \mathcal{S}\}.$$

THEOREM 4.2. *Let the subspace $V \subset \mathbb{C}^{g^m}$ be defined by (23) or (29). Then the Fourier transform of V is given by*

$$(30) \quad \widehat{V} := \{\widehat{x} \in \mathbb{C}^{g^m} : \widehat{x}(R)_i = R(\alpha)\widehat{x}(R)_i \quad \forall \alpha \in \Gamma_i, R \in \mathcal{R}, i \in \mathcal{S}\},$$

that is, each column of $\widehat{x}(R)_i$ lies in the space $M_R(\Gamma_i)$.

Proof. We will show that the Fourier transform $F[V]$ of the space V and the inverse Fourier transform $F^H[\widehat{V}]$ of \widehat{V} satisfy the following inclusions:

$$(31) \quad F[V] \subset \widehat{V},$$

$$(32) \quad F^H[\widehat{V}] \subset V.$$

From (32) it follows immediately that $\widehat{V} \subset F[V]$ and, together with (31), the assertion.

To show (31), consider a vector $x \in V$ and its Fourier transform $\widehat{x} = Fx$. For $i \in \mathcal{S}$, $R \in \mathcal{R}$, and $\alpha \in \Gamma_i$ we have

$$\begin{aligned} \widehat{x}(R)_i &= \sqrt{d_R/g} \sum_{\gamma \in \Gamma} R(\gamma^{-1}) x(\gamma)_i \\ &= \sqrt{d_R/g} \sum_{\gamma \in \Gamma} R(\gamma^{-1}) x(\gamma\alpha)_i \quad \text{by (29)} \\ &= \sqrt{d_R/g} \sum_{\beta \in \Gamma} R(\alpha\beta^{-1}) x(\beta)_i \quad \text{where } \beta = \gamma\alpha \\ &= \sqrt{d_R/g} R(\alpha) \sum_{\beta \in \Gamma} R(\beta^{-1}) x(\beta)_i \\ &= R(\alpha) \widehat{x}(R)_i. \end{aligned}$$

Hence $\hat{x} \in \hat{V}$ by (30).

To show (32), consider a vector $\hat{x} \in \hat{V}$ and an $\alpha \in \Gamma_i$, and denote its inverse Fourier transform by $x = F^H \hat{x}$. We have

$$\begin{aligned} x(\gamma\alpha)_i &= \sum_{R \in \mathcal{R}} \sqrt{d_R/g} \operatorname{tr} \{R(\gamma\alpha) \hat{x}(R)_i\} && \text{by (14)} \\ &= \sum_{R \in \mathcal{R}} \sqrt{d_R/g} \operatorname{tr} \{R(\gamma) R(\alpha) \hat{x}(R)_i\} \\ &= \sum_{R \in \mathcal{R}} \sqrt{d_R/g} \operatorname{tr} \{R(\gamma) \hat{x}(R)_i\} && \text{by (30)} \\ &= x(\gamma)_i && \text{by (14),} \end{aligned}$$

and $x \in V$ by (29). This completes the proof. \square

The Fourier transform does not change the dimension of the subspaces, thus by comparing the dimensions of V with \hat{V} we obtain

$$(33) \quad \sum_{i \in \mathcal{S}} \sum_{R \in \mathcal{R}} d_R m_R(\Gamma_i) = n.$$

By using a calculation similar to the above proof, it is easy to see that the Fourier transform of the space

$$V_i = \{x \in \mathbb{C}^{gm} : x(\gamma\alpha)_i = x(\gamma)_i, x(\gamma)_j = 0 \quad \forall j \in \mathcal{S} \setminus \{i\}, \alpha \in \Gamma_i, \gamma \in \Gamma\},$$

is given by

$$\hat{V}_i := \{\hat{x} \in \mathbb{C}^{gm} : \hat{x}(R)_i = R(\alpha) \hat{x}(R)_i, \hat{x}(R)_j = 0 \quad \forall j \in \mathcal{S} \setminus \{i\}, \alpha \in \Gamma_i, R \in \mathcal{R}\}.$$

Comparing the dimensions of V_i with its transform \hat{V}_i gives

$$(34) \quad \sum_{R \in \mathcal{R}} d_R m_R(\Gamma_i) = [\Gamma : \Gamma_i],$$

which can be viewed as a generalization of (6).

