Estimates for Piecewise Linear Approximations of Implicitly Defined Manifolds

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Abstract. Recently several algorithms have been given for obtaining piecewise linear (PL) approximations of manifolds M which are implicitly defined. We derive error estimates and global approximation results for such algorithms.

1. Introduction

Recently several algorithms have been given for obtaining piecewise linear (PL) approximations of manifolds M which are implicitly defined as a solution set of an equation $H(x) = 0$, where $0 \in \operatorname{range} H$ and $H : \mathbb{R}^{N+K} \to \mathbb{R}^N$ is a smooth map. In this paper we derive error estimates and global approximation results for the algorithms given in [2] and [5]. Analogous results can also be obtained for the algorithms given in [3] and [7]. Many of the given error estimates have been obtained by Saigal[8] for the case $K = 0$. However for $K > 0$, the proofs need a different approach. Although our assumptions can be relaxed in several ways, we make them here in order to hold our discussion to a simple and unified form. Specifically we assume that 0 is a regular value of H i.e. the Jacobian $H'(x)$ has maximal rank for $x \in H^{-1}(0)$, and hence $M = H^{-1}(0)$ is a smooth K-dimensional manifold, see e.g. [6]. Restrictions of space prevent us from fully describing the algorithms. For the present purpose it suffices however to describe their output.

Underlying any PL algorithm is a triangulation T of $\mathbb{R}^{N+K}$ and the PL map $H_T : \mathbb{R}^{N+K} \to \mathbb{R}^N$ which interpolates H on the vertices of T. The algorithm yields the set $H_T^{-1}(0)$ by a complementary pivoting process which generates simplices $\sigma \in T$ which are “transverse” to H. By this we mean that $H_T^{-1}(\vec{\varepsilon}) \cap \sigma \neq \emptyset$ for small $\varepsilon > 0$ where $\vec{\varepsilon} := (\varepsilon, \varepsilon^2, \ldots, \varepsilon^N)^t$. In the special case $K = 0$, such simplices are termed “completely labeled”, see e.g. [1]. It can be seen [4] that for almost all $\varepsilon > 0$, $\vec{\varepsilon}$ is a regular value of H_T, and hence, see [2] and [5], for almost all $\varepsilon > 0$, the family

$$M_T := \{ H_T^{-1}(\vec{\varepsilon}) \cap \sigma \;\big|\; \sigma \in T \text{ transverse } \}$$

is a K-dimensional PL manifold consisting of polytopes. Our aim is to estimate the quality with which M_T approximates M.

In the sequel we use the following notation and assumptions:

(1.1) The triangulation T of $\mathbb{R}^{N+K}$ is locally finite and has mesh size

$$\delta := \sup\{\operatorname{diam} \sigma \;\big|\; \sigma \in T\}\,.$$

The norm $||\cdot||_\infty$ is used throughout.

(1.2) For $\sigma \in T$, the thickness of σ is defined by $\theta(\sigma) := \rho/\delta$ where ρ is the radius of the largest ball in σ which is centered at the barycenter, and δ is the diameter of σ. The thickness of T is defined by $\theta(T) := \inf\{\theta(\sigma) \bigm| \sigma \in T\}$.

(1.3) $||H'(x)^+|| \le \kappa$ for all $x \in M$. Here A^+ denotes the Moore-Penrose inverse of A.

(1.4) $||H''(x)|| \le \alpha$ for all $x \in \mathbb{R}^{N+K}$.

(1.5) For $u, v \in \mathbb{R}^{N+K}$ we define the mean value

$$A(v,u) := \int_0^1 H''(u + \xi(v-u))2(1-\xi)d\xi\,,$$

and hence $||A(v,u)|| \le \alpha$ by (1.4).

In fact, the bounds (1.3)–(1.4) need only to hold in a convex region containing all of the points considered in the sequel. We remark also that it would be sufficient to assume that the Jacobian $H'(x)$ is Lipschitz continuous with constant α.

2. Approximation Estimates

Now we are prepared to state our approximation results. The estimates given in this section are known for the case $K = 0$, see [8].

(2.1) Proposition. *Let T be a triangulation of $\mathbb{R}^{N+K}$ of mesh size δ, and let $H, H_T : \mathbb{R}^{N+K} \to \mathbb{R}^N$ satisfy the assumptions of section 1. Then*

$$||H(x) - H_T(x)|| \le \frac{1}{2}\alpha\delta^2$$

for all $x \in \mathbb{R}^{N+K}$. In particular, $||H(x)|| \le \frac{1}{2}\alpha\delta^2$ if $x \in H_T^{-1}(0)$.

