An introduction to PL algorithms

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Abstract. We give a brief introduction to piecewise linear (PL) algorithms, also called complementary pivot or fixed point algorithms. Our approach is based on the fundamental presentation of Eaves [14], hence we describe the algorithms in the general setting of PL manifolds. In particular, we introduce the PL homotopy method of Eaves & Saigal [16]. The recently established class of variable dimension algorithms will be presented. We use a particular cone construction for handling the homotopy parameter. Special attention is given to convergence results. Numerical details of the algorithms can only be sketched. For a more detailed presentation of such algorithms and bibliographical remarks we refer to [4].

I. Introduction

1 The first and most prominent example of a PL algorithm was designed by Lemke & Howson [33] and Lemke [30] to calculate a solution of the linear complementarity problem. This algorithm played a crucial role in the development of subsequent PL algorithms. Linear complementarity problems arise in quadratic programming, bimatrix games, variational inequalities and economic equilibria problems, and numerical methods for their solution have been of considerable interest. Scarf [39] gave a numerically implementable proof of the Brouwer fixed point theorem, based upon Lemke's algorithm. Eaves [13] observed that a related class of algorithms can be obtained by considering PL approximations of homotopy maps. Thus the PL continuation methods began to emerge as a parallel to the classical imbedding or predictor-corrector methods.

2 The PL methods require no smoothness of the underlying equations and hence have, at least in theory, a more general range of applicability than classical imbedding methods. In fact, they can be used to calculate fixed points of set-valued maps. They are more combinatorial in nature and are closely related to the topological degree, see Peitgen & Siegberg [37]. PL continuation methods are usually considered to be less efficient than the predictor-corrector methods when the latter are applicable, especially in higher dimensions. The reasons for this lie in the fact that steplength adaptation and exploitation of special structure are more difficult to implement for PL methods.

II. PL manifolds

3 In order to give an idea of the flexibility which is possible and to describe the construction of PL algorithms for all kinds of special purposes, Eaves [14] has given a very elegant geometric approach to general PL methods which has strongly influenced the writing of this introduction, see also Eaves & Scarf [17]. Hence, we are going to cast the notion of PL algorithms into the general setting of subdivided manifolds which we will call **PL manifolds**.

4 Throughout this paper, we assume that $\mathcal{E}$ denotes some ambient finite dimensional Euclidean space which contains all points arising in the sequel. We denote the transposition of a column $x \in \mathcal{E}$ by x^*, and hence the scalar product in $\mathcal{E}$ is written as x^*y for $x, y \in \mathcal{E}$. A **half-space** η and the corresponding **hyperplane** $\partial\eta$ are defined by $\eta = \{y \in \mathcal{E} : x^*y \le \xi\}$ and $\partial\eta = \{y \in \mathcal{E} : x^*y = \xi\}$, respectively, for some $x \in \mathcal{E}$ with $x \ne 0$ and some $\xi \in \mathbf{R}$. A finite intersection of half-spaces is called a **cell**. If σ is a cell and $\tau \subset \sigma$ is convex, then we call τ a **face** of σ if the condition

(4.1) $$x, y \in \sigma,\ 0 < \lambda < 1,\ \lambda x + (1-\lambda)y \in \tau \quad \Longrightarrow \quad x, y \in \tau$$

holds for all $x, y \in \sigma$. Trivially, σ is a face of itself. For notational reasons, we also consider the empty set to be a face of σ. In the theory of convex sets, the above definition of a face coincides with that of an **extremal set**. By using separation theorems for convex sets, see e.g. [12] or [38], it can be shown that a subset $\tau \subset \sigma$, $\tau \ne \sigma$ is a face of σ if and only if there is a half-space ξ such that $\sigma \subset \xi$ and $\tau = \sigma \cap \partial\xi$. From this characterization it follows immediately that a face of a cell is again a cell. The **dimension** of a cell is the dimension of its affine hull, i.e. $\dim\sigma := \dim \operatorname{aff}\sigma$. In particular, the dimension of a singleton is 0 and the dimension of the empty set is -1. If the singleton $\{v\}$ is a face of σ, then v is called a **vertex** of σ. If τ is a face of σ such that $\dim\tau = \dim\sigma - 1$, then τ is called a **facet** of σ.

5 A **PL manifold** of dimension n is a system $\mathcal{M} \ne \emptyset$ of cells of dimension n such that the following conditions hold:

- **(5.1)** If $\sigma_1, \sigma_2 \in \mathcal{M}$, then $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2.
- **(5.2)** A cell τ of dimension $n-1$ can be a facet of at most two cells in $\mathcal{M}$.
- **(5.3)** The family $\mathcal{M}$ is locally finite i.e. any relatively compact subset of

  (5.4) $$|\mathcal{M}| := \bigcup_{\sigma \in \mathcal{M}} \sigma$$

  meets only finitely many cells $\sigma \in \mathcal{M}$.

We say that the PL manifold $\mathcal{M}$ **subdivides** the set $|\mathcal{M}|$. Let us further introduce the notation

(5.5) $$\mathcal{M}^k := \{\tau : \tau \text{ is a face of some } \sigma \in \mathcal{M} \text{ such that } \dim\tau = k\}$$

for $k = -1, 0, 1, 2, 3, \ldots$ and

(5.6) $$\mathcal{M}^\infty := \bigcup_{k=-1}^{\infty} \mathcal{M}^k\,,$$

We see that $\mathcal{M} = \mathcal{M}^n$ if n is the dimension of $\mathcal{M}$. From (5.2) it follows that a facet $\tau \in \mathcal{M}^{n-1}$ is common to at most two cells in $\mathcal{M}$. We therefore introduce the **boundary** $\Delta\mathcal{M} \subset \mathcal{M}^{n-1}$ of $\mathcal{M}$ as the system of facets $\tau \in \mathcal{M}^{n-1}$ which are common to exactly one cell of $\mathcal{M}$. Generally, we cannot expect $\Delta\mathcal{M}$ to again be a PL manifold. However, this is true for subdivisions of convex sets:

6 Theorem. *Let $\mathcal{M}$ be a PL manifold of dimension $n > 0$ such that $|\mathcal{M}|$ is convex. Then $\Delta\mathcal{M}$ is a PL manifold of dimension $n-1$ without boundary.*

7 Proof. The proof is rather technical, and we give only a sketch of the main ideas. The theorem is trivial in case $n = 1$, and we hence assume $n > 1$. If $\sigma \subset \mathcal{E}$ is a set, we denote by $\operatorname{int}\sigma$ the **interior** of σ relative to the affine hull $\operatorname{aff}\sigma$. Note that the relative interior of a singleton is the singleton itself. From the properties (5.1)–(5.3) it is evident that the system

$$\textbf{(7.1)} \qquad \{\operatorname{int}\tau : \tau \in \mathcal{M}^\infty\}$$

is a pairwise disjoint subdivision of $|\mathcal{M}|$. Therefore we only have to show that a cell $\xi \in (\Delta\mathcal{M})^{n-2}$ is a common facet to exactly two cells in $\Delta\mathcal{M}$. Hence, let $\tau_1 \in \Delta\mathcal{M}$, and let $\xi \in \mathcal{M}^{n-2}$ be a facet of τ_1. We choose points $y_1 \in \operatorname{int}\tau_1$ and $z \in \operatorname{int}\xi$ and consider the path

$$b : \varepsilon \in \mathbf{R} \longmapsto (1-\varepsilon)y_1 + \varepsilon z \in |\mathcal{M}|\,.$$

Let $\sigma \in \mathcal{M}$ be the unique cell which contains τ_1, and let $x_0 \in \operatorname{int}\sigma$ be a chosen point. If x is any point in $|\mathcal{M}|$, then $(1-\lambda)x_0 + \lambda x \in |\mathcal{M}|$ for $0 \le \lambda \le 1$, and by the local finiteness (5.3) it follows that $(1-\lambda)x_0 + \lambda x \in \sigma$ for $\lambda > 0$ being sufficiently small. Hence, $\operatorname{aff}\sigma = \operatorname{aff}|\mathcal{M}|$, and $x_0 \in \operatorname{int}|\mathcal{M}|$. The function $\rho : \operatorname{aff}\sigma \to (0,\infty]$ defined by

$$\rho(x) := \sup\{\lambda > 0 : (1-\lambda)x_0 + \lambda x \in |\mathcal{M}|\}$$

is continuous. Such functions play a fundamental role in the theory of convex sets, see e.g. [12] or [38]. Actually, we need only the continuity of $\varepsilon \mapsto \rho\big(b(\varepsilon)\big)$ for ε in a neighborhood of 1. It is clear that $\rho\big(b(\varepsilon)\big) = 1$ for $0 \le \varepsilon \le 1$. Furthermore, since τ_1 is a boundary facet, a straightforward convexity argument shows that $\rho\big(b(\varepsilon)\big) \le 1$ for all $\varepsilon \in \mathbf{R}$. We now consider the projection of the path $b(\varepsilon)$ onto the boundary of $\mathcal{M}$:

$$c(\varepsilon) := \Big(1 - \rho\big(b(\varepsilon)\big)\Big)x_0 + \rho\big(b(\varepsilon)\big)b(\varepsilon)\,.$$

From the property of the above subdivision (7.1) we see that the following two assertions hold:

(7.2) For $\varepsilon < 1$ and ε sufficiently near 1 we have $c(\varepsilon) \in \operatorname{int}\tau_1$.

(7.3) There is a unique $\tau_2 \in \Delta\mathcal{M}$ such that for $\varepsilon > 1$ and ε sufficiently near 1 we have $c(\varepsilon) \in \operatorname{int}\tau_2$.