5. Reduction of the Extended System. In this section we will discuss how the results of the previous section can be applied to reduce the linear dependent rows and columns in the transformed extended matrix $\hat{A} = F A^T F^H$ which is given by equation (17). Recall that this matrix is singular, however, there is a unique solution to $\hat{A} \hat{x} = \hat{b}$ when both \hat{x} and \hat{b} are chosen from the subspace \hat{V} .

Consider one (block-) row of the reduced system $\hat{A}(R) \hat{x}(R) = \hat{b}(R)$. From equation (18) this row has the form

$$(35) \quad \sqrt{g/d} \sum_{j \in \mathcal{S}} \hat{a}(R)_{i,j} \hat{x}(R)_j = \hat{b}(R)_i,$$

where a^T is defined as the leading block-column of A^T , see (11). By Theorem 4.2 each column of the blocks $\hat{x}(R)_i$ and $\hat{b}(R)_i$ lies in the invariant subspace $M_R(\Gamma_i)$. Now let

$m_{R,i} = m_R(\Gamma_i) = \dim M_R(\Gamma_i)$, and let $U_R(\Gamma_i) \in \mathbb{C}^{d_R \times m_{R,i}}$ be a matrix whose columns consist of an orthonormal basis of $M_R(\Gamma_i)$, then there are $m_{R,i} \times d_R$ matrices $\xi(R)_i$ and $\eta(R)_i$ such that

$$\begin{aligned}\widehat{x}(R)_i &= U_R(\Gamma_i) \xi(R)_i, \\ \widehat{b}(R)_i &= U_R(\Gamma_i) \eta(R)_i.\end{aligned}$$

It is possible that the dimension of the invariant subspace $M_R(\Gamma_i)$ is zero, in which case $U_R(\Gamma_i)$, $\widehat{x}(R)_i$, $\widehat{b}(R)_i$, $\xi(R)_i$ and $\eta(R)_i$ vanish. If the isotropy subgroup is trivial, then the space $M_R(\Gamma_i)$ is all of \mathbb{C}^{d_R} and hence the obvious choice for an orthogonal basis in $U_R(\Gamma_i)$ is given by the identity matrix I_{d_R} .

Substituting the above representations of $\widehat{x}(R)_i$ and $\widehat{b}(R)_i$ into equation (35) gives a new linear system with right hand side $\eta(R)_i$ and unknowns $\xi(R)_i$:

$$(36) \quad \sqrt{g/d} \sum_{\substack{j \in \mathcal{S} \\ m_{R,j} \geq 1}} U_R(\Gamma_i)^H \widehat{a}(R)_{i,j} U_R(\Gamma_j) \xi(R)_i = \eta(R)_i,$$

where we have used the fact that $U_R(\Gamma_i)^H U_R(\Gamma_i) = I$. The index i must correspond to a nontrivial subspace, i.e., $m_{R,i} \geq 1$.

Setting $\tilde{a}(R)_{i,j} = U_R(\Gamma_i)^H \widehat{a}(R)_{i,j} U_R(\Gamma_j)$, we obtain for one block of the reduced system the matrix

$$(37) \quad \tilde{A}(R) = \sqrt{\frac{g}{d_R}} \begin{pmatrix} \tilde{a}(R)_{1,1} & \dots & \tilde{a}(R)_{1,m} \\ \vdots & & \vdots \\ \tilde{a}(R)_{m,1} & \dots & \tilde{a}(R)_{m,m} \end{pmatrix},$$

where rows and columns corresponding to trivial invariant subspaces have to be cancelled. Equation (36) simplifies to

$$(38) \quad \tilde{A}(R) \xi(R) = \eta(R).$$

These are the subproblems that have to be solved for each $R \in \mathcal{R}$.

If we collect all the block columns $\xi(R)$ into one long column ξ (and analogously for η), then we obtain a block diagonal system

$$\tilde{A} \xi = \eta,$$

where the matrix \tilde{A} resembles the transformed matrix of the fixed point free case, see (17)–(19), namely

$$(39) \quad \tilde{A} = \text{diag} \left\{ \tilde{A}(R) \right\}_{R \in \mathcal{R}},$$

where each block $\tilde{A}(R)$ is repeated d_R -times.

