Exploiting Symmetry in Boundary Element Methods

Eugene L. Allgower [1,4,5], Klaus Böhmer [2,6], Kurt Georg [1,2,5], and Rick Miranda [3,5]

May 1990, revised January 1991

Abstract

We consider linear operator equations $\mathcal{L}f = g$ in the context of boundary element methods, where the operator $\mathcal{L}$ is equivariant i.e., commutes with the actions of a given finite symmetry group. By introducing a generalization of Reynolds projectors, we construct a decomposition of the identity operator, which in turn allows us to decompose the problem $\mathcal{L}f = g$ into a finite number of symmetric subproblems. The data function g does not need to possess any symmetry properties. We show that analogous reductions are possible for discretizations. An explicit construction of the corresponding reduced system matrices is given. This effects a considerable reduction in the computational complexity. For example, in the case of the isometry group of the 3-cube, we reduce the computational complexity of a direct linear equation solver for full matrices by 99.65%. Specific decompositions of the identity are given for most of the significant finite isometry groups acting on $\mathbf{R}^2$ and $\mathbf{R}^3$.

Keywords. Boundary element method, linear equation solvers, symmetry groups, collocation methods, equivariant operators

AMSMOS. 65R20, 20B25, 45L10, 31C20, 65F05

[1]Partially supported by the National Science Foundation via grant # DMS-8947761
[2]Partially supported by the Deutsche Forschungsgemeinschaft
[3]Partially supported by the National Science Foundation via grant # DMS-8501007
[4]Partially supported by the Deutscher Akademischer Austauschdienst
[5]Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523
[6]Fachbereich Mathematik, Universität Marburg, W-3550 Marburg, Fed. Rep. of Germany

1 Introduction

Many problems in science and mathematics exhibit symmetry phenomena which may be exploited to effect a significant cost reduction in their numerical treatment. Usually the symmetry stems from the domain or body on which the problem is considered. The numerical treatment of problems such as partial differential equations and integral equations generally involves discretizations which ought (as far as possible) to incorporate or respect these symmetries in order to exploit the possibilities of reducing computational cost. The aim of this paper is to study some systematic techniques for exploiting symmetry in the numerical treatment of linear integral equations, with particular emphasis on boundary integral methods in three dimensions. However, it should be evident that the methods considered may also be applied more generally to other linear systems of equations.

Recent monographs [10], [11], [15] have shown the efficacy of applying group theoretical methods in the study of various problems having symmetry properties. Some efforts have also recently been made to bring such methods to bear in the numerical treatment of certain partial differential equations via finite differences and finite elements, see [1], [2], [7], [8], [9], [13]. The articles, e.g., [2], [7] have demonstrated that the use of discretizations of partial differential equations which are suitably adapted to respect symmetry properties yield highly useful decompositions which can reduce computational effort, improve the numerical conditioning of problems, and significantly facilitate the study of bifurcation behavior at singularities. The results in the present paper extend those in [2] by combining its ideas with the concepts of characters of groups.

In contrast to other recent papers which apply symmetry to reduce domains or obtain direct sum decompositions of the function spaces involved, our approach is to decompose the identity operator by introducing a generalization of Reynolds projectors which do not necessarily commute. This leads us to a method which is not based on a direct sum decomposition of the underlying function spaces. However, in some interesting cases our method does indeed give a direct sum decomposition as a byproduct, which illustrates its efficiency. This phenomenon is discussed in Section 6.

We will concentrate on the numerical aspects of solving linear equations derived from discretizations. In particular, we systematically generate symmetry-reduced linear subproblems which are easily implemented. From the solution of these subproblems, the full solution of the discrete linear system is immediately reconstructed. As a result we obtain considerable savings in the numerical effort of calculating this full solution. Although the method we are going to describe applies as well to finite difference or finite element methods for partial differential equations in general, we concentrate on boundary element methods since there the resulting linear systems generally involve full matrices and hence the above mentioned savings are particularly useful.