Proof: Suppose x lies in a simplex $\sigma \in T$ having vertices $\{v_i\}_{i=1}^{N+K+1}$. Then there exist $\gamma_i \ge 0$ such that $x = \sum \gamma_i v_i$ and $\sum \gamma_i = 1$. From Taylor's formula we have

$$H(v_i) = H(x) + H'(x)(v_i - x) + \frac{1}{2}A(v_i, x)[v_i - x, v_i - x]\,.$$

Multiplying these equations with the corresponding γ_i, summing and taking norms yields the assertion since $H_T(x) = \sum \gamma_i H(v_i)$ and the estimates (1.4)–(1.5) hold. ■

Our next result yields an estimate for H'.

(2.2) Proposition. *Let $\sigma \subset \mathbb{R}^{N+K}$ be an $(N+K)$-simplex having diameter δ and thickness θ. Let $H_\sigma(x)$ be the affine map which interpolates the values of H at the vertices of σ. Then $||H'(x) - H'_\sigma(x)|| \le \delta\alpha/\theta$ for all $x \in \sigma$.*

Proof: Let $\{v_i\}_{i=1}^{N+K+1}$ be the vertices of σ. From Taylor's formula we have

$$\begin{aligned} H'(x)(v_i - v_j) &= H'(x)(v_i - x) - H'(x)(v_j - x) = H(v_i) - H(v_j) \\ &- \frac{1}{2}A(v_i, x)[v_i - x, v_i - x] + \frac{1}{2}A(v_j, x)[v_j - x, v_j - x] \end{aligned} \qquad \mathbf{(2.3)}$$

for $i, j = 1, 2, \ldots, N+K+1$. From the definition of the PL approximation we have

$$H'_\sigma(x)(v_i - v_j) = H(v_i) - H(v_j)\,. \tag{2.4}$$

Subtracting (2.4) from (2.3) and using (1.4) yields

$$||\big(H'(x) - H'_\sigma(x)\big)(v_i - v_j)|| \le \alpha\delta^2\,. \tag{2.5}$$

By making appropriate convex combinations with (2.5), we obtain

$$||\big(H'(x) - H'_\sigma(x)\big)(u - v)|| \le \alpha\delta^2$$

for all $u, v \in \sigma$. From (1.2) we have that $\{u - v \mid u, v \in \sigma\}$ contains the ball with radius $\theta\delta$ and center zero. Thus the above estimate extends to the corresponding matrix norms

$$\theta\delta||\big(H'(x) - H'_\sigma(x)\big)|| \le \alpha\delta^2,$$

and the assertion follows. ■

The next result shows that every regular zero point of H can be approximated by transverse simplices.

(2.6) PROPOSITION. *Let $H(x) = 0$, and let $\sigma \subset \mathbb{R}^{N+K}$ be an $(N+K)$-simplex having barycenter x, diameter δ and thickness θ. If $\kappa\alpha\delta/\theta < \frac{1}{2}$, then σ is transverse.*

PROOF: We need to show that $H_\sigma(x) = \vec{\varepsilon}$ has a solution $x_\varepsilon \in \sigma$ when $\varepsilon > 0$ is sufficiently small. Since H_σ is an affine map, any "generalized Newton step"

$$x_\varepsilon := x - B\big(H_\sigma(x) - \vec{\varepsilon}\big)$$

will give a solution to $H_\sigma(x) = \vec{\varepsilon}$ if B is a right inverse of H'_σ. We will show that

$$||BH_\sigma(x)|| < \theta\delta\,. \tag{2.7}$$

Then by (1.2) $x_\varepsilon \in \sigma$ for sufficiently small $\varepsilon > 0$, which implies the assertion. By (2.2) we have $||H'(x) - H'_\sigma(x)|| \le \delta\alpha/\theta$. Hence by (1.3) and the hypothesis we have

$$||H'(x)^+\big(H'(x) - H'_\sigma(x)\big)|| \le \frac{\kappa\delta\alpha}{\theta} < \frac{1}{2}\,.$$

We can now define $B := \sum_{i=0}^{\infty}\Big(H'(x)^+\big(H'(x) - H'_\sigma(x)\big)\Big)^i H'(x)^+$. Multiplying the identity

$$H'_\sigma(x) = H'(x)\Big(\,\mathrm{Id} - H'(x)^+\big(H'(x) - H'_\sigma(x)\big)\Big)$$

from the right by B verifies that B is indeed a right inverse of H'_σ. Here and in the following Id denotes the identity matrix. We also easily see that the estimate

$$||B|| \le \frac{\kappa}{1 - \kappa\alpha\delta/\theta} < 2\kappa$$

holds. By (2.1) and the hypothesis we have

$$||H_\sigma(x)|| = ||H_\sigma(x) - H(x)|| \le \frac{1}{2}\alpha\delta^2\,.$$

Combining the last two estimates yields (2.7). ■

The next result shows that the PL manifold $M_T := H_T^{-1}(0)$ approximates $M := H^{-1}(0)$ quadratically in the mesh size.