From formula (33) it follows immediately that the size of \tilde{A} equals the size of the original matrix A .

6. Solving Equivariant Systems with Fixed Points. The above discussion leads us to the following description of the general Fourier Transformation (GFT) on the group Γ for solving a linear system $Ax = b$ with an equivariant matrix:

1. Find a selection of indices \mathcal{S} and reorder the indices such that the vector x can be written in the block form (9).
2. (a) Find the irreps of the group.
(b) Find the fixed points of the group action on the indices. For each isotropy subgroup Γ_i and for each irrep determine the matrix $U_R(\Gamma_i)$ whose columns are an orthonormal basis of the subspace $M_R(\Gamma_i)$, see the remarks following (28). Recall that $U_R(\Gamma_i) = I_{d_R}$ for a trivial isotropy subgroup $\Gamma_i = \{e\}$.
3. Extend the right hand side b to $b^{\mathcal{I}} = \mathcal{I}(b)$ and calculate the Fourier Transformation $\hat{b} = Fb^{\mathcal{I}}$.
4. Extend the matrix A to $A^{\mathcal{I}}$ and calculate the blocks $\hat{a}(R)_{i,j}$ according to (15) applied to $A^{\mathcal{I}}$. For each pair of indices $i, j \in \mathcal{S}$ calculate the product $\tilde{a}(R)_{i,j} = U_R^H(i) \hat{a}(R)_{i,j} U_R(\Gamma_j)$ and assemble the matrices in (37).
5. (a) Solve the subproblems (38) of the block diagonalized system.
(b) recover the vector \hat{x} by $\hat{x}(R)_i = U_R(\Gamma_i) \xi(R)_i$.
6. Calculate $x^{\mathcal{I}}$ from \hat{x} using the inverse Fourier Transformation: $x^{\mathcal{I}} = F^H \hat{x}$. The solution x is given by $x^{\mathcal{I}} = \mathcal{I}(x)$.

Let J be the matrix representation of the inclusion map \mathcal{I} with respect to the canonical bases of \mathbb{C}^n and \mathbb{C}^{gm} . Then it is easy to see that the extended matrix is given by $A^{\mathcal{I}} = J A J^H$, and its Fourier transform by $\hat{A} = F J A J^H F^H$. The cancellation of linearly dependent rows and columns described in Section 5 is equivalent to multiplying with the matrix $U \in \mathbb{C}^{mg \times n}$ given by

$$U = \text{diag} \{ \{ U_R(\Gamma_i) \}_{i \in \mathcal{S}} \}_{R \in \mathcal{R}} ,$$

where each diagonal block is repeated according to the dimension d_R and blocks corresponding to trivial subspaces ($m_{R,i} = 0$) have to be deleted.

Hence the reduced matrix \tilde{A} can be written in the form $\tilde{A} = Q A Q^H$ with the matrix Q being $Q = U^H F J$. Since each of the transformations U^H , F and J is length preserving and $Q \in \mathbb{C}^{n \times n}$, it follows that Q is unitary. Thus, also in the case of fixed points, the symmetry reduction method (on A) can be viewed as a unitary similarity transformation, which preserves the eigenvalue structures and is numerically very stable.

7. Symmetries of the Right Hand Side. It is important to remark that the symmetry reduction method described above does not assume that the right hand side of the linear system has symmetries. However, more computational savings are possible when this is the case. We will use the same techniques already developed for fixed points to further reduce the linear system.

The vector $b \in \mathbb{C}^n$ is called Γ_0 -symmetric when there is a subgroup $\Gamma_0 < \Gamma$ so that $\Pi_\alpha b = b$ for all $\alpha \in \Gamma_0$. Thus the extension $\mathcal{I}(b)$ of a Γ_0 -symmetric vector lies in the linear space

$$(40) \quad W := \{ x \in \mathbb{C}^{gm} : x(\alpha\gamma)_i = x(\gamma)_i \quad \forall i \in \mathcal{S}, \alpha \in \Gamma_0, \gamma \in \Gamma \} .$$

The equivariance of the matrix A readily implies that the solution of $Ax = b$ is Γ_0 -symmetric as well. Thus the problem lives in the lower dimensional subspace W and – similar to the discussion for fixed points – we have to study its image under the Fourier transform.