2 Equivariance

Let $\mathcal{M} \subset \mathbf{R}^n$ be an oriented submanifold and let $\mathcal{F}$ be a Banach space of functions on $\mathcal{M}$. Let $\Gamma \subset \mathbf{O}(n)$ be a finite group of orthogonal transformations on $\mathbf{R}^n$ which act as isometries of $\mathcal{M}$, i.e.,

$$T(\mathcal{M}) = \mathcal{M} \quad \text{for all} \quad T \in \Gamma\,.$$

The unit element of Γ is denoted by 1 and corresponds to the identity transformation. The action of the group Γ on $\mathcal{F}$ is defined in the following natural way:

Definition 1 If $f \in \mathcal{F}$ and $T \in \Gamma$, then we define a new function Tf by setting $(Tf)(x) := f(T^{-1}x)$ for $x \in \mathcal{M}$.

We make the following

Assumption 2 $T : \mathcal{F} \to \mathcal{F}$ is a continuous linear operator for all $T \in \Gamma$. This property will be easily obtained in the function spaces of interest.

The general problem we want to consider is solving the linear equation

$$\mathcal{L}f = g, \tag{1}$$

where $\mathcal{L} : \mathcal{F} \to \mathcal{F}$ is a continuous bijective linear operator, $g \in \mathcal{F}$ is a given data function, and $f \in \mathcal{F}$ is the unknown solution function. We will in particular study Galerkin and collocation methods for approximating f. To exploit the symmetry structure of $\mathcal{M}$, we use the following definition, see [11]:

Definition 3 Let $\mathcal{K} : \mathcal{F} \to \mathcal{F}$ be a continuous linear operator. We call $\mathcal{K}$ Γ-*equivariant* if $T\mathcal{K} = \mathcal{K}T$ for all $T \in \Gamma$.

The main interest of our paper is to study the case where the linear operator $\mathcal{L}$ in (1) is Γ-equivariant. Let us illustrate this definition with the example of an integral operator.

Example 4 Consider the Banach space $\mathcal{F} := C(\mathcal{M})$ of continuous functions on $\mathcal{M}$ and an integral operator $(\mathcal{K}f)(s) := \int_{\mathcal{M}} k(s,t)\, f(t)\, d\mathcal{M}(t)$ with a suitable kernel k. By definition we have

$$(T\mathcal{K}f)(s) = \int_{\mathcal{M}} k(T^{-1}s, t)\, f(t)\, d\mathcal{M}(t)\,.$$

Since T is an orthogonal transformation of $\mathbf{R}^n$, the element $d\mathcal{M}$ of area is invariant under T, thus

$$(\mathcal{K}Tf)(s) = \int_{\mathcal{M}} k(s,t)\, f(T^{-1}t)\, d\mathcal{M}(t) = \int_{\mathcal{M}} k(s, Tt)\, f(t)\, d\mathcal{M}(t).$$

Hence we obtain that the equivariance assumption, see Definition 3, amounts to the following equivalent conditions:

$$\begin{array}{rclllll} k(Ts,t) & = & k(s,T^{-1}t) & \text{for all} & s,t\in\mathcal{M} & \text{and} & T\in\Gamma, \quad \text{or} \\ k(Ts,Tt) & = & k(s,t) & \text{for all} & s,t\in\mathcal{M} & \text{and} & T\in\Gamma. \end{array} \tag{2}$$

Particular cases are the typical potential operators of boundary element methods which we discuss next.