We conclude that ξ is a common facet to exactly two cells in $\Delta\mathcal{M}$, namely τ_1 and τ_2. This proves the assertion. ∎

8 As we will see, PL algorithms perform steps from one cell into an adjacent cell across a common facet. Such steps are possible in a unique way because of property (5.2). More precisely, let us consider a PL manifold $\mathcal{M}$ of dimension $n > 0$, a cell $\sigma \in \mathcal{M}$ and a facet $\tau \in \mathcal{M}^{n-1}$ such that $\tau \subset \sigma$. Then we have two possible cases:

(8.1) If $\tau \notin \Delta\mathcal{M}$, then there exists a unique cell $\tilde{\sigma} \in \mathcal{M}$ such that $\tilde{\sigma} \neq \sigma$ and $\tau \subset \tilde{\sigma}$. We say that σ is **pivoted into** $\tilde{\sigma}$ **across** the facet τ.

(8.2) If $\tau \in \Delta\mathcal{M}$ then the cell $\sigma \in \mathcal{M}$ containing τ is unique, and a pivoting of σ across τ is not possible.

9 Example. For the implementation of PL algorithms, it is important to note that a PL manifold is "entered" into a computer by storing some data which describes a current cell σ and a current facet τ of σ, and by writing subroutines which handle the above pivoting steps. We illustrate this for the simple example where $\mathbf{R}^n$ is subdivided into cubes. Let $\mathbf{Z}$ denote the set of integers. For $p \in \mathbf{Z}^n$, $k \in \{1, 2, \ldots, n\}$ and $s \in \{+1, -1\}$ we define the cell

$$\sigma_p := \left\{ x \in \mathbf{R}^n : ||x - p||_\infty \leq \frac{1}{2} \right\}$$

and its facet

$$\tau_{p,k,s} := \left\{ x \in \sigma_p : x[k] = p[k] + \frac{s}{2} \right\} ,$$

where $p[k]$ denotes the k-th **co-ordinate** of p. Clearly, $\mathcal{M} := \{\sigma_p\}_{p \in \mathbf{Z}^n}$ is a PL manifold of dimension n which subdivides $\mathbf{R}^n$. It is evident that σ_p is pivoted into $\sigma_{\tilde{p}}$ across $\tau_{p,k,s}$ if we take $\tilde{p} = p + s\, e_k$. Here e_k denotes the k-th unit base vector of $\mathbf{R}^n$. The following pseudo code enables us to run through this manifold by pivoting across facets. It is assumed at each step that a decision has been made for determining across which facet the current cell has to be pivoted next. The code is very trivial and is given here only in order to familiarize the reader with the approach which will be adopted in this paper.

10 Pivoting cubes. *comment:*

`input` $p \in \mathbf{Z}^n$; *data characterizing initial cell*

`repeat`

  `enter` $s \in \{+1, -1\}$, $k \in \{1, \ldots, n\}$; *data characterizing facet*

  $p[k] \leftarrow p[k] + \frac{s}{2}$ *pivoting*

`until` pivoting is stopped.

11 A cell of particular interest is a **simplex** $\sigma = [v_1, v_2, \ldots, v_{n+1}]$ of dimension n which is defined as the convex hull of $n + 1$ affinely independent points $v_1, v_2, \ldots, v_{n+1} \in \mathcal{E}$. These points are the vertices of σ. If a PL manifold $\mathcal{M}$ of dimension n consists only of simplices, then $\mathcal{M}$ is called a **pseudo manifold** of dimension n. Such manifolds are of special importance, since PL methods simplify considerably, see e.g. [2], [3], [4], [25], [40],[42]. In this case, a pivoting step can also be described in a different way: Let $\sigma = [v_1, v_2, \ldots, v_{n+1}] \in \mathcal{M}$. Then a facet τ of σ is obtained by deleting one vertex, say the vertex

v_i for some $i \in \{1, 2, \ldots, n+1\}$. Thus we obtain the facet $\tau = [v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}]$ **lying opposite the vertex** v_i of σ. Again we consider two cases.

(11.1) If $\tau \notin \Delta\mathcal{M}$, then σ is pivoted across τ into

$$\tilde{\sigma} = [v_1, \ldots, v_{i-1}, \tilde{v}_i, v_{i+1}, \ldots, v_{n+1}]$$

where this one vertex v_i is replaced by a new vertex $\tilde{v}_i$. We say that the **vertex** v_i **of** σ **is pivoted into** $\tilde{v}_i$.

(11.2) If $\tau \in \Delta\mathcal{M}$ then the cell $\sigma \in \mathcal{M}$ containing τ is unique, and a pivoting of v_i is not possible.

If a pseudo manifold $\mathcal{T}$ subdivides a set M, then we also say that $\mathcal{T}$ triangulates M or that $\mathcal{T}$ is a **triangulation** of M. Some triangulations of $\mathbf{R}^n$ of practical importance were already considered by Coxeter [9] and Freudenthal [18], see also Todd [42]. Eaves [15] gave an overview of standard triangulations. An affine image of the simplest such triangulation can be generated by the following pivoting rules, see [1] or [10]:

12 Standard triangulation of $\mathbf{R}^n$. *comment:*

```
input [v_1, v_2, ..., v_{n+1}] ⊂ R^n;                  starting simplex
for k = 1,...,n do ρ(k) ← k + 1;                        cyclic right shift
ρ(n + 1) ← 1;
repeat
  enter i ∈ {1, 2, ..., n + 1};          index of vertex to be pivoted next
  v_i ← v_{ρ^{-1}(i)} − v_i + v_{ρ(i)}                  pivoting by reflection
until pivoting is stopped.
```

A triangulation of $\mathbf{R}^n$ which is preferable to 12 from various viewpoints has been introduced by Todd [42] but can also be obtained via certain reflection rules of Coxeter [9]:

13 Union Jack triangulation of $\mathbf{R}^n$. *comment:*

`input` $[v_1, v_2, \ldots, v_{n+1}] \subset \mathbf{R}^n$; *starting simplex*

`repeat`

  `enter` $i \in \{1, 2, \ldots, n+1\}$; *index of vertex to be pivoted next*

$$v_i \leftarrow \begin{cases} 2v_2 - v_1 & \text{for } i = 1, \\ 2v_n - v_{n+1} & \text{for } i = n+1, \\ v_{i-1} - v_i + v_{i+1} & \text{else;} \end{cases}$$ *pivoting by reflection*

`until` pivoting is stopped.

III. PL algorithms

14 Let $\mathcal{M}$ be a PL manifold of dimension $n+1$. We call $H : \mathcal{M} \to \mathbf{R}^n$ a **PL map** if

(14.1) $H : |\mathcal{M}| \to \mathbf{R}^n$ is a map;

(14.2) the restriction $H_\sigma : \sigma \to \mathbf{R}^n$ of H to σ is an affine map for all $\sigma \in \mathcal{M}$.

In this case, H_σ can be uniquely extended to an affine map $H_\sigma : \operatorname{aff}\sigma \to \mathbf{R}^n$. If we introduce the $(n+1)$-dimensional linear space $\operatorname{tng}\sigma := \{x-y : x, y \in \operatorname{aff}\sigma\}$ as the **tangent space** of σ, then the **Jacobian** $H'_\sigma : \operatorname{tng}\sigma \to \mathbf{R}^n$ is defined as the linear map which has the property $H'_\sigma(x-y) = H_\sigma(x) - H_\sigma(y)$ for $x, y \in \operatorname{aff}\sigma$. Note that H'_σ corresponds to an $n \times (n+1)$-matrix which has a one-dimensional kernel in case of non-degeneracy i.e. if its rank is maximal.

15 Let $\mathcal{M}$ be a PL manifold of dimension $n+1$ and $H : \mathcal{M} \to \mathbf{R}^n$ a PL map. A PL algorithm is essentially a method for following a polygonal path in $H^{-1}(0)$. To avoid degenaracies, we introduce a concept of regularity, see [14]. A point $x \in |\mathcal{M}|$ is called a **regular point** of H if

(15.1) x is not contained in any lower dimensional face $\tau \in \mathcal{M}^k$ for $k < n$;

(15.2) H'_σ has maximal rank n for all $\sigma \in \mathcal{M}^n \cup \mathcal{M}^{n+1}$ such that $x \in \sigma$.

A value $y \in \mathbf{R}^N$ is a **regular value** of H if all points in $H^{-1}(y)$ are regular. By definition, y is vacuously a regular value if it is not contained in the range of H. If a point is not regular it is called **singular**. Analogously, if a value is not regular it is called singular. A standard perturbation technique, see e.g. [11], [14], [37], is used in order to prove a Sard type theorem:

16 Perturbation lemma. *Let $H : \mathcal{M} \to \mathbf{R}^n$ be a PL map where $\mathcal{M}$ is a PL manifold of dimension $n+1$. Then for any relatively compact subset $C \subset |\mathcal{M}|$ there are at most finitely many $\varepsilon > 0$ such that $C \cap H^{-1}(\vec{\varepsilon})$ contains a singular point of H. Consequently, $\vec{\varepsilon}$ is a regular value of H for almost all $\varepsilon > 0$. Here we use the notation*

$$\vec{\varepsilon} := \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^n \end{pmatrix}.$$

