Exploiting Symmetry in BEM[2]

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Abstract: Classical integral operators usually display invariance with respect to orthogonal transformations. If the domain of an operator equation is symmetric with respect to some orthogonal transformations, then appropriate discretizations of the operator equation lead to system matrices which are equivariant with respect to a group of permutations. This property can be exploited to design efficient methods for solving the discrete problem. A generalization of the finite Fourier transform for arbitrary finite groups is used for this purpose. Some nodes of the discretization may be left invariant under some actions. This leads to complications in the numerical treatment which have recently been overcome. Often an operator equation is considered over a domain which is only nearly symmetric with respect to certain orthogonal transformations. Then the Fourier transform techniques on a related symmetrized problem can be used to obtain efficient preconditioners. This is especially important for ill-conditioned problems, e.g., integral equations of the first kind. In this paper we illustrate these techniques with a very simple two dimensional example, and show what are the reductions which take place when the rich symmetries of a three dimensional cube are exploited.

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1. Introduction

In many cases, operator equations exhibit symmetries which can be described by the action of a certain group Γ. Under appropriate discretizions of such equations, these symmetries can be exploited to considerably reduce the computational cost of solving them. We illustrate this reduction for the case of a typical boundary integral equation of the second kind. The approach obviously generalizes to many other cases.

The first symmetry reduction approaches along these lines have been given by Stiefel and Fässler [10, 16] and Bossavit [7] for the case of finite differences and finite elements. The present paper gives a new unified presentation (giving several modified definitions of the Fourier transform) of the material in [2, 3, 5, 11, 12, 13, 17, 19], adapted to the situation and special needs of BEM. The regularization techniques discussed in Section 9 are new. A related but different approach via domain reduction has been given in [1, 8, 9].

We illustrate our ideas with one simple example which, however, exhibits the salient features of symmetry reduction. The reader should bear in mind that the power of the method can be much better exploited on more complicated higher dimensional problems and higher order symmetry groups. Proofs of assertions in this paper have been omitted. They will be presented in the forthcoming paper [4].

A systematic approach to our symmetry reduction method has been implemented and is available via the net:

```
ftp       ftp.math.colostate.edu
login:    anonymous
password: your_email_address
cd        pub/georg
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2. A Simple Example of a Domain Reduction via Symmetry

Let us illustrate the idea of domain reduction with a simple PDE (Poisson equation):

$$\begin{aligned} \Delta u &= f \quad \text{in} \quad D \\ u &= 0 \quad \text{on} \quad \partial D \end{aligned} \tag{1}$$

see Figure 1.

The domain D exibits the symmetries of a square S: it is invariant under the reflection F about the y-axis and under the counterclockwise rotation R by $\pi/2$. By combining these two orthogonal transformations, we obtain the *dihedral group*

$$\Gamma = \mathcal{D}_4 = \{\mathrm{I}, R, R^2, R^3, F, FR, FR^2, FR^3\} \tag{2}$$

which is the symmetry group of the square S and also leaves D invariant.

Next, let us explain how this group acts. If u is a function on D and γ is an orthogonal transformation in Γ, then we define the function γu by

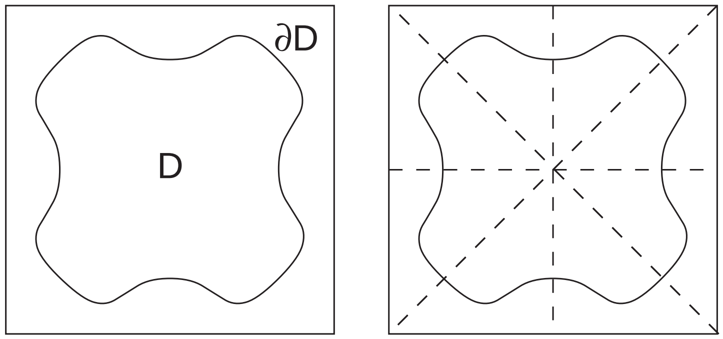

FIGURE 1. A Simple Symmetric Domain

setting $(\gamma u)(x) = u(\gamma^{-1}x)$. We say that the group Γ *acts* on a function space over D.

Now we see that the Laplace operator in (1) is *equivariant* under the actions of Γ: If $\gamma \in \Gamma$, then $\gamma\Delta u = \Delta\gamma u$, which can be easily checked since γ is an orthogonal transformation.

Let us first show how the equivariance under the reflection F can be exploited. In fact, this case is still so simple that it has been exploited in many applications without referring to group theory. We call a function u on D *even* under F if $Fu = u$, and *odd*, if $Fu = -u$. By setting

$$\begin{aligned} u_{\text{even}}(x,y) &= \tfrac{1}{\sqrt{2}}\,u(x,y) + \tfrac{1}{\sqrt{2}}\,u(-x,y) \\ u_{\text{odd}}(x,y) &= \tfrac{1}{\sqrt{2}}\,u(x,y) - \tfrac{1}{\sqrt{2}}\,u(-x,y) \end{aligned}$$

it is immediately seen that

$$u = \tfrac{1}{\sqrt{2}}\,u_{\text{even}} + \tfrac{1}{\sqrt{2}}\,u_{\text{odd}} \tag{3}$$

Now note that even and odd functions obey a natural boundary condition on the symmetry axis (y-axis). This leads immediately to a subdivision of (1) into the two PDEs on the reduced domain D_2

$$\begin{aligned} \Delta u_{\text{odd}} &= f_{\text{odd}} \quad \text{in} \quad D_2 \\ u_{\text{odd}} &= 0 \quad \text{on} \quad \partial D_2 \end{aligned} \tag{4}$$

and

$$\begin{aligned} \Delta u_{\text{even}} &= f_{\text{even}} \quad \text{in} \quad D_2 \\ u_{\text{even}} &= 0 \quad \text{on} \quad (\partial D) \cap D_2 \\ \tfrac{\partial}{\partial \nu}\,u_{\text{even}} &= 0 \quad \text{on} \quad (y\text{-axis}) \cap D \end{aligned} \tag{5}$$

see Figure 2.