Example 5 Let $\Omega\subset\mathbf{R}^3$ be a bounded domain, and let its boundary $\mathcal{S}$ have suitable regularity properties. On $\mathcal{F}=C(\mathcal{S})$ we define the single layer potential

$$(\mathcal{K}_1 f)(s)=\int_{\mathcal{S}}\frac{f(t)}{|s-t|}\,d\mathcal{S}(t)\,,$$

the double layer potential

$$(\mathcal{K}_2 f)(s)=\int_{\mathcal{S}}\frac{\langle t-s\,,\,\mathbf{n}(t)\rangle}{|s-t|^3}\,f(t)\,d\mathcal{S}(t)\,,$$

and on $\mathcal{F}=C(\Omega)$ we define the volume potential

$$(\mathcal{K}_3 f)(s)=\int_{\Omega}\frac{f(t)}{|s-t|}\,d\Omega(t)\,.$$

Here $|\cdot|$ denotes the euclidean norm in $\mathbf{R}^3$, and $\mathbf{n}$ denotes the outer normal vector. Let us note that all three operators are Γ-equivariant with respect to any isometry group $\Gamma\subset\mathbf{O}(3)$ of Ω. We demonstrate this only for the double layer potential operator, for the other operators the calculations are similar and in fact simpler. According to (2), all we have to show is that

$$\frac{\langle Tt-Ts\,,\,\mathbf{n}(Tt)\rangle}{|Ts-Tt|^3}=\frac{\langle t-s\,,\,\mathbf{n}(t)\rangle}{|s-t|^3}$$

for all $s,t\in\mathcal{S}$ and all $T\in\Gamma$. Since T is an orthogonal transformation on $\mathbf{R}^3$ with $T(\Omega)=\Omega$, we have that $\mathbf{n}(Tt)=T\mathbf{n}(t)$, and since the euclidean norm and the inner product are invariant under T, the above equation is obvious.

3 Twisted Reynolds Projectors

The central notion for our symmetry decompositions is the following

Definition 6 We call a complex-valued function $\chi:\Gamma\to\mathcal{C}$ a *character* if the following properties hold:

1. $\Gamma_\chi:=\{T\in\Gamma:\chi T\neq 0\}$ is a subgroup of Γ,
2. the restriction of χ to Γ_χ is a group homomorphism from Γ_χ into the multiplicative group of complex numbers with unit modulus.

We denote the order of the subgroup Γ_χ by $\#\chi$. The trivial character, denoted by τ, is obtained by setting $\tau(1) = 1$ and $\tau(T) = 0$ for $T \in \Gamma$ with $T \neq 1$.

More precisely, in group theory the above defined function would correspond to a character of degree 1 defined on a subgroup, see [12]. Our convention is adopted for convenience of notation. Hereafter, let χ denote a character on Γ.

Definition 7 We call a function $f \in \mathcal{F}$ χ-*symmetric* if

$$Tf = \chi(T^{-1})f$$

for all $T \in \Gamma_\chi$.

Note that the set $\mathcal{F}_\chi$ of χ-symmetric functions forms a closed linear subspace of $\mathcal{F}$. The following projection is a generalization of the usual Reynolds operator of a finite group acting on a vector space, which is the case when the character is identically one. We therefore call this operator a *twisted Reynolds projector.*

Definition 8 The twisted Reynolds projector with respect to χ is defined by

$$P_\chi := \frac{1}{\#\chi} \sum_{T \in \Gamma_\chi} \chi(T)T.$$

We show that P_χ is indeed a projection in the following lemma, extending results in [2] to a general setting.

Lemma 9 *Let us denote by $\mathcal{F}_\chi$ the closed linear subspace of χ-symmetric functions in $\mathcal{F}$. Then P_χ is a linear projection from $\mathcal{F}$ onto $\mathcal{F}_\chi$.*