17 Proof. The proof will be given by contradiction. Let us assume that there is a strictly decreasing sequence $\{\varepsilon_i\}_{i\in\mathbf{N}}$ of positive numbers, converging to zero, for which a bounded sequence $\{x_i\}_{i\in\mathbf{N}} \subset |\mathcal{M}|$ of singular points can be found such that the equations

$$H(x_i) = \vec{\varepsilon}_i \tag{17.1}$$

for $i \in \mathbf{N}$ are satisfied. For any subset $I \subset \mathbf{N}$ of cardinality $n+1$ we see that the $\{\vec{\varepsilon}_i\}_{i\in I}$ are affinely independent, and by (17.1) and the piecewise linearity (14.2) of H the $\{x_i\}_{i\in I}$ cannot all be contained in the same lower dimensional face $\tau \in \mathcal{M}^k$ for $k < n$. Since this holds for all index sets I, we use this argument repeatedly, and the local finiteness (5.3) of $\mathcal{M}$ permits us to find a strictly increasing function $\nu : \mathbf{N} \to \mathbf{N}$ (to generate a subsequence), and to find a face $\sigma \in \mathcal{M}^{n+1} \cup \mathcal{M}^n$ such that the subsequence $\{x_{\nu(i)}\}_{i\in\mathbf{N}}$ is contained in $\operatorname{int}\sigma$. But now we can again use the above argument: for an index set $I \subset \nu(\mathbf{N})$ of cardinality $n+1$ the $\{\vec{\varepsilon_i}\}_{i\in I}$ are affinely independent, and we conclude that H'_σ has maximal rank n. However, this means that all points $\{x_{\nu(i)}\}_{i\in\mathbf{N}}$ are regular, a contradiction to the choice of $\{x_i\}_{i\in\mathbf{N}}$. The last assertion of the perturbation lemma follows since $|\mathcal{M}|$ can be written as a countable union of relatively compact subsets. ∎

18 The last lemma enables us to confine ourselves to regular values. We now show that inverse images of regular values consist of polygonal paths whose vertices are always in the interior of some facet. More precisely:

19 Lemma. *Let $H : \mathcal{M} \to \mathbf{R}^n$ be a PL map where $\mathcal{M}$ is a PL manifold of dimension $n+1$, and let $\varepsilon > 0$ be such that $\vec{\varepsilon}$ is a regular value of H. We consider a facet $\tau \in \mathcal{M}^n$ and a cell $\sigma \in \mathcal{M}$. Then the following is true:*

(19.1) *If $\tau \cap H^{-1}(\vec{\varepsilon}) \neq \emptyset$, then $H^{-1}(\vec{\varepsilon})$ meets τ in exactly one point $a \in \operatorname{int}\tau$. Hence, facets are intersected* **transversely** *by $H^{-1}(\vec{\varepsilon})$.*

(19.2) *If $\sigma \cap H^{-1}(\vec{\varepsilon}) \neq \emptyset$, then one of the following three cases holds:*

Segment: *$H^{-1}(\vec{\varepsilon})$ meets exactly two facets τ_1, τ_2 of σ, and we have*

$$\tau_1 \cap H^{-1}(\vec{\varepsilon}) = \{a_1\}, \quad \tau_2 \cap H^{-1}(\vec{\varepsilon}) = \{a_2\},$$
$$(\operatorname{int}\sigma) \cap H^{-1}(\vec{\varepsilon}) = \{(1-\lambda)a_1 + \lambda a_2 : 0 < \lambda < 1\}.$$

Ray: *$H^{-1}(\vec{\varepsilon})$ meets exactly one facet τ of σ, and we have*

$$\tau \cap H^{-1}(\vec{\varepsilon}) = \{a_1\}, \quad a_2 \in \operatorname{int}\sigma,$$
$$(\operatorname{int}\sigma) \cap H^{-1}(\vec{\varepsilon}) = \{(1-\lambda)a_1 + \lambda a_2 : 0 < \lambda\}.$$

Line: *$H^{-1}(\vec{\varepsilon})$ meets no facet of σ, and we have*

$$a_1, a_2 \in \operatorname{int}\sigma, \quad a_1 \neq a_2,$$
$$(\operatorname{int}\sigma) \cap H^{-1}(\vec{\varepsilon}) = \sigma \cap H^{-1}(\vec{\varepsilon}) = \{(1-\lambda)a_1 + \lambda a_2 : \lambda \in \mathbf{R}\}.$$

20 Proof. "Ad (19.1)": Since $\vec{\varepsilon}$ is a regular value, (15.1) implies $\tau \cap H^{-1}(\vec{\varepsilon}) \subset \operatorname{int}\tau$. If $x_1, x_2 \in \tau \cap H^{-1}(\vec{\varepsilon})$, then $H'_\tau(x_1 - x_2) = 0$, and (15.2) implies $x_1 = x_2$. This proves assertion (19.1). "Ad (19.2)": By (19.1), $H^{-1}(\vec{\varepsilon})$ cannot intersect σ only in a boundary point. Hence $\sigma \cap H^{-1}(\vec{\varepsilon})$ is a 1-dimensional cell which has its boundary points in the boundary of σ. The assertion is now straightforward. ∎

21 Let us consider a connected component $\mathcal{C}$ of $H^{-1}(\vec{\varepsilon})$. From the above discussion it is clear that we can introduce a direction of traversing as orientation, and we can parametrize $\mathcal{C}$ with respect to, say, arc length so that we obtain a polygonal path

$$c : I \longrightarrow H^{-1}(\vec{\varepsilon})$$

which is defined on some suitable interval $I \subset \mathbf{R}$. We call a parameter $s \in I$ a **node** of c if $c(s)$ is in some facet $\tau \in \mathcal{M}^n$, we call s a **boundary node** if $c(s)$ is in some boundary facet $\tau \in \partial\mathcal{M}$, and we call s an **inner node** if $c(s)$ is in some inner facet $\tau \in \mathcal{M}^n \setminus \partial\mathcal{M}$. We say that the path c has a **boundary start** if it has a smallest node which is a boundary node, a **ray start** if it has a smallest node which is an inner node, and **no start** if it does not have a smallest node. Analogously, c has a **boundary termination** if it has a largest node which is a boundary node, a **ray termination** if it has a largest node which is an inner node, and **no termination** if it does not have a largest node. From the regularity of the value $\vec{\varepsilon}$ it is clear that the path c is locally injective, and hence it is either injective and has one of the above starts and terminations (which makes 9 cases), or it is cyclic. In the latter case, $I = \mathbf{R}$, and there is a number $p > 0$ such that $c(s) = c(\tilde{s})$ if and only if $s - \tilde{s}$ is an integer multiple of the period p. Furthermore, a cyclic path has no start and no termination and infinitly many nodes. Let us also note that there is a pathological case where the path c does not have any node at all. This corresponds to the case "line" of (19.2) and is of no interest for PL algorithms.

22 Completely labeled facets and transverse cells. Let $H : \mathcal{M} \to \mathbf{R}^n$ be a PL map where $\mathcal{M}$ is a PL manifold of dimension $n + 1$. Roughly speaking, a **PL algorithm** traces the polygonal path c discussed in the previous section from node to node. In order to also handle the case where zero is a singular value of H, we choose $\varepsilon > 0$ such that $\vec{\varepsilon}$ is a reguler value, and then we let ε tend to zero. This is possible via lemma 16. We are thus led to the following definition:

(22.1) A facet $\tau \in \mathcal{M}^n$ is called **completely labeled** with respect to H, if $\tau \cap H^{-1}(\vec{\varepsilon}) \neq \emptyset$ for all sufficiently small $\varepsilon > 0$.

(22.2) A cell $\sigma \in \mathcal{M}$ is called **transverse** with respect to H, if $\sigma \cap H^{-1}(\vec{\varepsilon}) \neq \emptyset$ for all sufficiently small $\varepsilon > 0$.

Since $\sigma \cap H^{-1}(\vec{\varepsilon})$ cannot contain any singular point for all sufficiently small $\varepsilon > 0$, see lemma 16, the three distinct cases of lemma 19 do not vary for $\varepsilon > 0$ tending to zero, provided that $\varepsilon > 0$ is sufficiently small. Hence, a transverse cell σ contains

(22.3) no completely labeled facet in case of a line,

(22.4) exactly one completely labeled facet in case of a ray,

(22.5) exactly two completely labeled facets in case of a segment.

For $\varepsilon > 0$ sufficiently small, a node of the above polygonal path c corresponds to a completely labeled facet which is intersected by c, and hence the PL algorithm traces such completely labeled facets. It is usually started either on the boundary, i.e. in a completely labeled facet $\tau \in \partial\mathcal{M}$, or on a ray, i.e. in a transverse cell $\sigma \in \mathcal{M}$ which has only one completely labeled facet. We are thus led to the following two generic versions of a PL algorithm.

23 Generic PL algorithm with boundary start. *comment:*

`input` $\tau_1 \in \partial\mathcal{M}$ completely labeled;	*starting facet*
find the unique $\sigma_1 \in \mathcal{M}$ such that $\tau_1 \subset \sigma_1$;	*starting cell*
for $i = 1, 2, 3, \ldots$ `do`	
`begin`	
`if` τ_i is the only completely labeled facet of σ_i `then` stop	*ray termination*
`else` find the unique completely labeled facet τ_{i+1} of σ_i such that $\tau_{i+1} \neq \tau_i$;	*PL step*
`if` $\tau_{i+1} \in \partial\mathcal{M}$ `then` stop	*boundary termination*

```
else pivot σ_i across τ_{i+1} into σ_{i+1}            pivoting step
end.
```

24 Generic PL algorithm with ray start. *comment:*

```
input σ_1 ∈ M transverse                              starting cell
   which possesses exactly one completely labeled facet;
find the unique facet τ_2 of σ_1                      initial PL step
   which is completely labeled;
for i = 2, 3, ... do
   begin
   if τ_i ∈ ∂M then stop                              boundary termination
   else pivot σ_{i-1} across τ_i into σ_i;            pivoting step
   if τ_i is the only completely labeled
      facet of σ_i then stop                          ray termination
   else find the unique completely labeled facet      PL step
      τ_{i+1} of σ_i such that τ_{i+1} ≠ τ_i
   end.
```

IV. Numerical considerations

25 From a numerical point of view, two steps have to be efficiently implemented. The **pivoting step** finds a new cell which is adjacent to a given cell across a given facet. Usually, the current cell is stored in the computer via some characteristic data, and a pivoting step is implemented by describing how this data is changed. In order to do this, a facet has to be specified at each step. Usually, this facet will be completely labeled. The pseudo codes 10 and 12 are simple examples. See [15] and [42] for further implementations of pivoting rules.