More generally, by exploiting all actions of the group Γ, it can be seen that the Poisson equation (1) can be decomposed into 4 PDEs on the reduced domain D_8 and two PDEs on the reduced domain D_4, see Figure 2. The boundaries of D_4 and D_8 that are interior to D are symmetry axes of the group, and they carry homogeneous Dirichlet or Neuman conditions as in equations (4–5). The solutions on the reduced domains are then extended to D via their symmetry property and combined to give the total solution.

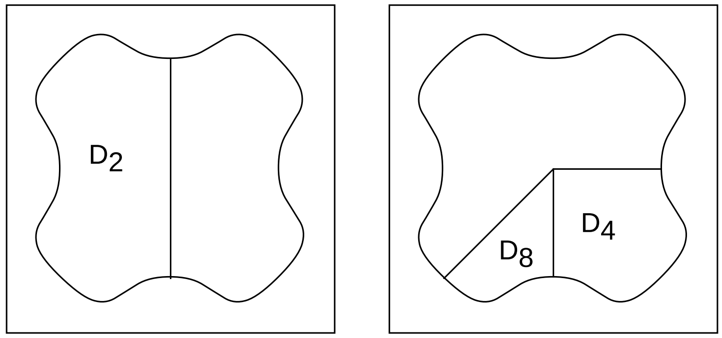

FIGURE 2. Domain Reduction

We note that no assumptions are made concerning the function f. See [1] for a more extensive discussion of domain reduction via symmetry.

3. Symmetry Reduction of Discretization

In the previous section we exploited the action of the symmetry group Γ to obtain PDEs on a reduced domain. Of course, these equations still need to be discretized to find a numerical approximation of the solution.

A different approach is to discretize the equations first and then exploit the action of the group. In particular for BEM, this seems to be the better approach. The rest of the paper will therefore be devoted to describing this method. For illustration we use the same simple example as before, but now look at the Laplace's equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad D \\ u &= f \quad \text{on} \quad \partial D. \end{aligned}$$

The reader should keep in mind, however, that the method is intended for much more complex situations.

We begin with the case that the discretization permits a group action without fixed points, since then the approach is easier to describe. Later we will drop this restriction.

By introducing the density function w on the boundary ∂D via

$$u(x) = \tfrac{1}{2\pi} \int_{\partial D} \tfrac{\partial}{\partial \nu_y} \log \|x - y\| \, w(y) \, d\mu_y \,,$$

where μ indicates the line element, it is well known that the density function satisfies the following boundary integral equation

$$\tfrac{1}{2} w(x) + \tfrac{1}{2\pi} \int_{\partial D} \tfrac{\partial}{\partial \nu_y} \log \|x - y\| \, w(y) \, d\mu_y = -f(x). \tag{6}$$

The simplest discretization of this integral equation consists of a boundary element method with constant elements. In our example, let us divide the square S into $n = 16$ pieces S_i in a regular fashion, i.e., by partitioning the sides of the boundary ∂S into intervals of equal length as indicated in Figure 3. The σ_i are the intersection of the S_i with the boundary ∂D. Furthermore, we choose collocation points P_i as midpoints of the σ_i, defined in an obvious way by further bisecting the intervals in the partitioning of ∂S.

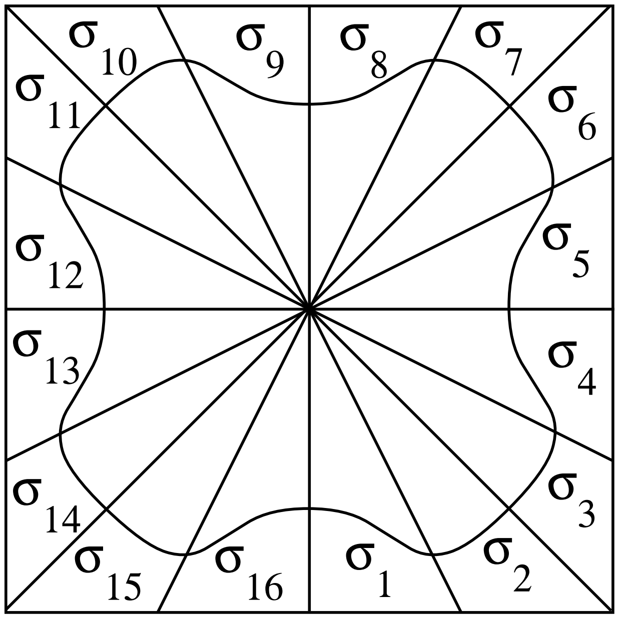

FIGURE 3. Constant Boundary Elements

By introducing the system matrix

$$A(i,k) := \tfrac{1}{2}\delta_k^i + \tfrac{1}{2\pi}\int_{\sigma_k} \tfrac{\partial}{\partial \nu_y} \log ||P_i - y||\, d\mu_y \tag{7}$$

and the column

$$b(i) := -f(P_i),$$

we obtain the collocation equation approximating the BIE (6):

$$Ax = b$$

where $x(i) \approx w(P_i)$.