Proof. Let $S \in \Gamma_\chi$. From

$$\begin{aligned} SP_\chi &= \frac{1}{\#\chi} \sum_{T \in \Gamma_\chi} \chi(T)ST = \frac{1}{\#\chi} \sum_{T \in \Gamma_\chi} \chi\big(S^{-1}T\big)T \\ &= \chi(S^{-1}) \frac{1}{\#\chi} \sum_{T \in \Gamma_\chi} \chi(T)T = \chi(S^{-1})P_\chi \end{aligned}$$

we obtain that the image of P_χ is contained in $\mathcal{F}_\chi$. On the other hand, let $h \in \mathcal{F}_\chi$. Then

$$P_\chi h = \frac{1}{\#\chi} \sum_{T \in \Gamma_\chi} \chi(T)Th = \frac{1}{\#\chi} \sum_{T \in \Gamma_\chi} \chi(T)\chi(T^{-1})h = h$$

shows that P_χ acts as the identity on $\mathcal{F}_\chi$. Both properties together are equivalent to the assertion. **QED**

If $\mathcal{L} : \mathcal{F} \to \mathcal{F}$ is Γ-equivariant, then $\mathcal{L}$ commutes with all such twisted Reynolds operators. Furthermore, if we denote by $\mathcal{L}_\chi$ the restriction of the map

$\mathcal{L}$ on $\mathcal{F}_\chi$, then this commutativity induces a restricted map $\mathcal{L}_\chi : \mathcal{F}_\chi \to \mathcal{F}_\chi$, which is bijective, since $\mathcal{L}$ is bijective. In other words, solving the equation

$$\mathcal{L}_\chi f_\chi = P_\chi g \tag{3}$$

for f_χ gives us the projected solution

$$f_\chi = P_\chi f \tag{4}$$

where f solves (1). The objective now is to reconstruct the solution f from certain of these particular χ-symmetric solutions f_χ. In order to do this, we need the existence of the following decomposition. Special cases of such decompositions involving real characters were introduced in [2] to obtain symmetry reductions for boundary value problems.

Definition 10 Let $\mathcal{X}$ be a finite set of characters over Γ. We call a choice of coefficients $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$ a *decomposition of the identity* if

$$\sum_{\chi\in\mathcal{X}} \gamma_\chi \chi = \tau$$

holds. Here τ denotes the trivial character, see Definition 6.

The following is now an immediate consequence of the Definition 8 of the Reynolds projectors.

Lemma 11 *If $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$ is a decomposition of the identity, then*

$$\sum_{\chi\in\mathcal{X}} (\#\chi\, \gamma_\chi)\, P_\chi = \text{Identity.}$$

Note that this does not necessarily imply that the symmetry spaces $\mathcal{F}_\chi$ form a direct sum of $\mathcal{F}$. However this holds in some interesting cases, see Section 6. Several relevant examples of decompositions will be discussed in Section 5. An immediate consequence of (3)–(4) is the following

Proposition 12 *Fix $g \in \mathcal{F}$. Let $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$ be a decomposition of the identity, let $g_\chi = P_\chi g$, and let $\{f_\chi\}_{\chi\in\mathcal{X}}$ be the solutions of the χ-symmetric subproblems $\mathcal{L}_\chi f_\chi = g_\chi$. Then*

$$f := \sum_{\chi\in\mathcal{X}} (\#\chi\, \gamma_\chi)\, f_\chi$$

is the solution of $\mathcal{L} f = g$.

The above decomposition suggests that considerable reductions of the computational effort in numerically solving (1) may be obtained by breaking down a generally non-symmetric problem (1) into the χ-symmetric subproblems (3). In order to accomplish this, the discretization method clearly has to be constructed in such a way that the given symmetry is respected. This will be outlined next.

4 Symmetry Respecting Discretizations

Let us illustrate our ideas via the example of collocation methods. Galerkin methods are treated in a very similar way. To motivate the discussion, let us return to the potential operators described in Example 5. A typical integral equation of the second kind is obtained by setting

$$\mathcal{L} := 1 - \frac{1}{2\pi}\,\mathcal{K}_2$$

and considering (1) over $\mathcal{F} = C(\mathcal{S})$. This is a boundary integral equation formulation of Poisson's equation, see, e.g., [5]. A typical discretization involves a triangulation of $\mathcal{S}$ and a finite element type approach with respect to this triangulation. Hence, we consider a set of basis functions $\{\phi_j\}_{j\in J} \subset \mathcal{F}$ and a set of collocation points $\{s_i\}_{i\in I} \subset \mathcal{S}$ of equal cardinality. Typically, the basis functions are generated by piecewise polynomial functions defined on charts corresponding to the triangulation having support on only a few triangles, and the collocation points usually are generated from the nodes of the triangulation. If the domain Ω admits the symmetry group Γ, it is natural to require that this symmetry is also respected by the triangulation and consequently by the choice of basis and collocation points.