Note that the group elements γ and α are switched in the characterizations of the subspaces V in (29) and W in (40).

THEOREM 7.1. *Let the subspace $W \subset \mathbb{C}^{gm}$ be defined by (40). Then the Fourier transform of W is given by*

$$\widehat{W} := \left\{ \widehat{x} \in \mathbb{C}^{gm} : \widehat{x}(R)_i = \widehat{x}(R)_i R(\alpha), \quad \forall \alpha \in \Gamma_0, R \in \mathcal{R}, i \in \mathcal{S} \right\},$$

that is, each column of $(\widehat{x}(R)_i)^H$ lies in the space $M_R(\Gamma_0)$.

The proof is a modification of the proof of Theorem 4.2.

Taking into account possible fixed points of the group action, we see that the components of the transforms \widehat{x} and \widehat{b} have the following representation

$$\begin{aligned} \widehat{x}(R)_i &= U_R(\Gamma_i) \xi(R)_i U_R^H(\Gamma_0), \\ \widehat{b}(R)_i &= U_R(\Gamma_i) \eta(R)_i U_R^H(\Gamma_0), \end{aligned}$$

where $U_R(\Gamma_0)$ is an orthonormal basis of $M_R(\Gamma_0)$, and $\xi(R)_i, \eta(R)_i$ are $m_R(\Gamma_i) \times m_R(\Gamma_0)$ matrices. Proceeding as in Section 5 yields the following reduced linear systems

$$(41) \quad \widetilde{A}(R) \xi(R) = \eta(R),$$

where

$$\widetilde{A}(R) = \left(U_R^H(\Gamma_i) \widehat{a}(R)_{i,j} U_R(\Gamma_j) \right)_{i,j}$$

for the case that the dimensions of the subspaces $M_R(\Gamma_0)$, $M_R(\Gamma_i)$ and $M_R(\Gamma_j)$ are positive. The matrices $\widetilde{A}(R)$ are unchanged from the case without symmetries of the right hand side. The advantage of considering symmetries is that the number of right hand sides is reduced in the subsystems and fewer linear systems have to be solved.

If b remains invariant under the full symmetry group Γ , then it follows from the orthogonality relation (7) that all projectors $P_{R,\Gamma}$ vanish except for the projector belonging to the unit representation. Thus only one subsystem remains to be solved.

8. An Illustration. In this section we give a simple illustration of the method. As example we choose a boundary value problem on an equilateral triangle T whose symmetry group is the dihedral group $\Gamma = D_3$, and an extremely coarse discretization. We have chosen this domain because it has the simplest possible non-abelian symmetry group. The next section discusses a more complicated group, namely the symmetry of the 3-cube. It is not our goal here to demonstrate the efficiency of the symmetry reduction techniques — we just want to illustrate it; for more serious calculations with high discretizations we refer to [5].

Let us consider the boundary element method for approximating Laplace's equation

$$(42) \quad \Delta u = 0 \quad \text{on } T;$$

$$(43) \quad u = g \quad \text{on } B = \partial T.$$

If the solution u is written in the form

$$u(x) = \frac{1}{2\pi} \int_B \ln|x-y| w(y) \lambda_B(dy),$$

where λ_B denotes Lebesgue measure on B (here: the line element), then it is well-known that the unknown function w solves the integral equation of the first kind

$$\frac{1}{2\pi} \int_B \ln|x-y| w(y) \lambda_B(dy) = g(x), \quad x \in B.$$

A standard approach for solving the above equation is the collocation method, see, e.g., [6]. The solution w is approximated by a linear combination of basis functions $w \approx \sum_{i \in \mathcal{N}} c_i w_i$. If the coefficients c_i are chosen such that the approximation satisfies the integral equation at given collocation points $\{x_i\}_{i \in \mathcal{N}} \subset B$, then this yields the linear system $Ax = b$, defined by