It is immediately seen that the action of the group Γ induces an action of a permutation group on the indices:

$$\gamma P_i = P_{\gamma i} \quad \text{and} \quad \gamma \sigma_i = \sigma_{\gamma i}.$$

We denote the permutation induced by the group element $\gamma \in \Gamma$ again by γ and also denote the induced permutation group by Γ. For example:

$$\begin{aligned} F : i &\mapsto 17 - i \\ R : i &\mapsto i + 4 \quad \text{mod } 16 \\ FR : i &\mapsto 13 - i \quad \text{mod } 16. \end{aligned} \tag{8}$$

Note that FR is the reflection about the diagonal $y = x$.

4. Equivariant Matrices

It can be checked that the system matrix A defined in (7) satisfies the following two equivalent conditions

$$
\begin{aligned}
(9) \qquad & A(\gamma i, k) = A(i, \gamma^{-1} k), && \text{for all } i, k \text{ and all } \gamma, \\
& A(\gamma i, \gamma k) = A(i, k), && \text{for all } i, k \text{ and all } \gamma.
\end{aligned}
$$

If (9) holds we say that the matrix A is *equivariant* with respect to the permutation group Γ.

For the previous example, equivariance is easily checked. More generally, if a discretization respects the symmetry structure of the operator equation, and if the kernel of the integral operator only depends on the distance between x and y, then the system matrix typically is equivariant. Hence, the symmetry reduction method presented here applies to many classical equations.

It follows immediately that for calculating and storing the matrix A, we only need to handle a selected number of columns (or rows) of A. By a *selection* of indices we mean a minimal subset $\mathbb{S} \subset \{ 1, 2, \ldots, n \}$ such that each index $i \in \{ 1, 2, \ldots, n \}$ can be written as

$$
(10) \qquad i = \gamma k \quad \text{with} \quad \gamma \in \Gamma \text{ and } k \in \mathbb{S}\,.
$$

In the case of our simple example, see Figure 3, we have $n = 16$, and $\mathbb{S} = \{ 1, 2 \}$ could be chosen as a selection of indices.

The present situation is particularly easy to handle since the action of the permutation group Γ is *fixed point free*: If $\gamma i = i$ for any $\gamma \in \Gamma$ and any $i \in \{ 1, 2, \ldots, n \}$, then $\gamma = \mathrm{I}$. This implies that in (10) each $i \in \{ 1, 2, \ldots, \mathrm{n} \}$ uniquely determines γ and k.

5. Generalized Fourier Transform: Fixed Point Free Case

In order to introduce a generalization of the finite Fourier transform, we need to introduce the following concepts. A *representation* of the group Γ is a map r which associates with each group element γ a unitary matrix $r(\gamma)$ such that $r(\mathrm{I}) = \mathrm{I}$ and $r(\gamma_1 \gamma_2) = r(\gamma_1) r(\gamma_2)$. The *dimension* $\dim r$ refers to the size of the unitary matrices. The representation is called *irreducible* if it has no proper invariant subspaces (uniformly in γ). Two representations are called *equivalent* if they differ only by a similarity transformation (uniformly in γ). A maximal set of non-equivalent irreducible representations is called a *complete list* $\mathcal{R}$ of representations for Γ. It turns out that this is a very important characteristic of a group, and for many groups which are important in applications a complete list of irreducible representations can be found in standard algebra books such as [15]. Some complete lists for symmetry groups have been implemented in our software, see [14].

It is important to realize that for the applications we have in mind (i.e., symmetry groups of three dimensional geometrical objects), the dimension of an irreducible representation is always 1, 2, or 3, with only one exception: the symmetry group of the icosahedron has one irreducible representation

γ	r_1	r_2	r_3	r_4	r_5
1	1	1	1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
R	1	1	-1	-1	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
R^2	1	1	1	1	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
R^3	1	1	-1	-1	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
F	1	-1	1	-1	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
FR	1	-1	-1	1	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
FR^2	1	-1	1	-1	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
FR^3	1	-1	-1	1	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

TABLE 1. Irreducible representations for $\mathcal{D}_4$

of dimension 5. In any event, we want to emphasize that the dimension of r is always very small.

For our special case that Γ is the symmetry group of the square (2), a complete list of irreducible representations is given in Table 1. Note that there are four irreducible representations of dimension 1, and one of dimension 2. Also note that for this special case all numbers are real (as opposed to complex). The latter would be different for the symmetry group of an equilateral triangle.

The following two results can be found in any textbook on group representation theory: Let $\mathcal{R}$ be a complete list of representations for Γ. For fixed $r \in \mathcal{R}$ and $1 \leq i, j \leq \dim r$ we can view $r_{i,j}(\gamma)$ as a column in $\mathbb{C}^{|\Gamma|}$. Recall that $|\Gamma|$ denotes the order (i.e., the number of elements) of Γ. Then

$$\sqrt{\tfrac{\dim r}{|\Gamma|}}\; r_{i,j}(\gamma) \quad \text{for } r \in \mathcal{R} \text{ and } 1 \leq i, j \leq \dim r$$

is an orthonormal basis of $\mathbb{C}^{|\Gamma|}$. Furthermore,

$$\sum_{r \in \mathcal{R}} d_r^2 = |\Gamma|.$$

Let us refer to these facts as the *orthogonality relations* for groups. As an example, the reader may check these orthogonality relations in Table 1.