26 The **PL step** finds the second completely labeled facet of a transverse cell or decides that a second such facet does not exist. This is usually a much more time-consuming step than the pivoting step since it involves some numerical linear algebra as is typical for linear programming methods. Let us consider an example. We assume that a cell of dimension $n+1$ is given by

(26.1)
$$\sigma := \{x \in \mathbf{R}^{n+1} : Lx \geq c\},$$

where $L : \mathbf{R}^{n+1} \to \mathbf{R}^m$ is a linear map and $c \in \mathbf{R}^m$ a given value. Furthermore, let us assume that

(26.2)
$$\tau_i := \{x \in \mathbf{R}^{n+1} : Lx \geq c,\ e_i^* Lx = e_i^* c\},$$

for $i = 1, 2, \ldots, m$ characterizes all the facets of σ. Here e_i denotes the i-th **unit base vector** in $\mathbf{R}^m$ and ' $*$ ' denotes **transposition**. On the cell σ, the PL map $H : \mathcal{M} \to \mathbf{R}^n$ reduces to an affine map, and hence there is a linear map $A : \mathbf{R}^{n+1} \to \mathbf{R}^n$ and a vector $b \in \mathbf{R}^n$ such that

(26.3)
$$\sigma \cap H^{-1}(0) = \{x \in \mathbf{R}^{n+1} : Ax = b,\ Lx \geq c\}.$$

Let τ_i be completely labeled. If we exclude degeneracies, then $\tau_i \cap H^{-1}(0) = \{x_0\}$ is a singleton, and there is a unique vector t in the one-dimensional kernel $A^{-1}(0)$ such that $e_i^* t = -1$. Furthermore, we have $e_j^* L x_0 > e_j^* c$ for $j = 1, \ldots, m$, $j \neq i$, and hence $x_0 - \lambda t \in \operatorname{int} \sigma$ for small $\lambda > 0$. If (26.3) is a ray, cf. (19.2), then $e_j^* L(x_0 - \lambda t) > e_j^* c$ for all $\lambda > 0$. Otherwise we have $e_j^* t > 0$ for at least one index j, and since we are excluding degeneracies, the minimization

$$\textbf{(26.4)} \qquad k := \arg\min \left\{ \frac{e_j^*(Lx_0 - c)}{e_j^* t} : j = 1, \ldots, m,\ e_j^* t > 0 \right\}$$

yields the unique completely labeled facet τ_k of σ with $k \neq i$. For

$$\lambda_0 := \frac{e_k^*(Lx_0 - c)}{e_k^* t} > 0$$

we obtain $\sigma \cap H^{-1}(0) = \{x_0 - \lambda t : 0 \leq \lambda \leq \lambda_0\}$. Minimizations such as (26.4) are typical for linear programming, and the numerical linear algebra can be efficiently handled by standard routines. Successive LP steps can often make use of previous matrix factorizations via update methods [22]. In case of a pseudo manifold $\mathcal{M}$ where the cell σ is a simplex, it is convenient to handle the numerical linear algebra with respect to the barycentric co-ordinates based on the vertices of σ, and the equations become particularly simple, see e.g. [2], [4], [42] for details.

V. PL homotopy algorithms

27 Let us show how the above ideas can be used to approximate zero points of a map $G : \mathbf{R}^n \to \mathbf{R}^n$ by applying PL methods to an appropriate homotopy map. Eaves [13] presented the first such method. A restart method based on somewhat similar ideas was developed by Merrill [34]. As an example of a PL homotopy method we will take the algorithm of Eaves & Saigal [16]. To insure success of the algorithms, we will follow a presentation [21] which uses a quite general boundary condition extending somewhat that of [34].

28 Let us first introduce some notations. For $x \in \mathbf{R}^n$ we denote by $\mathcal{U}(x)$ the **system of neighborhoods** of x. By $\overline{\text{co}}(X)$ we denote the closed convex hull of a set $X \subset \mathbf{R}^n$. By $\mathbf{R}^n_\Sigma$ we denote the system of compact convex non-empty subsets of $\mathbf{R}^n$. We call the map $G : \mathbf{R}^n \to \mathbf{R}^n$ **asymptotically linear** if the following three conditions hold:

(28.1) G is **locally bounded** i.e. each point $x \in \mathbf{R}^n$ has a neighborhood $U \in \mathcal{U}(x)$ such that $G(U)$ is a bounded set.

(28.2) G is **differentiable at** ∞ i.e. there exists a linear map $G'_\infty : \mathbf{R}^n \to \mathbf{R}^n$ such that $||x||^{-1} ||G(x) - G'_\infty x|| \to 0$ for $||x|| \to \infty$.

(28.3) G'_∞ is nonsingular.

If a map $G : \mathbf{R}^n \to \mathbf{R}^n$ is locally bounded, then we can define its **set-valued hull** $G_\Sigma : \mathbf{R}^n \to \mathbf{R}^n_\Sigma$ by setting

$$\textbf{(28.4)} \qquad G_\Sigma(x) := \bigcap_{U \in \mathcal{U}(x)} \overline{\text{co}} \Big(G(U) \Big).$$

It is not difficult to see that G_Σ is upper semi-continuous, and that G is continuous at x if an only if $G_\Sigma(x)$ is a singleton. By using a degree argument [24] on the set-valued homotopy

$$\textbf{(28.5)} \qquad H_\Sigma(x,\lambda) := (1-\lambda)G'_\infty x + \lambda G_\Sigma(x)\,,$$

it can be seen that G_Σ has at least one zero point i.e. a point $\bar{x}$ such that $0 \in G_\Sigma(\bar{x})$.

29 It is known [36], [37] that degree arguments in nonlinear analysis are essentially constructive. We describe how the above situation can be implemented by a PL homotopy method. Let us introduce the following canonical projections $\pi_n : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ and $\pi_1 : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by setting $\pi_n(x,\lambda) = x$ and $\pi_1(x,\lambda) = \lambda$. It is possible to construct a pseudo manifolds $\mathcal{M}$ which triangulates $\mathbf{R}^n \times [0,\infty)$ and has the following **refining properties**:

(29.1) For each $\sigma \in \mathcal{M}$ there is an integer $i \geq 0$ which we call the **level** of σ such that $\pi_1(\sigma) \subset [i, i+1]$.

(29.2) There is a number $0 < \rho < 1$ which we call the **rate of refinement** of $\mathcal{M}$ such that for each $\sigma \in \mathcal{M}$ we have $\operatorname{diam} \pi_n(\sigma) \leq \delta\rho^i$ where i is the level of σ and $\delta > 0$ is a constant which characterizes the initial mesh of the triangulation.

The first such triangulation was proposed by Eaves [13]. Todd [42] gave a triangulation with refining factor $1/2$. Subsequently, many triangulations with arbitrary refining factors were developed, see [15].

30 We now construct a PL homotopy for an asymptotically linear map $G : \mathbf{R}^n \to \mathbf{R}^n$. First we define $h : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}^n$ by setting

$$\textbf{(30.1)} \qquad h(x,\lambda) := \begin{cases} G'_\infty(x - x_1) & \text{for } \lambda = 0, \\ G(x) & \text{for } \lambda > 0. \end{cases}$$

Then we choose a pseudo manifold $\mathcal{M}$ which triangulates $\mathbf{R}^n \times [0,\infty)$ and has the refining properties (29.1)–(29.2). Since each cell in $\mathcal{M}$ is a simplex, there is a unique PL map $H : \mathcal{M} \to \mathbf{R}^n$ which coincides with h on all vertices: if $\sigma = [v_1, \ldots, v_{n+2}] \in \mathcal{M}$, see section 11, and if a point $u \in \sigma$ is expanded into its **barycentric co-ordinates** $u = \sum_{i=1}^{n+2} c_i v_i$, then $H(u) = \sum_{i=1}^{n+2} c_i h(v_i)$.

31 Eaves-Saigal algorithm. We continue to consider the above refining pseudo manifold $\mathcal{M}$ and the PL map H generated by the asymptotically linear map G. According to theorem 6 it is clear that the boundary $\Delta\mathcal{M}$ is a pseudo manifold which triangulates the sheet $\mathbf{R}^n \times \{0\}$. If we assume that the starting point $u_1 := (x_1, 0)$ is in the interior of a facet $\tau_1 \in \Delta\mathcal{M}$, then it is immediately clear that τ_1 is the only completely labeled facet of $\Delta\mathcal{M}$. Hence, the PL algorithm started in τ_1 can not terminate in the boundary, and since all cells of $\mathcal{M}$ are compact, it cannot terminate in a ray. Hence, it has no termination, see program 23. Thus the PL algorithm generates a sequence $\tau_1, \tau_2, \ldots$ of completely labeled facets in $\mathcal{M}^n$. Let us also consider the polygonal path generated by the PL algorithm. This path is characterized by the nodes $(x_1,\lambda_1), (x_1,\lambda_2), \ldots$ such that (x_i,λ_i) is the unique zero point of the PL map H in τ_i for $i = 1, 2, \ldots$. We call $\bar{x} \in \mathbf{R}^n$ an **accumulation point** of the algorithm if some subsequence of $x_1, x_2, \ldots$ converges to $\bar{x}$.

32 Lemma. *The set A of accumulation points of algorithm 31 is compact, connected and non-empty. Each point $\bar{x} \in A$ is a zero point of G_Σ i.e. we have $0 \in G_\Sigma(\bar{x})$.*

33 Proof. From the construction of the PL map H it follows that

$$\lim_{||x|| \to \infty} ||x||^{-1} ||H(x, \lambda) - G'_\infty x|| = 0$$

uniformly for $\lambda \in [0, \infty)$. Since G'_∞ is non-singular, $H(x_i, \lambda_i) = 0$ implies that the sequence x_i is bounded. Hence the set A is non-empty and compact.