Recently, automatic surface triangulations have been introduced in [4], see also [3]. Basic to these methods is a given triangulation $\mathcal{T}$ on $\mathbf{R}^3$. If this triangulation $\mathcal{T}$ respects the symmetry group Γ, then it is easy to see that also the generated triangulation of the surface respects Γ (assuming, of course, that the surface $\mathcal{S}$ itself respects Γ). For this reason it is desirable to take a basic triangulation $\mathcal{T}$ of $\mathbf{R}^3$ which has many symmetries, such as the "Union Jack" triangulation in [14]. An alternative approach would be to triangulate only a "fundamental domain" of $\mathcal{S}$ and subsequently apply the group action to generate a triangulation of $\mathcal{S}$. These techniques will be investigated in general elsewhere.

Returning to our general context, see Definitions 1–3, let us suppose that the manifold $\mathcal{M}$ admits an isometry group Γ, and that the linear isomorphism $\mathcal{L}$ in (1) is Γ-equivariant. We consider a collocation method for (1) based on a choice of basis functions $\{\phi_j\}_{j\in J} \subset \mathcal{F}$ and a choice of collocation points $\{s_i\}_{i\in I}$ of equal cardinality, such that interpolation is possible i.e., the matrix $\phi_j(s_i)$ has nonzero determinant. Following standard techniques we can write the collocation equation as

$$Ac = b, \tag{5}$$

where the entries of the square matrix A are given by

$$A[i,j] := (\mathcal{L}\phi_j)(s_i)\,, \tag{6}$$

the column b is obtained from the data by $b[i] := g(s_i)$, and the unknown column c represents the coefficients of the approximate solution, i.e., $f \approx \sum_{j\in J} c[j]\phi_j$. The standard assumption in collocation methods is that A is a nonsingular matrix.

In order to exploit the symmetry, we need to make the following assumption which signifies that the chosen configuration for the collocation method respects the symmetry structure. This will usually be easy to satisfy in practical cases if the triangulation is chosen as discussed above.

Assumption 13 For any $T \in \Gamma$, we assume

1. $\{T\phi_j\}_{j\in J}$ is a permutation of $\{\phi_j\}_{j\in J}$,
2. $\{Ts_i\}_{i\in I}$ is a permutation of $\{s_i\}_{i\in I}$.

For the purpose of exploiting this symmetry in a numerical application, these permutations must be implemented in some way. Therefore Assumption 13 allows us to view the symmetry group Γ as a permutation group acting on the index sets J and I via the equations

$$\begin{array}{rclll} T\phi_j & = & \phi_{Tj} & \text{for} & j \in J,\ T \in \Gamma, \\ Ts_i & = & s_{Ti} & \text{for} & i \in I,\ T \in \Gamma, \end{array} \tag{7}$$

i.e., $Tj = k \Leftrightarrow T\phi_j = \phi_k$, etc. These are the permutations which must be implemented in a symmetry exploiting approach of the collocation method. We note that the actions of Γ on the index sets J and I may be different. These actions in turn induce actions on the column vectors c and b by setting

$$(Tc)[j] := c[T^{-1}j], \quad (Tb)[i] := b[T^{-1}i] \quad \text{for} \quad j \in J,\ i \in I,\ T \in \Gamma. \tag{8}$$

We denote by $\mathcal{C}$ and $\mathcal{B}$ the spaces of columns indexed over the sets J and I, respectively. Clearly, the above actions of the group Γ on $\mathcal{C}$ or $\mathcal{B}$ can be described by permutation matrices.