$$A(i, j) = \frac{1}{2\pi} \int_B \ln|x_i - y| w_j(y) \lambda_B(dy),$$

$$x(i) = c_i, \quad b(i) = g(x_i).$$

Let us arrange the collocation points as suggested in Figure 1 and choose piecewise linear basis functions w_i such that

$$(44) \quad w_i(x_j) = \delta_{i,j}.$$

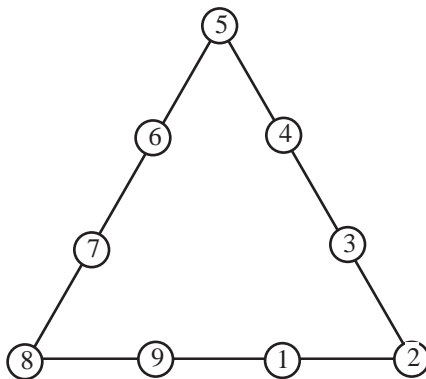


FIG. 1. *The Discretization of an Equilateral Triangle*

The symmetry group of the equilateral triangle is generated by the flip ϕ across the vertical axis and the counterclockwise rotation ρ of 120° . The group induces a group action on the indices, for the generators we have

$$\rho i = [(i + 2) \bmod 9] + 1,$$

$$\phi i = 10 - i.$$

A selection of indices consists of $\mathcal{S} = \{1, 2\}$. For the isotropy subgroups of this selection we obtain

$$\begin{aligned}\Gamma_1 &= \{e\}, \\ \Gamma_2 &= \{e, \rho\phi\}.\end{aligned}$$

Using a change of variables in the integrations, it is straightforward to show that the discretization matrix is equivariant.

To construct the GFT, a complete set of irreducible representations are needed, as given in Table 1.

	e	ρ	ρ^2	ϕ	$\rho\phi$	$\rho^2\phi$
R_1	1	1	1	1	1	1
R_2	1	1	1	-1	-1	-1
R_3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

TABLE 1
Irreducible Representations of D_3

To obtain the symmetry reduction, we have to read the projectors P_{R,Γ_2} from Table 1.

$$P_{R_1,\Gamma_2} = 1, \quad P_{R_2,\Gamma_2} = 0, \quad P_{R_3,\Gamma_2} = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}.$$

Therefore the dimensions of the invariant subspaces are given by

$$m_{R_1,2} = 1, \quad m_{R_2,2} = 0, \quad m_{R_3,2} = 1.$$

Moreover, we have $m_{R,1} = d_R$ for all R , because the isotropy subgroup of index 1 is trivial. Thus the discretized 9×9 -matrix reduces to a block diagonal system of the shape sketched in Figure 2, where the two 3×3 -blocks belonging to representation R_3 are identical.

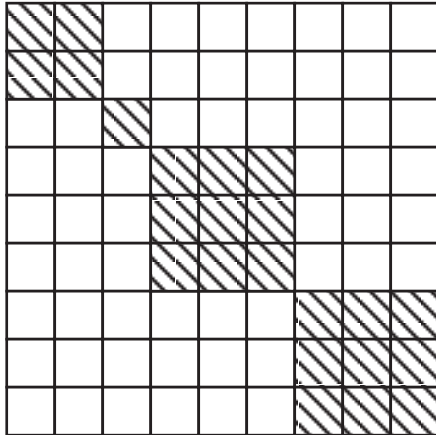
Now suppose that the right hand side is symmetric with respect to reflections across the y -axis, i.e., $b(\phi\gamma)_i = b(\gamma)_i$ for all $i \in \{1, 2\}$ and $\gamma \in D_3$. Hence, $\Gamma_0 = \{e, \phi\}$, the projectors into the invariant subspaces are given by

$$P_{R_1,\Gamma_0} = 1, \quad P_{R_2,\Gamma_0} = 0, \quad P_{R_3,\Gamma_0} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and their dimensions are

$$m_{R_1}(\Gamma_0) = 1, \quad m_{R_2}(\Gamma_0) = 0, \quad m_{R_3}(\Gamma_0) = 1.$$

Thus the reduced system (41) consists of one 2×2 -block for the first representation and one 3×3 -block for the third representation.