Let us assume that A can be written as a disjoint union of two non-empty compact sets A_1 and A_2. Then $\operatorname{dist}(A_1, A_2) > 0$, and $\liminf_{i \to \infty} \operatorname{dist}(x_i, A_j) = 0$ for $j = 1, 2$. On the other hand, $\lim_{i \to \infty} \operatorname{dist}(x_i, A) = 0$, and the refining property 29 implies that $\lim_{i \to \infty} ||x_i - x_{i+1}|| = 0$. This leads to a contradiction, and hence A is connected.

Since a PL manifold is locally bounded and the projections $\pi_n(\tau_i)$ stay in a bounded set, it follows that the level of the τ_i tends to ∞ for $i \to \infty$. Hence, for i sufficiently large, the definition 30 of H and the fact that the facets τ_i are completely labeled implies that $0 \in \operatorname{co} G\big(\pi_n(\tau_i)\big)$. Since a point $\bar{x} \in A$ is an accumulation point of the sequence x_i, and since $\lim_{i \to \infty} \operatorname{diam} \pi_n(\tau_i) = 0$, we have that for each neighborhood $U \in \mathcal{U}(\bar{x})$ there is an i such that $\operatorname{co} G\big(\pi_n(\tau_i)\big) \subset G(U)$. Intersecting over all $U \in \mathcal{U}(\bar{x})$ and applying (28.4) gives $0 \in G_\Sigma(\bar{x})$. ∎

34 Corollary. *If the set-valued hull G_Σ has only isolated zero points, then the sequence x_i generated by algorithm 31 converges to a zero point of G_Σ.*

35 Example. As a simple example, we consider the situation of the celebrated Brouwer fixed point theorem [5]. Let $F : C \to C$ be a continuous map an a convex, compact, non-empty subset $C \subset \mathbf{R}^n$ with non-empty interior. We define an asymptotically linear map $G : \mathbf{R}^n \to \mathbf{R}^n$ by setting

(35.1)
$$G(x) := \begin{cases} x - F(x) & \text{for } x \in C, \\ x - x_1 & \text{for } x \notin C. \end{cases}$$

Here, a point x_1 in the interior of C is used as a starting point. The above PL algorithm generates a point $\bar{x} \in \mathbf{R}^n$ such that $0 \in G_\Sigma(\bar{x})$. If $\bar{x} \notin C$, then $G_\Sigma(\bar{x}) = \{\bar{x} - x_1\}$, but $\bar{x} \neq x_1$ implies that this case is impossible. If $\bar{x}$ is an interior point of C, then $G_\Sigma(\bar{x}) = \{\bar{x} - F(\bar{x})\}$, and hence $\bar{x}$ is a fixed point of F. If $\bar{x}$ is in the boundary ∂C, then $G_\Sigma(\bar{x})$ is the convex hull of $\bar{x} - x_1$ and $\bar{x} - F(\bar{x})$, and hence $\bar{x} = (1 - \lambda) x_1 + \lambda F(\bar{x})$ for some $0 \leq \lambda \leq 1$. But $\lambda < 1$ would imply that $\bar{x}$ is an interior point of C, and hence we have $\lambda = 1$, and again $\bar{x}$ is a fixed point of F. Hence, the above PL homotopy algorithm generates a fixed point of F in either case. Many similar asymptotically linear maps can be constructed which correspond to important nonlinear problems, see e.g. [4].

VI. Index and orientation

36 Nearly all PL manifolds $\mathcal{M}$ which are of importance for practical implementations, are orientable. If $\mathcal{M}$ is orientable and of dimension $n+1$, and if $H : \mathcal{M} \to \mathbf{R}^n$ is a PL map, then it is possible to introduce an index for the PL solution manifold $H^{-1}(0)$ which has important invariance properties and occasionally yields some useful information, see [14], [17], [23], [32], [41], [43]. It should be noted that this index is closely related to the topological index which is a standard tool in topology and nonlinear analysis, see [35], [36], [37]. There are many ways to introduce the index for the present case. Our discussion follows [2], [4].

37 We begin with some basic definitions. Let F be a linear space of dimension k. An **orientation** of F is a function or : $F^k \to \{-1, 0, 1\}$ such that the following conditions hold:

(37.1) $\text{or}(b_1, \ldots, b_k) \neq 0$ if and only if $b_1, \ldots, b_k$ are linearly independent.
(37.2) $\text{or}(b_1, \ldots, b_k) = \text{or}(c_1, \ldots, c_k) \neq 0$ if and only if the transformation matrix between $b_1, \ldots, b_k$ and $c_1, \ldots, c_k$ has positive determinant.

It is clear from the basic facts of linear algebra that any finite dimensional linear space permits exactly two orientations.

A cell σ of dimension k is oriented by orienting its tangent space $\text{tng}(\sigma)$. Such an orientation or_σ of σ **induces an orientation** $\text{or}_{\tau,\sigma}$ on a facet τ of sigma by the following convention:

(37.3)
$$\text{or}_{\tau,\sigma}(b_1, \ldots, b_{k-1}) := \text{or}_\sigma(b_1, \ldots, b_k)$$

whenever b_k points from τ into the cell σ. It is routine to check that the above definition of $\text{or}_{\tau,\sigma}$ verifies the conditions (37.1)–(37.2).

If $\mathcal{M}$ is a PL manifold of dimension $n+1$, then an **orientation of** $\mathcal{M}$ is a choice of orientations $\{\text{or}_\sigma\}_{\sigma \in \mathcal{M}}$ such that

(37.4)
$$\text{or}_{\tau,\sigma_1} = -\,\text{or}_{\tau,\sigma_2}$$

for each facet $\tau \in \mathcal{M}^N$ which is adjacent to two different cells $\sigma_1, \sigma_2 \in \mathcal{M}$. By making use of the standard orientation

(37.5)
$$\text{or}\left(b_1, \ldots, b_{n+1}\right) := \text{sign}\det\left(b_1, \ldots, b_{n+1}\right)$$

of $\mathbf{R}^{n+1}$, it is clear that any PL manifold of dimension $n+1$ which subdivides a subset of $\mathbf{R}^{n+1}$ is oriented in a natural way. But many oriented PL manifolds are known which are less trivial.

38 If $H : \mathcal{M} \to \mathbf{R}^n$ is a PL map on a PL manifold of dimension $n+1$ such that zero is a regular value of H, then it is clear that the system

$$\ker H := \{\sigma \cap H^{-1}(0)\}_{\sigma \in \mathcal{M}}$$

is a 1-dimensional PL manifold which subdivides the solution set $H^{-1}(0)$. For the case that $\mathcal{M}$ is oriented, the orientation of $\mathcal{M}$ and the natural orientation of $\mathbf{R}^n$ induce a **natural orientation of** $\ker H$. Namely, for $\xi \in \ker H$, $v \in \operatorname{tng}(\xi)$ and $\sigma \in \mathcal{M}$ such that $\xi \subset \sigma$, the definition

(38.1) $$\operatorname{or}_\xi(v) := \operatorname{or}_\sigma(b_1, \ldots, b_n, v) \ \operatorname{or}\big(H'_\sigma b_1 \,, \ \ldots \ , \ H'_\sigma b_n\big)$$

is independent of the special choice of $b_1, \ldots, b_n \in \operatorname{tng}(\sigma)$, provided the $b_1, \ldots, b_n$ are linearly independent. Clearly, an orientation of the 1-dimensional manifold $\ker H$ is just a rule which indicates a direction for traversing each connected component of $\ker H$. Keeping this in mind, we now briefly indicate why the above definition indeed yields an orientation for $\ker H$. Let $\tau \in \mathcal{M}^n$ be a facet which meets $H^{-1}(0)$ and does not belong to the boundary $\Delta\mathcal{M}$, let $\sigma_1, \sigma_2 \in \mathcal{M}$ be the two cells adjacent to τ, and let $\xi_j := H^{-1}(0) \cap \sigma_j \in \ker H$ for $j = 1, 2$. If $b_1, \ldots, b_n$ is a basis of $\operatorname{tng}(\tau)$, and if $v_j \in \operatorname{tng}(\xi_j)$ points from τ into σ_j, then from condition (37.4) it follows that

$$\operatorname{or}_{\sigma_1}(b_1, \ldots, b_n, v_1) = -\operatorname{or}_{\sigma_2}(b_1, \ldots, b_n, v_2)\,,$$

and hence (38.1) implies that

$$\operatorname{or}_{\xi_1}(v_1) = -\operatorname{or}_{\xi_2}(v_2)\,,$$

which is exactly the right condition in the sense of (37.4) to ensure that the manifold $\ker H$ is oriented.