(2.8) PROPOSITION. *Let* $x \in \mathbb{R}^{N+K}$ *be such that* $\operatorname{dist}(x, M) < (\kappa\alpha)^{-1}$*. Let* $w \in M$ *be a nearest point to* x *i.e.* $||x - w|| = \operatorname{dist}(x, M)$*. If* $H_T(x) = 0$*, then* $||x - w|| \leq \kappa\alpha\delta^2$*.*

PROOF: Since w solves $\min_w \{\, ||x - w|| \;\big|\; H(w) = 0 \,\}$, we have that $(x - w)$ is orthogonal to $\ker\big(H'(w)\big)$. From Taylor's formula we have

$$H(x) - H(w) = H'(w)(x - w) + \frac{1}{2}A(x, w)[x - w, x - w]\,.$$

Since the Moore-Penrose inverse performs the inversion orthogonally to $\ker\big(H'(w)\big)$, we have

$$H'(w)^+ H(x) = x - w + \frac{1}{2}H'(w)^+ A(x, w)[x - w, x - w]\,.$$

Applying (2.1) and the assumptions (1.3)–(1.4) yields

$$||x - w|| \leq \frac{1}{2}\kappa\alpha\delta^2 + \frac{1}{2}\kappa\alpha||x - w||^2 \leq \frac{1}{2}\kappa\alpha\delta^2 + \frac{1}{2}||x - w||\,,$$

and the assertion follows. ■

3. Global Estimates

Up to now our approximation estimates have been of a local nature. In order to obtain global approximation results we need to apply more sophisticated tools and technical arguments. For our purpose the continuous Newton method seems to be a suitable tool. We consider the autonomous ODE

$$\dot{x} = -H'(x)^+ H(x)\,. \tag{3.1}$$

If an initial point x_0 for (3.1) is sufficiently near the manifold $M = H^{-1}(0)$, then the flow initiating at x_0 has an exponentially asymptotic limit $x_\infty \in M$, and the map $x_0 \mapsto x_\infty$ is smooth, see [9]. Analogously, if zero is a regular value of H_T and the meshsize of T is sufficiently small, then we may consider the flow defined by

$$\dot{x} = -H_T'(x)^+ H_T(x)\,. \tag{3.2}$$

By some technical arguments one can show that for an initial point x_0 sufficiently near the manifold $M_T = H_T^{-1}(0)$, the flow (3.2) initiating at x_0 has an exponentially asymptotic limit $x_\infty \in M_T$, and the map $x_0 \mapsto x_\infty$ is absolutely continuous.

(3.3) PROPOSITION. *If* $x_0 \in M$ *and the ratio* δ/θ *is sufficiently small, then there exists a transverse* $\sigma \in T$ *such that* $\operatorname{dist}(x_0, \sigma) \leq \kappa\alpha\delta^2$*.*

SKETCH OF PROOF: Consider the initial value problem (3.2) with initial value x_0. A full initial Newton step is given by $-H_T'(x_0)^+ H_T(x_0)$. From (2.1) we obtain the estimate $||H_T(x_0)|| \leq \frac{1}{2}\alpha\delta^2$. From (2.2) and (1.3) we obtain $||H_T'(x_0)^+|| \approx ||H'(x_0)^+|| \leq \kappa$. Thus a rough bound for the full initial Newton step is given by $\frac{1}{2}\kappa\alpha\delta^2$. Hence to obtain the assertion we estimate $||x_0 - x_\infty||$ by twice this steplength. ■

The algorithms in [2] and [5] generate connected components of the PL manifold M_T. The following result assures that such a connected component approximates the entire manifold M if it is compact and connected.

(3.4) PROPOSITION. *Let zero be a regular value of H and H_T. Let $C \subset M$ be a compact connected subset (possibly all of M). Then for any triangulation T with δ/θ sufficiently small, there is a connected compact PL submanifold $C_T \subset M_T$ such that for every $x_0 \in C$ there is an $x_\infty \in C_T$ for which $||x_0 - x_\infty|| < \kappa\alpha\delta^2$ holds.*

SKETCH OF PROOF: Consider the continuous Newton map $x_0 \in C \mapsto x_\infty \in M_T$ introduced above. Since the continuous image of a compact and connected set is also compact and connected, the PL submanifold

$$C_T := \{M_\sigma \mid \sigma \in T \text{ and } x_\infty \in \sigma \text{ for some } x_0 \in C\}$$

is compact and connected. Now the assertion follows from the estimates in (3.3). ■

It is now clear from the preceding discussion that if M is compact and connected, then a connected component of M_T approximates M globally and quadratically for sufficiently small meshsize, provided the measure of thickness of T stays bounded away from zero.

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