We are now in a position to define the generalized Fourier transform for the case that the action of the permutation group Γ is fixed point free: Given a column $x \in \mathbb{C}^n$, we define the Fourier transform

$$\hat{x}(r,k) := \sqrt{\tfrac{\dim r}{|\Gamma|}} \sum_{\gamma\in\Gamma} x(\gamma k)\, r(\gamma^{-1}) \quad \text{for } r \in \mathcal{R} \text{ and } k \in \{\, 1, 2, \dots, \mathrm{n} \,\}. \tag{11}$$

Note that $\hat{x}(r,k)$ is a matrix that has the same size as r, hence it is a number only for the case that the dimension of r is 1.

The following symmetry relation holds for the Fourier transform:

$$\hat{x}(r, \gamma k) = r(\gamma)\hat{x}(r,k). \tag{12}$$

Therefore, it is sufficient to describe $\hat{x}(r,\gamma k)$ for indices k in a selection $\mathbb{S}$, since the other components are generated automatically by the latter formula.

For our method also the inverse Fourier transform is of importance:

$$x(k) = \sum_{r\in\mathcal{R}} \sqrt{\tfrac{\dim r}{|\Gamma|}}\ \operatorname{trace} \hat{x}(r,k). \tag{13}$$

This formula can be obtained by exploiting the orthogonality relations.

Let us look at a very simple example. If the group $\Gamma = \{\, \mathrm{I}, F \,\}$ is generated by just one reflection F, and if F acts on $\mathbb{R}^8$ via the permutation $F : k \mapsto 9 - k$, then a complete list of irreducible representations is given in

γ	r_1	r_2
I	1	1
F	1	-1

,

and the Fourier transform

$$\begin{aligned} \hat{x}(r_1,k) &= \tfrac{1}{\sqrt{2}}\, x(k) + \tfrac{1}{\sqrt{2}}\, x(9-k) \\ \hat{x}(r_2,k) &= \tfrac{1}{\sqrt{2}}\, x(k) - \tfrac{1}{\sqrt{2}}\, x(9-k) \end{aligned}$$

amounts to nothing else but an even and odd decomposition as in (3).

6. The Symmetry Reduction Method: Fixed Point Free Case

Recall that we were considering a linear system $Ax = b$ where the system matrix A was equivariant with respect to Γ. If the action of the permutation group Γ is fixed point free, then it can be shown that we obtain the following equation for the Fourier transforms:

$$\sqrt{\tfrac{|\Gamma|}{\dim r}} \sum_{l\in\mathbb{S}} \hat{A}_l(r,k)\ \hat{x}(r,l) = \hat{b}(r,k) \quad \text{for } k \in \mathbb{S} \text{ and } r \in \mathcal{R}, \tag{14}$$

where A_l denotes the l-th column of A.

Note that the index r in the above system is not used in the summation, hence the system has a block diagonal structure: For each $r \in \mathcal{R}$, we have a linear system of equations in the transformed unknowns $\hat{x}(r,l)$. In fact, since the components $\hat{x}(r,l)$ are small matrices of the size of r, we can view the system as block structured, where each block has the size of r.

It can be shown that the number of unknowns does not change (of course). However, the new system can be solved with much less computational expense, since it has a block diagonal structure. We also note that it can be shown that all blocks are invertible if and only if A is invertible.

Going back to our simple example of Figure 3, the original BEM generated a system $Ax = b$ with 16 unknowns. Recall that in this case we have four irreducible representations of dimension 1 and one of dimension 2. After the Fourier transformation, the system splits into four systems of the form

$$\begin{bmatrix} \hat{a} & \hat{a} \\ \hat{a} & \hat{a} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \hat{b} \end{bmatrix}$$

and one system of the form

$$\begin{bmatrix} \hat{a} & \hat{a} & \hat{a} & \hat{a} \\ \hat{a} & \hat{a} & \hat{a} & \hat{a} \\ \hat{a} & \hat{a} & \hat{a} & \hat{a} \\ \hat{a} & \hat{a} & \hat{a} & \hat{a} \end{bmatrix} \begin{bmatrix} \hat{x} & \hat{x} \\ \hat{x} & \hat{x} \\ \hat{x} & \hat{x} \\ \hat{x} & \hat{x} \end{bmatrix} = \begin{bmatrix} \hat{b} & \hat{b} \\ \hat{b} & \hat{b} \\ \hat{b} & \hat{b} \\ \hat{b} & \hat{b} \end{bmatrix}.$$

Here the symbols $\hat{a}$ and $\hat{b}$ denote the various entries that are obtained via the Fourier transformation of the matrix A and of the right hand side b, respectively, and the $\hat{x}$ denote the various unknowns to be solved for. Note again that we still have 16 unknowns in this example. Table 2 describes the symmetry reduction in a schematic way. The number n of elements has to be divisible by $|\Gamma| = 8$.

Total number of original unknowns	$n = 16$
Total number of transformed unknowns	$n = 16$
Irreducible representation producing a subsystem	Size of transformed unknowns in this subsystem
$r_1 \ldots, r_4$	$(n/8 , 1) = (2 , 1)$
r_5	$(n/4 , 2) = (4 , 2)$

Table 2. Symmetry Reduction for $\mathcal{D}_4$ Without Fixed Points

Let us now summarize our method of symmetry reduction: Suppose that a discretization of a boundary integral equation with geometric symmetries leads to a linear system of equations $Ax = b$ with the (full) system matrix of size (n, n) which is equivariant under a group of permutations Γ obtained from a group of symmetries (orthogonal transformations). Assume further that the group of permutations Γ acts fixed point free. Then the method consists of the following steps:

Method 1.