39 Let $H : \mathcal{M} \to \mathbf{R}^n$ be a PL map on an oriented PL manifold of dimension $n+1$. Given a facet τ of a cell $\sigma \in \mathcal{M}$, we can define an index by setting

(39.1) $$\operatorname{index}_{\tau,\sigma}(H) := \operatorname{or}_{\tau,\sigma}(b_1, \ldots, b_n) \ \operatorname{or}(H'_\sigma b_1, \ldots, H'_\sigma b_n)$$

if τ is completely labeled with respect to H, and $\operatorname{index}_{\tau,\sigma}(H) := 0$ else. It is clear that this definition is independent of the special choice of the basis $b_1, \ldots, b_n$ of $\operatorname{tng}(\tau)$. Furthermore, if zero is a regular value of H, τ is completely labeled and $\xi := H^{-1}(0) \cap \sigma \in \ker H$, then (38.1) implies that $\operatorname{index}_{\tau,\sigma}(H) = 1$ if and only if a positively oriented vector in $\operatorname{tng}\xi$ points from τ into σ. By possibly using the perturbation technique 22, the results of section 38 immediately yield

(39.2) $$\operatorname{index}_{\tau_1,\sigma}(H) = -\operatorname{index}_{\tau_2,\sigma}(H)$$

for the case that τ_1 and τ_2 are two different completely labeled facets of a cell σ (see the PL step in algorithms 23–24), and

(39.3) $$\operatorname{index}_{\tau,\sigma_1}(H) = -\operatorname{index}_{\tau,\sigma_2}(H)$$

for the case that τ is a completely labeled facet of two different cells σ_1 and σ_2 (see the pivoting step in algorithms 23–24). A case of special importance is a facet τ in the boundary $\Delta\mathcal{M}$, since then we do not have to specify the cell σ which contains τ since σ is unique. If the PL algorithm 23 starts on the boundary in a completely labeled facet τ_1 and stops again in the boundary in a completely labeled facet τ_k, then the above formulae imply that

(39.4) $$\text{index}_{\tau_1}(H) = -\,\text{index}_{\tau_k}(H)$$

holds. Hence, for a compact PL manifold (i.e. if $|\mathcal{M}|$ is compact) where only a boundary start or termination is possible, we obtain the following celebrated index formula

(39.5) $$\sum_{\tau\in\Delta\mathcal{M}} \text{index}_{\tau}(H) = 0\,.$$

VII. Lemke's algorithm

40 The first and most prominent example of a PL algorithm was designed by Lemke [30], [33] to calculate a solution of the linear complementarity problem. We present the Lemke algorithm as an example of a PL algorithm since it played a crucial role in the development of subsequent PL algorithms. Let us consider the following **linear complementarity problem**: Given an affine map $g : \mathbf{R}^n \to \mathbf{R}^n$, find an $x \in \mathbf{R}^n$ such that

(40.1) $$x \in \mathbf{R}^n_+; \quad g(x) \in \mathbf{R}^n_+; \quad x^* g(x) = 0.$$

Here $\mathbf{R}_+$ denotes the set of non-negative real numbers, and in the sequel we also denote the set of positive real numbers by $\mathbf{R}_{++}$. If $g(0) \in \mathbf{R}^n_+$, then $x = 0$ is a trivial solution to the problem. Hence this trivial case is always excluded and the additional assumption

(40.2) $$g(0) \notin \mathbf{R}^n_+$$

is made. Linear complementarity problems arise in quadratic programming, bimatrix games, variational inequalities and economic equilibria problems, and numerical methods for their solution have been of considerable interest [6], [7], [8], [31]. For $x \in \mathbf{R}^n$ we introduce the positive part $x_+ \in \mathbf{R}^n_+$ by setting $e_i^* x_+ := \max\{e_i^* x, 0\}$, $i = 1, \ldots, n$ and the negative part $x_- \in \mathbf{R}^n_+$ by $x_- := (-x)_+$. The following formulae are then obvious: $x = x_+ - x_-$, $(x_+)^*(x_-) = 0$. The next proposition is not difficult to prove and reduces the linear complementarity problem (40.1) to a zero point problem in a simple way:

41 Proposition. *Under the assumptions of section 32, let us define $f : \mathbf{R}^n \to \mathbf{R}^n$ by $f(z) := g(z_+) - z_-$. If x is a solution of the linear complementarity problem (40.1), then $z := x - g(x)$ is a zero point of f. Conversely, if z is a zero point of f, then $x := z_+$ solves (40.1).*

42 The advantage which f provides is that it is obviously a PL map if we subdivide $\mathbf{R}^n$ into orthants. This is the basis for our description of Lemke's algorithm. For a fixed $d \in \mathbf{R}^n_{++}$ we define the homotopy $H : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}^n$ by

(42.1) $$H(x,\lambda) := f(x) + \lambda d\,.$$

For a given subset $I \subset \{1,2,\dots,n\}$ we introduce the complement $I' := \{1,2,\dots,n\} \setminus I$. Furthermore we introduce the power set

(42.2) $$\mathcal{P}_n := \Big\{ I : I \subset \{1,2,\dots,n\} \Big\}\,.$$

Then an orthant in $\mathbf{R}^n \times [0,\infty)$ can be written in the form

(42.3) $$\sigma_I := \{\, (x,\lambda) : \lambda \ge 0,\ e_i^* x \ge 0 \text{ for } i \in I,\ e_i^* x \le 0 \text{ for } i \in I' \,\}\,,$$

and the family

(42.4) $$\mathcal{M} := \{\, \sigma_I \,\}_{I \in \mathcal{P}_n}$$

is a PL manifold (of dimension $n+1$) which subdivides $\mathbf{R}^n \times [0,\infty)$. Furthermore it is clear that $H : \mathcal{M} \to \mathbf{R}^n$ is a PL map since $x \mapsto x_+$ switches its linearity character only at the hyperplanes $\{x \in \mathbf{R}^n : e_i^* x = 0\}_{i=1,2,\dots n}$.

43 Let us assume for simplicity that zero is a regular value of H. We note however, that the case of a singular value is treated in the same way by using the perturbation techniques described in section 22. Lemke's algorithm is started on a ray: if $\lambda > 0$ is sufficiently large, then

$$\big(-g(0) - \lambda d\big)_+ = 0 \quad \text{and} \quad \big(-g(0) - \lambda d\big)_- = g(0) + \lambda d \in \mathbf{R}^N_{++}\,,$$

and consequently

$$H\big(-g(0) - \lambda d\,,\,\lambda\big) = f\big(-g(0) - \lambda d\big) + \lambda d = g(0) - \big(g(0) + \lambda d\big) + \lambda d = 0\,.$$

Hence, the ray $\xi \in \ker H$ defined by

$$\lambda \in [\lambda_0, \infty) \longmapsto -g(0) - \lambda d \in \sigma_\emptyset$$

$$\text{for} \quad \lambda_0 := \max_{i=1,\dots,N} \frac{-g(0)[i]}{d[i]}$$

is used (for decreasing λ-values) for the ray start. This ray is usually called the **primary ray**, and all other rays in $\ker H$ are called **secondary rays**. Note that $\lambda_0 > 0$ by assumption (40.2). Since the PL manifold $\mathcal{M}$ consists of the orthants of $\mathbf{R}^n \times [0,\infty)$, it is finite, and there are only two possibilities:

(43.1) The algorithm terminates on the boundary $|\partial\mathcal{M}| = \mathbf{R}^n \times \{0\}$ at a point $(z,0)$. Then z is a zero point of f, and proposition 41 implies that z_+ solves the linear complementarity problem (40.1).

(43.2) The algorithm terminates on a secondary ray. Then it can be shown [6] that (40.1) has no solution if the Jacobian g' belongs to a certain class of matrices.

44 Let us illustrate the use of index and orientation by showing that the algorithm generates a solution in the sense of (43.1) under the assumption that all principle minors of the Jacobian g' are positive. Note that the Jacobian g' is a constant matrix since g is affine.

For $\sigma_I \in \mathcal{M}$, see (42.3)–(42.4), we immediately calculate the Jacobian

$$H'_{\sigma_I} = (f'_{\sigma_I}\,,\, d)\,,$$

(44.1)
$$\text{where} \quad f'_{\sigma_I} e_i = \begin{cases} g' e_i & \text{for } i \in I, \\ e_i & \text{for } i \in I'. \end{cases}$$

If $\xi \in \ker H$ is a solution path in σ_I, then formula (38.1) yields

$$\mathrm{or}_\xi(v) = \operatorname{sign}\det f'_{\sigma_I}\ \ \mathrm{or}_{\sigma_I}(e_1, \ldots, e_n, v)\,,$$

and since $\mathrm{or}_{\sigma_I}(e_1, \ldots, e_n, v) = \operatorname{sign} v^* e_{n+1}$ by the standard orientation in $\mathbf{R}^{n+1}$, we have that $\det f'_{\sigma_I}$ is positive or negative if and only if the λ-direction is increasing or decreasing, respectively, while ξ is traversed according to its orientation. It is immediately seen from (44.1) that $\det f'_{\sigma_I}$ is obtained as a **principle minor of** g' i.e. by deleting all columns and rows of g' with index $i \in I'$ and taking the determinant of the resulting matrix (where the determinant of the "empty matrix" is assumed to be 1). Since we start in the negative orthant $\sigma_\emptyset$ where the principle minor is 1, we see that the algorithm traverses the primary ray against its orientation, because the λ-values are initially decreased. Hence, the algorithm continues to traverse $\ker H$ against its orientation. For the important case that all principle minors of g' are positive, the algorithm must continue to decrease the λ-values and thus stops in the boundary $|\Delta\mathcal{M}| = \mathbf{R}^n \times \{0\}$. Hence, in this case the algorithm finds a solution. Furthermore, it is clear that this solution is unique, since $\ker H$ can contain no other ray than the primary ray.

VIII. Variable dimension algorithms

45 In recent years, a new class of PL algorithms has attracted considerable attention. They are called **variable dimension algorithms** since they all start from a single point, a zero dimensional simplex, and successively generate simplices of varying dimension, until a completely labeled simplex is found. Numerical results [27] indicate that these algorithms improve the computational efficiency of PL homotopy methods. The first variable dimension algorithm is due to Kuhn [28]. However, this algorithm had the disadvantage that it could only be started from a vertex of a large triangulated standard simplex S, and therefore PL homotopy algorithms were preferred. By increasing the sophistication of Kuhn's algorithm considerably, van der Laan & Talman [29] developed an algorithm which could start from any point inside S. It soon became clear, see Todd [44], that this algorithm could be interpreted as a homotopy algorithm. Numerous other variable dimension algorithms were developed. Two unifying approaches have been given, one due to Kojima & Yamamoto [26], the other due to Freund [19]–[20]. We present here a modified version of the first approach. The modification consists of introducing a cone construction for dealing with the homotopy parameter. In a special case, this construction was also used by Kojima & Yamamoto, see their lemma 5.13.