FIG. 2. Block diagonalization resulting from the symmetry group D_3

9. Numerical Example. In this section we demonstrate the reduction method discussed in this article on a more involved example. We consider a boundary reformulation (using the double layer potential operator) of Laplace's equation (43), where T now denotes the 3-cube. The integral equation is

$$\left(4\pi - \Omega(x)\right)w(x) - \int_B \frac{\partial}{\partial \nu(y)} \frac{1}{\|x - y\|} w(y) \lambda_B(dy) = 4\pi g(x), \quad x \in B,$$

where $\Omega(x)$ denotes the interior solid angle of B at x , the outer normal is denoted by ν , λ_B denotes Lebesgue measure on B (here: the surface element), and w is the unknown density function on B , see, e.g., Atkinson [7].

We have used Atkinson's collocation package BIEPACK [8] for the discretization and the calculation of the system matrix. This package generates a triangulation of the boundary surface from a given initial triangulation and some other information and sets the collocation points in the vertices and the centers of edges of the triangles. The basis functions are piecewise quadratic and satisfy the Lagrange condition (44) at the collocation points. The triangulation generated for this example consists of 96 triangles and is shown in Figure 3.

The discretized system is a 194×194 -matrix which is dense and equivariant with respect to the symmetry group Γ of the 3-cube which has order 48.

Clearly, a possible selection of indices consists of the nine numbered points as seen in Figure 3. Since the discretization is fairly coarse, almost all indices have non-trivial isotropy subgroups which are listed below.

$$\begin{aligned} \Gamma_1 &\cong D_4 \\ \Gamma_2 = \Gamma_3 = \Gamma_4 &\cong Z_2 \\ \Gamma_5 &\cong D_3 \\ \Gamma_6 &\cong Z_2 \\ \Gamma_7 &= \{e\} \\ \Gamma_8 &\cong Z_2 \\ \Gamma_9 &\cong Z_2 \otimes Z_2 \end{aligned}$$

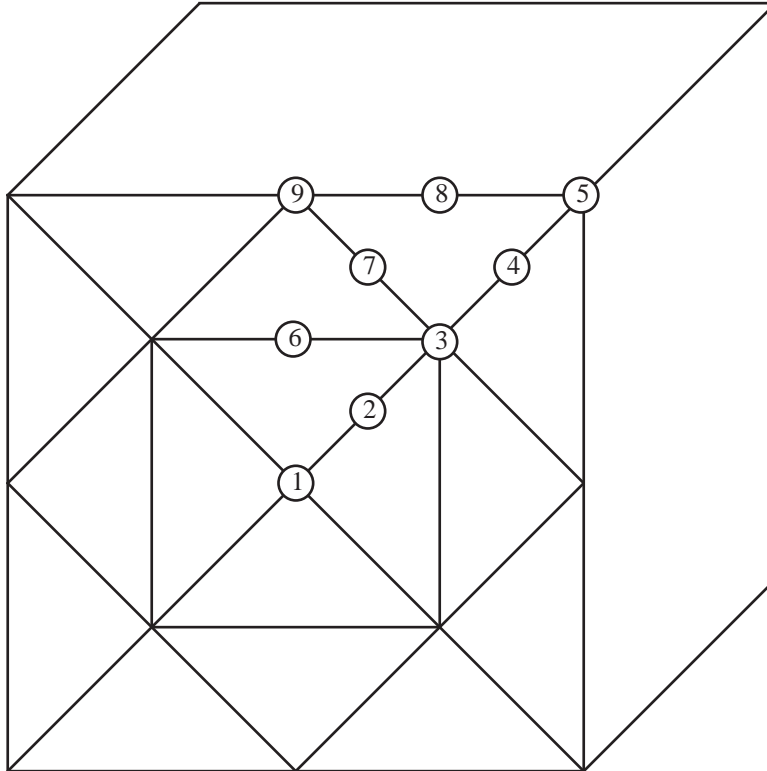


FIG. 3. *Symmetry respecting discretization of the 3-cube, only the triangulation of the front side and the selection of collocation points is shown.*

The block reduction depends on the irreducible representations of Γ . For the case discussed here there are four one-dimensional, two two-dimensional and four three-dimensional non-equivalent irreps. The block diagonalization we have calculated is shown in Figure 4.