1. Generate a selection of columns of the system matrix A, and generate the column b.
2. Perform the Fourier transform (11) on the selected columns of A and on b.
3. Solve the reduced systems of linear equations (14), one system for each irreducible representation $r \in \mathcal{R}$.
4. Use the inverse Fourier transform (13) and the symmetry relations (12) to retrieve the unknowns x.

As a general rule, the larger the symmetry group, the greater is the reduction in computational expense. In [5] we reported a dramatric reduction in computational expense for the case of the symmetry group of a three dimensional cube associated with a three dimensional exterior boundary value problem solved by BEM. In this case, the discretization was performed via collocation with constant elements which led to a fixed point free permutation group of order 48. In the next sections we will see how to handle the case where fixed points are present.

7. Generalized Fourier Transform: General Case

If a BEM discretization is performed not with constant but instead with higher order elements, then this naturally leads to permutation groups Γ that have fixed points. Let us introduce the so-called *isotropy subgroups* which leave a certain index i fixed:

$$\Gamma_i := \{ \gamma \in \Gamma \, : \, \gamma i = i \} . \tag{15}$$

As an example, we now consider a piecewise linear BEM collocation method for (6). For simplicity, let us assume that the square S has sides of length 4, and let s denote a counter clockwise arclength parameter on ∂S which counts modulo 16, such that $s = 3$ indicates the lower right position, etc., see Figure 4.

Let φ indicate the map that associates with each point on ∂S the corresponding point on ∂D obtained by intersecting with the line going through the center. Then $P_i = \varphi(i)$ are the collocation points of the method, and basis functions τ_i are defined on ∂D via

$$\tau_i(\varphi(s)) := \begin{cases} 1 - |i - s| & \text{for } |i - s| \le 1, \\ 0 & \text{else.} \end{cases}$$

Again an element γ of the group Γ defines a permutation of indices via

$$\gamma P_i = P_{\gamma i} \quad \text{and} \quad (\gamma \tau_i)(x) := \tau_i(\gamma^{-1} x) = \tau_{\gamma i}(x)$$

We again denote the permutation induced by the group element $\gamma \in \Gamma$ by γ. For example:

$$\begin{array}{rcll} F : i & \mapsto & 18 - i & \mod 16 \\ R : i & \mapsto & i + 4 & \mod 16 \\ FR : i & \mapsto & 14 - i & \mod 16. \end{array} \tag{16}$$

Note the difference between (8) and (16) for the reflections F and FR.

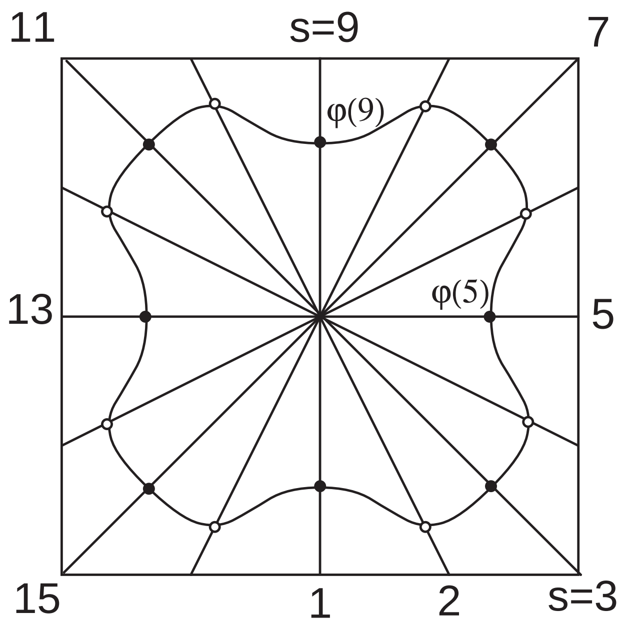

FIGURE 4. Collocation Points Fixed Under Symmetry

The even numbered collocation points (white dots) are fixed only under the identity, however the odd numbered collocation points (black dots) have non-trivial isotropy subgroups, namely

$$\Gamma_1 = \Gamma_9 \ = \{\, \mathrm{I}, F \,\} \qquad \Gamma_3 = \Gamma_{11} = \{\, \mathrm{I}, FR^3 \,\}$$
$$\Gamma_5 = \Gamma_{13} = \{\, \mathrm{I}, FR^2 \,\} \qquad \Gamma_7 = \Gamma_{15} = \{\, \mathrm{I}, FR \,\}$$

generated by the four possible reflections, respectively. For example, P_{11} is invariant under the reflection FR^3.

We again introduce a corresponding system matrix

$$A(i,k) := \tfrac{1}{2}\delta_k^i + \tfrac{1}{2\pi}\int_{\partial D} \tfrac{\partial}{\partial \nu_y} \log ||P_i - y||\, \tau_k(y)\, d\mu_y \tag{17}$$

and the column

$$b(i) := -f(P_i)$$

to obtain the collocation equation

$$Ax = b$$

where $\sum_i x(i)\tau_i \approx w$ approximates the solution of (6). It is again easy to show that A is equivariant in the sense of (9). This means that only a selected number of columns of A needs to be generated, the rest is automatic.

Here a selection of indices $\mathbb{S}$ is defined as in (10). In the example of Figure 4, the set $\mathbb{S} = \{\, 1, 2, 3 \,\}$ would be a reasonable selection.