46 Before we can give a description of these algorithms, we introduce the notion of a primal-dual pair of PL manifolds due to Kojima & Yamamoto [26]. In fact, we only need a special case. Let $\mathcal{P}$ and $\mathcal{D}$ be two PL manifolds of dimension n. We call $(\mathcal{P}, \mathcal{D})$ a **primal-dual pair** if there is a bijective map

$$\tau \in \mathcal{P}^k \longmapsto \tau^d \in \mathcal{D}^{n-k}, \quad k = 0, 1, \ldots, n,$$

such that

(46.1)
$$\tau_1 \subset \tau_2 \quad \Leftrightarrow \quad \tau_2^d \subset \tau_1^d$$

holds for all $\tau_1 \in \mathcal{P}^{k_1}$ and $\tau_2 \in \mathcal{P}^{k_2}$.

47 We will deal with a homotopy parameter via the following cone construction. Throughout the rest of this paper, ω denotes a point which is affinely independent from all cells under consideration. The introduction of ω is only formal and may be obtained by e.g. increasing the dimension of the ambient finite dimensional Euclidean space $\mathcal{E}$ introduced in section 4. If σ is a cell, then

(47.1)
$$\langle \omega, \sigma \rangle := \Big\{ (1-\lambda)\omega + \lambda x : x \in \sigma, \ \lambda \geq 0 \Big\}$$

denotes the cone containing σ with vertex ω. Clearly, $\langle \omega, \sigma \rangle$ is again a cell and $\dim \langle \omega, \sigma \rangle = \dim \sigma + 1$. If $H : \sigma \to \mathbf{R}^k$ is an affine map, then the affine extension $\langle \omega, H \rangle : \langle \omega, \sigma \rangle \to \mathbf{R}^k$ is defined by

(47.2)
$$\langle \omega, H \rangle \big((1-\lambda)\omega + \lambda x \big) := \lambda H(x)$$

for $x \in \sigma$ and $\lambda \geq 0$. If $\mathcal{M}$ is a PL manifold of dimension n, then

(47.3)
$$\langle \omega, \mathcal{M} \rangle := \Big\{ \langle \omega, \sigma \rangle \Big\}_{\sigma \in \mathcal{M}}$$

is a PL manifold of dimension $n+1$, and a PL map $H : \mathcal{M} \to \mathbf{R}^k$ is extended to a PL map $\langle \omega, H \rangle : \langle \omega, \mathcal{M} \rangle \to \mathbf{R}^k$.

48 We will be interested below in rays traversing a cone $\langle \omega, \sigma \rangle$, and we therefore collect some formulae. A ray in $\langle \omega, \sigma \rangle$ is given as

$$\Big\{ (1-\varepsilon) z_1 + \varepsilon z_2 : \varepsilon \geq 0 \Big\} \subset \langle \omega, \sigma \rangle\,,$$
$$\text{where} \quad z_j = (1-\lambda_j)\omega + \lambda_j x_j\,, \quad j = 1, 2$$

for some suitable $\lambda_1, \lambda_2 \geq 0$ and $x_1, x_2 \in \sigma$. A simple calculation using the affine independence of ω yields

$$(1-\varepsilon) z_1 + \varepsilon z_2 = (1-\lambda_\varepsilon)\omega + \lambda_\varepsilon x_\varepsilon\,,$$
$$\text{where} \quad \lambda_\varepsilon = (1-\varepsilon)\lambda_1 + \varepsilon \lambda_2$$
$$\text{and} \quad x_\varepsilon = \frac{(1-\varepsilon)\lambda_1 x_1 + \varepsilon \lambda_2 x_2}{\lambda_\varepsilon}\,.$$

Since $\lambda_\varepsilon \geq 0$ for all $\varepsilon \geq 0$, it follows that $\lambda_2 \geq \lambda_1$. This leaves two cases to consider:

(48.1) $$\lambda_2 > \lambda_1 \geq 0 \quad \Rightarrow \quad \lim_{\varepsilon \to \infty} x_\varepsilon = \frac{\lambda_2 x_2 - \lambda_1 x_1}{\lambda_2 - \lambda_1} \in \sigma\,,$$

(48.2) $$\lambda_2 = \lambda_1 > 0 \quad \Rightarrow \quad x_1 \neq x_2\,, \quad x_\varepsilon = (1-\varepsilon)x_1 + \varepsilon x_2 \in \sigma \quad \text{for} \quad \varepsilon \geq 0\,.$$

The second case is only possible if the cell σ is unbounded.

49 Let $\mathcal{T}$ and $\mathcal{M}$ be manifolds of dimension n. We call $\mathcal{T}$ a **refinement** of $\mathcal{M}$ if for all $\sigma \in \mathcal{M}$ the restricted PL manifold $\mathcal{T}_\sigma := \{\xi : \xi \in \mathcal{T},\ \xi \subset \sigma\}$ subdivides σ.

50 We are now in a position to introduce primal-dual manifolds. Let $(\mathcal{P}, \mathcal{D})$ be a primal-dual pair of n-dimensional PL manifolds, and let $\mathcal{T}$ be a refinement $\mathcal{P}$. Then

(50.1) $$\mathcal{T} \otimes \mathcal{D} := \{\xi \times \tau^d : k \in \{0, 1, \ldots, N\},\ \xi \in \mathcal{T}^k,\ \tau \in \mathcal{M}^k,\ \xi \subset \tau\}$$

is an n-dimensional PL manifold with empty boundary. A proof of this and related results was given by Kojima & Yamamoto [26]. We call $\mathcal{T} \otimes \mathcal{D}$ the **primal-dual manifold** generated by $\mathcal{T}$ and $\mathcal{D}$. An essential part of the proof consists of discussing the possible pivoting steps. Let $\xi \times \tau^d \in \mathcal{T} \otimes \mathcal{D}$ with $k = \dim \xi$ as above, and let κ be a facet of $\xi \times \tau^d$. We now describe the pivoting of $\xi \times \tau^d$ across the facet κ, see (8.1), i.e. we have to find a cell $\eta \in \mathcal{T} \otimes \mathcal{D}$ such that $\eta \neq \xi \times \tau^d$ and $\kappa \subset \eta$. There are three possible cases:

(50.2) Increasing the dimension. Let $\kappa = \xi \times \sigma^d$ such that $\sigma \in \mathcal{M}^{k+1}$ contains τ. Then there is exactly one $\rho \in \mathcal{T}^{k+1}$ such that $\xi \subset \rho$ and $\rho \subset \sigma$. This is a consequence of the fact that $\mathcal{T}$ refines $\mathcal{P}$ and is not difficult to prove. Then $\eta := \rho \times \sigma$ is the desired second cell. In this case the dimension k of the primal cell ξ is increased when performing the pivoting step.

(50.3) Decreasing the dimension. Let $\kappa = \delta \times \tau^d$ such that $\delta \in \mathcal{T}^{k-1}$ is a facet of ξ. If $\delta \subset \partial\tau$, then there exists exactly one facet $\nu \in \mathcal{M}^{k-1}$ of τ such that $\delta \subset \nu$, and $\eta := \delta \times \nu^d$ is the desired second cell. In this case the dimension k of the primal cell ξ is decreased when performing the pivoting step.

(50.4) Keeping the dimension. Let $\kappa = \delta \times \tau^d$ such that $\delta \in \mathcal{T}^{k-1}$ is a facet of ξ. If $\delta \not\subset \partial\tau$, then there exists exactly one cell $\xi' \in \mathcal{T}^k$ such that $\xi' \neq \xi$, $\xi' \subset \tau$ and $\delta \subset \xi'$. This is again a consequence of the fact that $\mathcal{T}$ refines $\mathcal{P}$ and is not difficult to prove. Now $\eta := \xi' \times \tau$ is the desired second cell. In this case the dimension k of the primal cell ξ is left invariant when performing the pivoting step.

The main point for practical purposes is that the above three different kinds of pivoting steps must be easy to implement on a computer. This is of course mainly a question of choosing a simple primal-dual pair $(\mathcal{P}, \mathcal{D})$ and either $\mathcal{T} = \mathcal{P}$ or some standard refinement $\mathcal{T}$ of $\mathcal{P}$ which can be handled well.

51 We now slightly modify the construction of primal-dual manifolds to include cones for the refinement $\mathcal{T}$ of the primal manifold:

(51.1) $$\langle \omega, \mathcal{T} \rangle \otimes \mathcal{D} := \{ \langle \omega, \xi \rangle \times \tau^d : k = 0, 1, \ldots, n,\ \xi \in \mathcal{T}^k,\ \tau \in \mathcal{M}^k,\ \xi \subset \tau \} .$$

If $\dim \xi = k > 0$, then the facets of $\langle \omega, \xi \rangle$ are simply the $\langle \omega, \rho \rangle$ where $\rho \in \mathcal{T}^{k-1}$ is a facet of ξ, and it is readily seen that the pivoting steps (50.2)–(50.4) apply. The only exception is the case $\dim \xi = k = 0$. In this case it follows that $\xi = \tau$, and ξ is a vertex of the primal manifold $\mathcal{P}$, but $\langle \omega, \xi \rangle$ is a ray which has one vertex, namely $\{\omega\}$. Hence, we now have a boundary

(51.2) $$\Delta(\langle \omega, \mathcal{T} \rangle \otimes \mathcal{D}) = \{ \{\omega\} \times \{v\}^d : \{v\} \in \mathcal{P}^0 \} .$$

Clearly, such a boundary facet $\{\omega\} \times \{v\}^d$ belongs to the $(n+1)$-dimensional cell $\langle \omega, \{v\} \rangle \times \{v\}^d \in \langle \omega, \mathcal{T} \rangle \otimes \mathcal{D}$. We will later see that such boundary facets are used for starting a PL algorithm. This corresponds to starting a homotopy method on the trivial level $\lambda = 0$ at the point v. We will now apply the above concept of primal-dual manifolds in order to describe some PL algorithms.