10. Cost of the Transformation. We have discussed an algorithm for the block diagonalization of matrices that commute with a permutation representation of a symmetry group. It is obvious that it is more efficient to solve a linear system or an eigenvalue problem in the transformed space, as only a number (which depends on the irreps of the group) of small systems have to be solved. In the following we will show that the cost for this transformation is small as compared to solving the reduced systems. We recall that n denotes the size of the matrix, g the group order, and m the size of the selection. The complexity of the algorithm also depends on the number n_F of indices in the index set \mathcal{N} with a non-trivial isotropy group; clearly n_F is given by

$$(45) \quad n_F = \sum_{\substack{j \in \mathcal{S} \\ \Gamma_j \neq \{e\}}} [\Gamma : \Gamma_j].$$

The transformation of a vector in (13) and (14) can be achieved in $g^2 m$ flops, and the block diagonalization (19) in $g^4 m^2$ flops (floating point operations in the sense of Golub and van Loan [17]). These numbers can be reduced (down to $m \mathcal{O}(g \log g)$ and $m^2 \mathcal{O}(g \log g)$ for abelian groups) when the fast Fourier transform on the symmetry

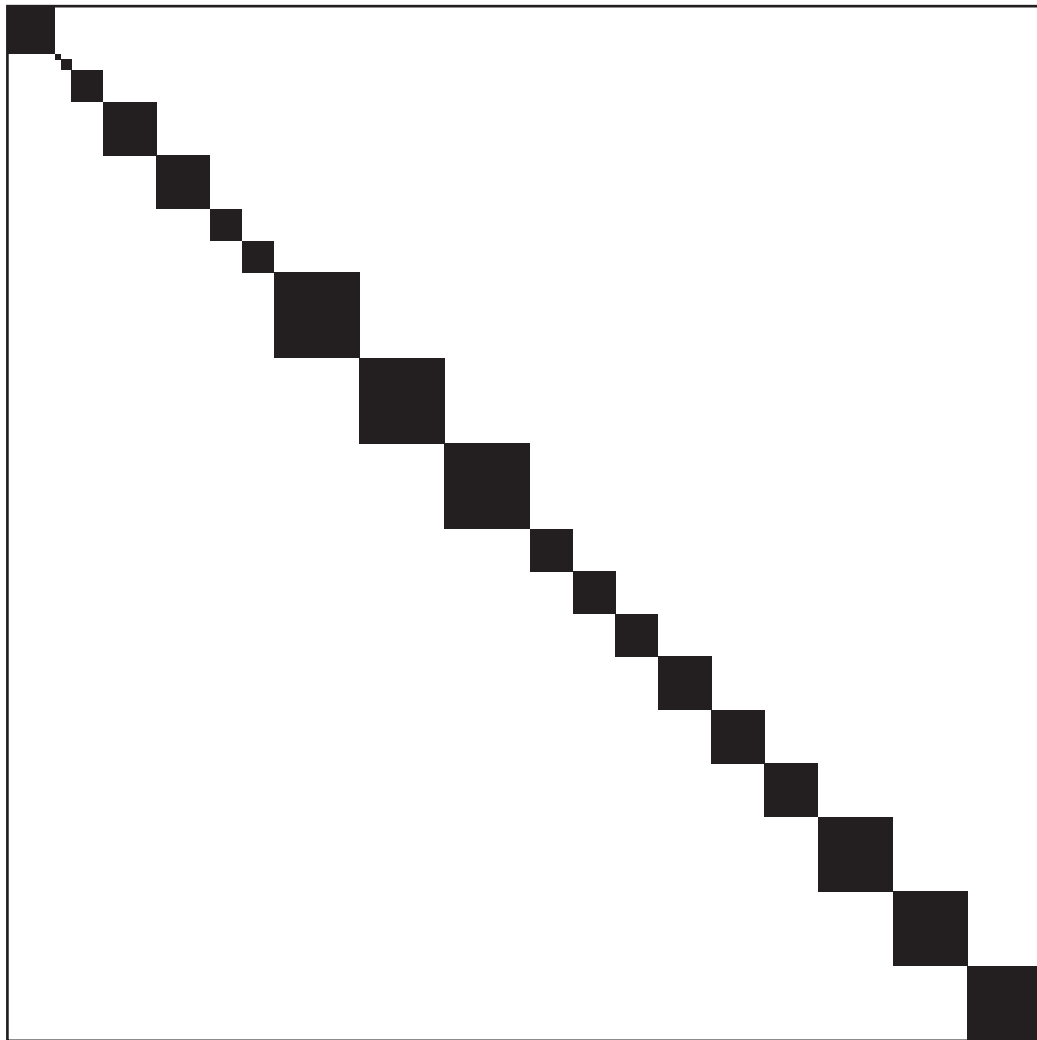


FIG. 4. Block diagonalization of a 194×194 matrix which is equivariant with respect to the symmetry of a 3-cube. The dimensions of the blocks are as follows: 9, 1, 2, 6, 2×10 , 2×6 , 3×16 , 3×8 , 3×10 , and 3×14 .

group is applied; see [13] and the literature cited therein. However, we have not yet investigated fast transforms; the main reason is that for applications we have in mind the group is relatively small ($g \ll n$).

It remains to estimate the extra numerical expense caused by fixed points. For the applications we have in mind the symmetry groups are finite subgroups of the orthogonal matrix groups $\mathbf{O}(2)$ or $\mathbf{O}(3)$ whose irreps usually don't have degrees larger than three (only the icosahedral group has an irrep of degree five). Thus finding the orthogonal bases $U_R(\Gamma_i)$ is trivial and has to be done only for those isotropy subgroups which occur. Also note that several indices $i \in \mathcal{S}$ may have the same isotropy subgroup. The effort to calculate the orthonormal bases is small and does not increase when the mesh of the discretization is refined.