However, the decomposition of (10) is not unique any more, and in general only

$$|\,\mathbb{S}\,| \cdot |\Gamma| \geq n$$

holds, with strict inequality if a fixed point exists. This complicates the Fourier transform considerably, and in the following sections we are going to introduce two different ways for accommodating our method to the case that fixed points occur.

8. Handling Fixed Points via Projections

A first approach is to introduce for each irreducible representation in $\mathcal{R}$ and each index k the following projection matrix:

$$P_{r,k} := \tfrac{1}{|\Gamma_k|} \sum_{\gamma \in \Gamma_k} r(\gamma).$$

Note that $P_{r,k}$ is of the same size as r. Since $r(\gamma)$ is a unitary matrix, r a representation, and Γ_k a group, it can be easily checked that $P_{r,k}$ is a projection matrix. The rank of the projection matrix is given by

$$m_{r,k} := \operatorname{rank} P_{r,k} = \operatorname{trace} P_{r,k} \ .$$

Let $a_{r,k}$ be a matrix whose columns are an orthonormal basis for the column space of $P_{r,k}$. For example, $a_{r,k}$ could be obtained via a singular value decomposition of $P_{r,k}$. Note that $a_{r,k}$ is a matrix of size $(\dim r\,,\, m_{r,k})$. Furthermore, $a_{r,k} a^*_{r,k} = P_{r,k}$ and $a^*_{r,k} a_{r,k} = \mathrm{I}$. Since the dimensions $\dim r$ are always very small for the applications (BEM) we have in mind, the generation of $a_{r,k}$ is only a small computational expense (overhead).

We now use the matrices $a_{r,k}$ to modify the definition of the Fourier transform (11):

$$\hat{x}(r,k) := \sqrt{\tfrac{\dim r}{|\Gamma|}}\ a^*_{r,k} \sum_{\gamma \in \Gamma} x(\gamma k)\, r(\gamma^{-1}) \tag{18}$$

for $r \in \mathcal{R}$ and $k \in \{\,1, 2, \ldots, \mathrm{n}\,\}$.

The symmetry relation (12) now takes the form:

$$\hat{x}(r, \gamma k) = \left(a^*_{r,k}\, r(\gamma)\, a_{r,k}\right)\ \hat{x}(r,k). \tag{19}$$

Again, it is sufficient to describe $\hat{x}(r, \gamma k)$ for indices k in a selection $\mathbb{S}$, since the other components are generated automatically by the latter formula.

For the inverse Fourier transform we obtain the following modification of (13):

$$x(k) = \sum_{r \in \mathcal{R}} \sqrt{\tfrac{\dim r}{|\Gamma|}}\ \operatorname{trace}\left(a_{r,k}\, \hat{x}(r,k)\right). \tag{20}$$

Let us now consider how the linear system $Ax = b$ is transformed if the system matrix A is equivariant with respect to Γ. If the action of the permutation group Γ has fixed points, then it can be shown that the following modification of (14) holds:

$$\sqrt{\tfrac{|\Gamma|}{\dim r}} \sum_{l \in \mathbb{S}} \tfrac{1}{|\Gamma_l|}\, \hat{A}_l(r,k)\ a_{r,l}\ \hat{x}(r,l) = \hat{b}(r,k) \quad \text{for } k \in \mathbb{S} \text{ and } r \in \mathcal{R}, \tag{21}$$

where A_l denotes the l-th column of A.

Again we have a linear system of equations in the transformed unknowns $\hat{x}(r, l)$ which has block diagonal structure, and it can be shown that the number of unknowns has not changed (as should be the case). The new system can be solved with much less computational expense. All blocks of (21) are invertible if and only if A is invertible.

Going back to our simple example of Figure 4, the symmetry reduction analogous to Table 2 is shown in Table 3.

Total number of original unknowns	$n = 16$
Total number of transformed unknowns	$n = 16$
Irreducible representation producing a subsystem	Size of transformed unknowns in this subsystem
r_2	$(\,n/8 - 1\,,\ 1\,) = (\,1\,,\ 1\,)$
r_3, r_4	$(\,n/8\,,\ 1\,) = (\,2\,,\ 1\,)$
r_1	$(\,n/8 + 1\,,\ 1\,) = (\,3\,,\ 1\,)$
r_5	$(\,n/4\,,\ 2\,) = (\,4\,,\ 2\,)$

TABLE 3. Symmetry Reduction for $\mathcal{D}_4$ via Projection

Let us again summarize the method of symmetry reduction: Suppose that a discretization of a boundary integral equation with geometric symmetries leads to a linear system of equations $Ax = b$ with the (full) system matrix of size (n, n) which is equivariant under a group of permutations Γ obtained from a group of symmetries (orthogonal transformations). Now the action of the group of permutations Γ is allowed to have fixed points. Then the method consists of the same steps as Method 1, except that the steps refer to different equations and transformations:

Method 2.

1. Generate a selection of columns of the system matrix A, and generate the column b.
2. Perform the Fourier transform (18) on the selected columns of A and on b.
3. Solve the reduced systems of linear equations (21), one system for each irreducible representation $r \in \mathcal{R}$.
4. Use the inverse Fourier transform (20) and the symmetry relations (19) to retrieve the unknowns x.

If an index k is not invariant under a symmetry, then $\Gamma_k = \{\,\mathrm{I}\,\}$, and $P_{r,k}$ is the identity matrix for all $r \in \mathcal{R}$. This is (of course) not an interesting case, since it does not contribute to the modifications which we have discussed in this section. Note also that if no index is fixed, then Method 2 reduces to Method 1.