52 Lemke's algorithm revisited. We consider again the linear complementarity problem (40.1) and introduce a primal-dual pair $(\mathcal{P}, \mathcal{D})$ by defining for $I \subset \{1, 2, \ldots, n\}$ and $I' := \{1, 2, \ldots, n\} \setminus I$ the primal and dual faces

(52.1) $$\alpha_I := \{ x \in \mathbf{R}^n : e_i^* x \geq 0 \text{ for } i \in I,\ e_i^* x = 0 \text{ for } i \in I' \} ,$$
$$\alpha_I^d := \alpha_{I'} .$$

The primal and dual manifolds consist of just one cell: $\mathcal{P} = \mathcal{D} = \{\mathbf{R}^n_+\}$. We now define a PL map $H : \mathcal{P} \otimes \mathcal{D} \times [0, \infty) \longrightarrow \mathbf{R}^n$ by $H(x, y, \lambda) := y - g(x) - \lambda d$ where $d \in \mathbf{R}^N_{++}$ is fixed. Note that the variables x and y are placed into complementarity with each other by the construction of $\mathcal{P} \otimes \mathcal{D}$, and hence a more complex definition of H as in (42.1) is not necessary. For sufficiently large $\lambda > 0$ the solutions of $H(x, y, \lambda) = 0$ are given by the primary ray $(x, y, \lambda) = (0\,,\, g(0) + \lambda d\,,\, \lambda)$. Here the PL algorithm following $H^{-1}(0)$ is started in negative λ-direction. If the level $\lambda = 0$ is reached, a solution $H(x, y, 0) = 0$ solves the LCP since the complementarity $x \in \mathbf{R}^n_+$, $y = g(x) \in \mathbf{R}^n_+$, $x^* y = 0$ holds by the construction of $\mathcal{P} \otimes \mathcal{D}$.

53 As a typical representative of the class of variable dimension algorithms we choose the **octrahedral algorithm** of Wright [45], since numerical experiments indicate that it performs favorably [27], and since it can be described in a reasonably simple way. Let us point out, however, that similar arguments hold for many other algorithms where the refinement $\mathcal{T}$ of the primal manifold $\mathcal{P}$ is a pseudo manifold which triangulates $\mathbf{R}^n$, and where the dual manifold $\mathcal{D}$ subdivides a compact subset of $\mathbf{R}^n$, see [4], [19], [20], [26], [27], [29].

We denote by $\Sigma := \{+1, 0, -1\}^n \setminus \{0\}$ the set of all nonzero sign vectors. For two vectors $s, t \in \Sigma$ we introduce the relation

(53.1) $$s \prec p \quad :\Longleftrightarrow \quad \forall_{i=1,\ldots,n} \Big(e_i^* s \neq 0 \Rightarrow e_i^* s = e_i^* p \Big).$$

Then we define a primal-dual pair $(\mathcal{P},\mathcal{D})$ of n-dimensional manifolds by introducing the following duality:

$$\alpha_0 := \{0\}, \quad \alpha_0^d := \{\, y \in \mathbf{R}^n : ||y||_1 \le 1 \,\},$$

and for $s \in \Sigma$ we consider

(53.2)
$$\alpha_s := \left\{ \sum_{\substack{p\in\Sigma \\ s\prec p}} \lambda_p \, p \ : \ \lambda_p \ge 0 \right\},$$
$$\alpha_s^d := \{\, y \in \mathbf{R}^n : ||y||_1 \le 1,\ s^* y = 1 \,\}.$$

Hence, the primal manifold $\mathcal{P}$ subdivides $\mathbf{R}^n$ into $2n$ cones centered around the unit base vectors $\pm e_i$ for $i = 1, 2, \ldots, n$, and the dual manifold $\mathcal{D}$ just consists of the unit ball with respect to the $||.||_1$-norm. We easily check that

$$y \in \alpha_s^d,\ s \prec p \quad \Rightarrow \quad y^* p \ge 0$$

and hence

(53.3)
$$(x, y) \in \mathcal{P} \otimes \mathcal{D} \quad \Rightarrow \quad x^* y \ge 0\,.$$

We now consider a pseudo manifold $\mathcal{T}$ which is a refinement of $\mathcal{P}$, for example it is easy to see that the Union Jack triangulation 13 of $\mathbf{R}^n$ has this property.

54 Our aim is to find an approximate zero point of an asymptotically linear map $G : \mathbf{R}^n \to \mathbf{R}^n$. To do this, we first need to introduce the **PL approximation** $G_{\mathcal{T}}$ of G with respect to the pseudo manifold $\mathcal{T}$, see also section 30. In fact, there is a unique PL map $G_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ such that $G(v) = G_{\mathcal{T}}(v)$ holds for all vertices $v \in \mathcal{T}^0$. If $\sigma = [v_1, \ldots, v_{n+1}] \in \mathcal{T}$, and if a point $u \in \sigma$ is expanded into its barycentric co-ordinates $u = \sum_{i=1}^{n+1} c_i v_i$, then $G_{\mathcal{T}}(u) = \sum_{i=1}^{n+1} c_i G(v_i)$. It is clear that $G_{\mathcal{T}}$ is also asymptotically linear and $G'_{\mathcal{T}}(\infty) = G'(\infty)$. A homotopy $\tilde{H} : \mathcal{T} \otimes \mathcal{D} \times [0, \infty) \to \mathbf{R}^N$ is introduced by setting

(54.1)
$$\tilde{H}(x, y, \lambda) := G'(\infty) y + \lambda G_{\mathcal{T}}(x)\,.$$

Here, for simplicity, $y = 0$ plays the role of a starting point. Unfortunately, $\tilde{H}$ is not PL. Hence, we use the cone construction to identify $\tilde{H}$ with a PL map $H : \langle \omega, \mathcal{P} \rangle \otimes \mathcal{D} \longrightarrow \mathbf{R}^n$ by collecting the variables in a different way:

(54.2)
$$H(z, y) := G'(\infty) y + \langle \omega, G_{\mathcal{T}} \rangle (z)\,.$$

For $z = \omega$, which corresponds to $\lambda = 0$, there is exactly one solution of $H(z, y) = 0$, namely $(z, y) = (\omega, 0)$. Hence $H^{-1}(0)$ intersects the boundary $\Delta(\langle \omega, \mathcal{P} \rangle \otimes \mathcal{D})$ in just one point. This is the starting point for our PL algorithm which traces $H^{-1}(0)$.

55 Let us first show that there is a constant $C > 0$ such that $\tilde{H}(x, y, \lambda) = 0$ implies $||x|| < C$. Indeed, otherwise we could find a sequence $\big\{(x_k, y_k, \lambda_k)\big\}_{k=1,2,\ldots} \subset H^{-1}(0)$ such that $\lim_{k\to\infty} ||x_k|| = \infty$. It follows from $\tilde{H}(x_k, y_k, \lambda_k) = 0$ and (54.1) that

(55.1) $$\lambda_k^{-1} y_k + G'(\infty)^{-1} G_{\mathcal{T}}(x_k) = 0\,.$$

If we multiply this equation from the left with x_k^* and divide by $||x_k||^2$, the asymptotic linearity of $G_{\mathcal{T}}$ yields

(55.2) $$\lim_{k\to\infty} ||x_k||^{-2} x_k^* G'(\infty)^{-1} G_{\mathcal{T}}(x_k) = 1\,,$$

and the boundedness $||y_k|| \le 1$ implies that

(55.3) $$x_k^* G'(\infty)^{-1} G_{\mathcal{T}}(x_k) > 0$$

for all sufficiently large k, and by (55.1), (55.3) we have that $x_k^* y_k > 0$ for all sufficiently large k, which is a contradiction to (53.3).

56 Now section 55 implies that the algorithm can only traverse finitely many cells, and since the solution on the boundary $\partial(\langle \omega, \mathcal{T}\rangle \otimes \mathcal{D})$ is unique, it can only terminate in a ray

$$\Big\{\big((1-\varepsilon)z_1 + \varepsilon z_2\,,\ (1-\varepsilon)y_1 + \varepsilon y_2\big) : \varepsilon \ge 0\Big\} \subset \langle \omega, \tau\rangle \times \alpha_I^d \in \langle \omega, \mathcal{T}\rangle \otimes \mathcal{D}\,,$$

where $\tau \in \mathcal{T}^k$ such that $\tau \subset \alpha_I$ and $k = \#I$. We use the notation and remarks of section 48. It follows from

(56.1) $$H\big((1-\varepsilon)z_1 + \varepsilon z_2\,,\ (1-\varepsilon)y_1 + \varepsilon y_2\big) = 0$$

and (54.1)–(54.2) that

(56.2) $$(1-\varepsilon)y_1 + \varepsilon y_2 + \lambda_\varepsilon G'(\infty)^{-1} G_{\mathcal{T}}(x_\varepsilon) = 0 \quad \text{for} \quad \varepsilon \ge 0\,.$$

Since the k-cell τ is bounded, we only have to consider the case $\lambda_2 > \lambda_1 \ge 0$, see (48.1). Dividing equation (56.1) by $\varepsilon > 0$ and letting $\varepsilon \to \infty$ yields

$$G_{\mathcal{T}}(x) = 0\,, \quad \text{where} \quad x := \frac{\lambda_2 x_2 - \lambda_1 x_1}{\lambda_2 - \lambda_1} \in \tau$$

is the desired approximate zero point of G.

57 Concluding remarks. By the above discussions we wanted to illustrate that the concept of primal-dual manifolds $\mathcal{P} \otimes \mathcal{D}$ enables a unifying description of many variable dimension algorithms. One class of such algorithms are the homotopy methods where the homotopy parameter caused our cone construction. An important feature of primal-dual manifolds is that a complementarity property of the variables (x, y) may be incorporated into the construction of $\mathcal{P} \otimes \mathcal{D}$ so that this property need not to be assumed by extra conditions or constructions. This is a very convenient trick for dealing with complementarity problems or related questions, and was illustrated here for the case of the linear complementarity problem in section 52, but many more applications have been considered, see the literature cited in [4].

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