This situation is a discrete analogue to Definition 1. This analogy carries further as we will now show. First, from the Γ-equivariance of the operator $\mathcal{L}$ we see that

$$\begin{array}{rcl} A[i,Tj] & = & (\mathcal{L}\phi_{Tj})(s_i) = (\mathcal{L}T\phi_j)(s_i) = (T\mathcal{L}\phi_j)(s_i) \\ & = & (\mathcal{L}\phi_j)(T^{-1}s_i) = (\mathcal{L}\phi_j)(s_{T^{-1}i}) = A[T^{-1}i, j]. \end{array}$$

Hence we see that the matrix A has the following two equivalent symmetry properties

$$\begin{array}{rclll} A[i,Tj] & = & A[T^{-1}i,j] & \text{for all} & i \in I\ , j \in J, \text{ and } T \in \Gamma, \quad \text{or} \\ A[Ti,Tj] & = & A[i,j] & \text{for all} & i \in I\ , j \in J, \text{ and } T \in \Gamma. \end{array} \tag{9}$$

The analogy to (2) is obvious. Furthermore,

$$\begin{array}{rcl} (TAc)[i] & = & (Ac)[T^{-1}i] = \sum_{j\in J} A[T^{-1}i,j]c[j] = \sum_{j\in J} A[i,Tj]c[j] \\ & = & \sum_{j\in J} A[i,j]c[T^{-1}j] = \sum_{j\in J} A[i,j]Tc[j] = (ATc)[i], \end{array}$$

shows that A can be viewed as a Γ-equivariant linear bijection from $\mathcal{C}$ onto $\mathcal{B}$. Let us again emphasize that the actions of Γ on the column spaces $\mathcal{C}$ and $\mathcal{B}$ may be different.

Continuing our analogy, we can also consider the twisted Reynolds projectors as projections onto the corresponding χ-symmetric column spaces

$$P_\chi : \mathcal{C} \to \mathcal{C}_\chi, \quad \text{or} \quad P_\chi : \mathcal{B} \to \mathcal{B}_\chi \,. \tag{10}$$

Note that $b[Ti] = \chi(T)b[i]$ for $b \in \mathcal{B}_\chi$, and similarly for columns in $\mathcal{C}_\chi$. We omit details, which are the same as in the above Banach space context. In fact, the only difference in the statements is that the points do not vary over a manifold, but over a finite index set.

For the mechanics of the linear algebra involved in efficiently solving the system (5) via χ-symmetric decompositions we now introduce some useful devices.

Definition 14 For $i \in I$ we define the *orbit* with respect to the subgroup Γ_χ by $\mathcal{O}_\chi(i) := \{Ti : T \in \Gamma_\chi\}$. These orbits lead to a *partition* of I: $I/\chi := \{\mathcal{O}_\chi(i) : i \in I\}$. We call the orbits in I/χ the orbits of I with respect to χ. Analogously, $J/\chi := \{\mathcal{O}_\chi(j) : j \in J\}$ gives a partition of J via its orbits with respect to χ. A subset $I_\chi \subset I$ is called a *selection* from I/χ if the map $i \in I_\chi \mapsto \mathcal{O}(i) \in I/\chi$ is bijective, i.e., we select from each orbit exactly one index $i \in I_\chi$. Analogous definitions are made with respect to J.

If one has a fundamental domain for the action of the group in question, one easily obtains selections from the orbits by choosing basis functions and collocation points related to this fundamental domain. Note that in applications, the generation of the entries of the discretization matrix A defined via (6) usually requires costly numerical quadratures. An important conclusion from (9) is that it is not necessary to construct all of A in this way. In fact, if the entries $A[i,j]$ are known for all $i \in I$ and a selection of indices $j \in J$ from the orbits in J with respect to the whole group Γ, then the remaining entries of A can be formally obtained via (9).