9. Handling Fixed Points via Regularization

It is possible to avoid the need of the factorization $a_{r,k}\, a^*_{r,k} = P_{r,k}$ in the previous section at the cost of augmenting the number of unknowns in the symmetry reduced equations to be solved. In the discretizations of BEM for which this approach is intended usually only a few nodes are invariant under some symmetry, and therefore the addition of just a few unknowns may well outweigh the overhead generated by dealing with these factorizations. Let us now outline this approach.

We introduce a different modification for the definition of the Fourier transform (11):

$$\hat{x}(r,k) := \sqrt{\tfrac{\dim r}{|\Gamma|}} \sum_{\gamma\in\Gamma} \tfrac{1}{\sqrt{|\Gamma_k|}}\, x(\gamma k)\, r(\gamma^{-1}) \tag{22}$$

for $r \in \mathcal{R}$ and $k \in \{1, 2, \ldots, \mathrm{n}\}$. The symmetry relation (12) remains the same.

For the inverse Fourier transform we obtain the following modification of (13):

$$x(k) = \sqrt{|\Gamma_k|} \sum_{r\in\mathcal{R}} \sqrt{\tfrac{\dim r}{|\Gamma|}}\ \operatorname{trace}\left(a_{r,k}\, \hat{x}(r,k)\right). \tag{23}$$

Let us again consider how the linear system $Ax = b$ is transformed if the system matrix A is equivariant with respect to Γ. If the action of the permutation group Γ has fixed points, then it can be shown that the following modification of (14) holds:

$$\sqrt{\tfrac{|\Gamma|}{\dim r}} \sum_{l\in\mathbb{S}} \tfrac{1}{\sqrt{|\Gamma_l|}}\, \hat{A}_l(r,k)\ \hat{x}(r,l) = \hat{b}(r,k) \tag{24}$$

for $k \in \mathbb{S}$ and $r \in \mathcal{R}$, where A_l denotes the l-th column of A.

Again we have a linear system of equations in the transformed unknowns $\hat{x}(r,l)$ which has block diagonal structure, but it can be shown that (if there are fixed points) the number of unknowns has been augmented. However it is not difficult to see that the above system is always singular.

In order to remedy this fact, we first observe that the symmetry condition (12) implies that

$$(\mathrm{I} - P_{r,l})\hat{x}(r,l) = 0.$$

Hence, we do not change the solutions by making the following modification of (24):

$$\sqrt{\tfrac{|\Gamma|}{\dim r}} \sum_{l\in\mathbb{S}} \left(\tfrac{1}{\sqrt{|\Gamma_l|}}\, \hat{A}_l(r,k) + \epsilon\, \delta_l^k\, (\mathrm{I} - P_{r,l}) \right) \hat{x}(r,l) = \hat{b}(r,k) \tag{25}$$

for $k \in \mathbb{S}$ and $r \in \mathcal{R}$. In fact, all blocks of (25) are invertible (for $\epsilon \neq 0$) if and only if A is invertible. Hence, we may regard (25) as a regularization of (24). The parameter ϵ should be chosen so that the condition number of the blocks is not artificially large.

Going back to our simple example of Figure 4, the symmetry reduction is now described in Table 4 which should be compared to Table 3. For increasing n, the number of additional unknowns obviously is of decreasing significance.

Total number of original unknowns	$n = 16$
Total number of transformed unknowns	$n + 8 = 24$
Irreducible representation producing a subsystem	Size of transformed unknowns in this subsystem
$r_1 \ldots, r_4$	$(\ n/8+1\ ,\ 1\ ) = (\ 3\ ,\ 1\ )$
r_5	$(\ n/4+2\ ,\ 2\ ) = (\ 6\ ,\ 2\ )$

Table 4. Symmetry Reduction for $\mathcal{D}_4$ via Regularization

Let us again summarize the method of symmetry reduction by regularization: Suppose that a discretization of a boundary integral equation with geometric symmetries leads to a linear system of equations $Ax = b$ with the (full) system matrix of size (n, n) which is equivariant under a group of permutations Γ obtained from a group of symmetries (orthogonal transformations). Now the action of the group of permutations Γ is allowed to have fixed points. Then the method consists of the same steps as Methods 1–2, except that the steps refer to different equations and transformations:

Method 3.

1. Generate a selection of columns of the system matrix A, and generate the column b.
2. Perform the Fourier transform (22) on the selected columns of A and on b.
3. Solve the reduced systems of linear equations (25), one system for each irreducible representation $r \in \mathcal{R}$.
4. Use the inverse Fourier transform (23) and the symmetry relations (12) to retrieve the unknowns x.

If an index k is not invariant under a symmetry, then $\Gamma_k = \{\,\mathrm{I}\,\}$, and $P_{r,k}$ is the identity matrix for all $r \in \mathcal{R}$. This is (of course) not an interesting case, since it does not contribute to the modifications which we have discussed in this section. Note also that if no index is fixed, then also Method 3 reduces to Method 1.

10. The Symmetry Reduction Method as a Preconditioner

In applications it is often the case that an operator equation under investigation only displays approximate symmetries which may be flawed by either less important terms in the equation and/or imperfections in the geometrical shape, see, e.g., Figure 5.