It remains to estimate the effort to calculate the matrix products

$$\tilde{a}(R)_{i,j} = U_R^H(\Gamma_i) \hat{a}(R)_{i,j} U_R(\Gamma_j).$$

These transformations have to be done for all $R \in \mathcal{R}$ but only for indices $i, j \in \mathcal{S}$ with non-trivial isotropy subgroups, because otherwise $U_R(\{e\}) = I_{d_R}$. The right matrix multiplication takes $d_R^2 m_R(\Gamma_j)$ flops, and the left matrix multiplication $d_R m_R(\Gamma_j) m_R(\Gamma_i)$ flops. Adding up all right matrix multiplications we obtain, using equations (45) and (34):

$$\begin{aligned} \sum_{j \in \mathcal{S}: \Gamma_j \neq \{e\}} \sum_{i \in \mathcal{S}} \sum_{R \in \mathcal{R}} d_R^2 m_R(\Gamma_j) &\leq m d_{\max} \sum_{j \in \mathcal{S}: \Gamma_j \neq \{e\}} d_R m_R(\Gamma_j) \\ &= m d_{\max} \sum_{j \in \mathcal{S}: \Gamma_j \neq \{e\}} [\Gamma: \Gamma_j] \\ &= m d_{\max} n_F. \end{aligned}$$

Here d_{\max} denotes the maximal degree of the irreps. A similar calculation (estimating $m_R(\Gamma_i) m_R(\Gamma_j) < d_R m_R(\Gamma_j)$) yields a bound for the left matrix multiplications:

$$\sum_{i \in \mathcal{S}: \Gamma_i \neq \{e\}} \sum_{j \in \mathcal{S}} \sum_{R \in \mathcal{R}} d_R m_R(\Gamma_j) m_R(\Gamma_i) \leq m d_{\max} n_F.$$

Thus the extra expense for fixed points is not larger than $2 m d_{\max} n_F$. It is typical for discretization methods that fixed points are only caused by discretization points lying in some lower dimensional linear subspaces. Therefore we may assume that $n_F \ll n$, and the bound we gave is much smaller than n^2 .

Summarizing, we have found that the numerical effort to transform an equivariant linear system into a block diagonal system compares to approximately one to two matrix-vector multiplications. Let us also mention that formulas (15) and (37) imply that any existing sparsity patterns of the matrix can be used.

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