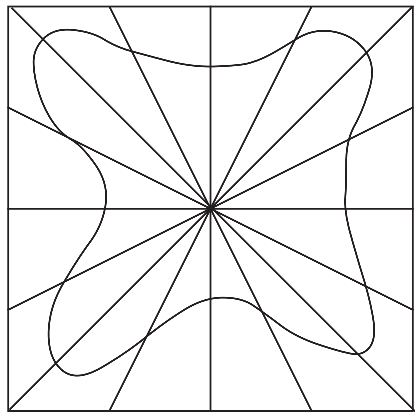

FIGURE 5. Nearly $\mathcal{D}_4$-Symmetric Domain

In this case, it has been shown in the PhD thesis [18, 19] that an equivariant preconditioner can be constructed which is efficient and improves convergence drastically, in particular for equations with bad conditioning (such as integral equations of the first kind).

In such cases, the action of a permutation group is evident from the discretization, however the equivariance conditions (9) are not quite satisfied by the system matrix A. The *equivariant part*

$$A_\Gamma(i,k) := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} A(\gamma i, \gamma k)$$

of A is a very good preconditioner. In fact, it is easily seen that A_Γ is the equivariant matrix nearest to A in the Frobenius norm. Our symmetry reduction method asserts that the action A_Γ^{-1} can be implemented at a low computational expense.

11. An Example: The 3-D Cube

The symmetry reduction method has been studied in some detail by the authors and co-workers for the case that the domain under consideration displays the symmetry of the three dimensional cube, e.g., BEM for an exterior boundary value problem, see [4, 5, 13, 14, 18, 19]. Let us mention two examples of a discretization of the cube surface via BEM, namely collocation with constant elements, and collocation with quadratic elements. In the latter case, the numerical codes in [6] can be used to obtain the entries of the system matrix and right hand side.

For constant elements, let us consider a coarse triangulation as sketched in Figure 6 where only a selection of elements is shown, i.e., the remaining elements are obtained by applying the 48 symmetries of the cube.

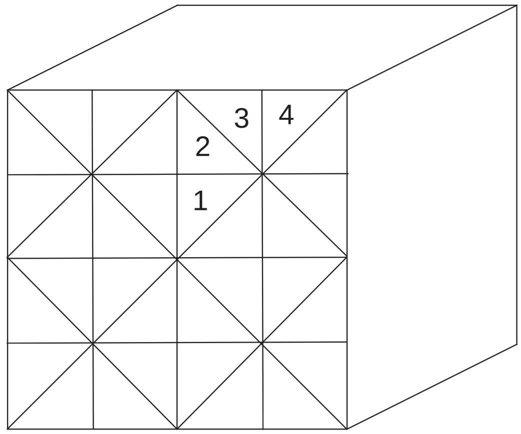

FIGURE 6. Fixed Point Free Cube Symmetry

The action of the group Γ is fixed point free in this case. We can use the selection $\mathbb{S} = \{1, \ldots, 4\}$. The symmetry reduction of the method is analyzed in Table 5.

Total number of original unknowns	192
Total number of transformed unknowns	192
Irreducible representation producing a subsystem	Size of transformed unknowns in this subsystem
$r_1 \ldots, r_4$	(4 , 1)
r_5, r_6	(8 , 2)
$r_7, \ldots, r_{10}$	(12 , 3)

TABLE 5. Symmetry Reduction for the Cube Without Fixed Points

Let us now consider the case of quadratic elements mentioned above. The collocation points are indicated in Figure 7 where only a selection of collocation points is shown, i.e., the remaining points are obtained by applying the 48 symmetries of the cube. Note that only one selected point, namely the one numbered 7, is not fixed under any symmetry, i.e., has a trivial isotropy group.

In Table 6 the symmetry reduction via projections is portrayed.

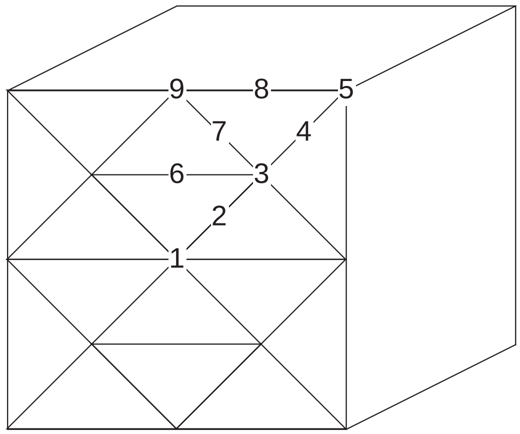

FIGURE 7. Cube Symmetry With Fixed Points

Total number of original unknowns	194
Total number of transformed unknowns	194
Irreducible representation producing a subsystem	Size of transformed unknowns in this subsystem
r_1	(9 , 1)
r_2	(1 , 1)
r_3	(2 , 1)
r_4	(6 , 1)
r_5	(10 , 2)
r_6	(6 , 2)
r_7	(16 , 3)
r_8	(8 , 3)
r_9	(10 , 3)
r_{10}	(14 , 3)

TABLE 6. Symmetry Reduction for the Cube via Projection

If regularization is used, then the number of unknowns changes drastically, see Table 7. However, let us note here that the discretization as indicated here for a surface (displaying the symmetries of a cube) is extremely coarse. The same effect as in Table 4 takes place: if the discretization is refined, then the number of additional unknowns becomes insignificant vis-à-vis the overall number of unknowns, in particular when regarding the reduced equations.

Total number of original unknowns	194
Total number of transformed unknowns	432
Irreducible representation producing a subsystem	Size of transformed unknowns in this subsystem
$r_1, \ldots, r_4$	(9 , 1)
r_5, r_6	(18 , 2)
$r_7, \ldots, r_{10}$	(27 , 3)

Table 7. Symmetry Reduction for the Cube via Regularization

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