Our next task is to characterize the χ-symmetric column spaces $\mathcal{B}_\chi$ and $\mathcal{C}_\chi$, see (10), by "essential" orbits.

Definition 15 We call an index $i \in I$ χ-*essential* if, for $T \in \Gamma_\chi$, $Ti = i$ implies $\chi(T) = 1$. Analogous definitions are made with respect to J.

Note that if i is χ-essential, then so is Ti for all $T \in \Gamma_\chi$. Therefore, the property of being χ-essential is inherited by the orbit $\mathcal{O}_\chi(i)$. We will therefore speak also of χ-essential orbits.

For the next lemma we recall that the symmetry space $\mathcal{B}_\chi$ is a linear subspace of the column space $\mathcal{B}$ which is defined over the index set I.

Lemma 16 *If $b \in \mathcal{B}_\chi$ and i is not χ-essential, then $b[i] = 0$. An analogous statement holds for $\mathcal{C}_\chi$.*

Proof. Since i is not χ-essential, there is a $T \in \Gamma_\chi$ such that $Ti = i$, but $\chi(T) \neq 1$. Therefore $b[i] = b[Ti] = \chi(T)b[i]$, forcing $b[i] = 0$.

Lemma 17 *Let I'_χ be a selection of indices $i \in I$ from the χ-essential orbits $\mathcal{I}'_\chi$. Let us denote by $\mathcal{B}'_\chi$ the column space indexed over I'_χ. Then the restriction map $R_\chi : \mathcal{B}_\chi \to \mathcal{B}'_\chi$ given by the inclusion $I'_\chi \subset I$ is a linear bijection. The analogous statement holds for the index set J and the corresponding column space $\mathcal{C}$.*

Proof. The linearity of R_χ is clear. Let us assume that $b \in \mathcal{B}_\chi$ such that $R_\chi b = 0$. If k is χ-essential, then $k \in \mathcal{O}(i)$ with $i \in I'_\chi$. $R_\chi b = 0$ implies that $b[i] = 0$. Furthermore $k = Ti$ for some $T \in \Gamma_\chi$. Hence by the symmetry of b, we have that $b[k] = b[Ti] = \chi(T)b[i] = 0$. If k is not χ-essential, then $b[k] = 0$ by Lemma 16. This shows that the restriction map R_χ is injective.

To show the surjectivity, let us assume that $b \in \mathcal{B}'_\chi$ is given. Hence $b[k]$ is only defined for indices $k \in I'_\chi$. We want to extend $b[k]$ for $k \in I$ in such a way that the resulting column belongs to $\mathcal{B}_\chi$ i.e., has the desired symmetry properties. We do this extension on each orbit separately. If k is not χ-essential, then we set $b[k] = 0$. If k is χ-essential, then $k \in \mathcal{O}_\chi(i)$ for a uniquely selected $i \in I'_\chi$. Furthermore, there is a $T \in \Gamma_\chi$ such that $Ti = k$. Hence we extend by symmetry: $b[k] = b[Ti] = \chi(T)b[i]$. Let us show that this extension is independent of the special choice of T. In fact, if $S \in \Gamma_\chi$ is a different group element such that $Si = k$, then $S^{-1}Ti = i$, and since i is also χ-essential, it follows that $\chi(S^{-1}T) = 1$ or $\chi(S) = \chi(T)$. It is easy to check that the extended column indeed has the required symmetry property. **QED**

The last part of the preceding proof contained the following argument: Let $j \in J$ be χ-essential, and let $S, T \in \Gamma_\chi$. Then $Tj = Sj$ implies $\chi(S) = \chi(T)$. Therefore, the next definition, which is needed for technical reasons, is independent of the particular choice of T.

Definition 18 Let $j \in J$ be χ-essential, and let $j' \in \mathcal{O}_\chi(j)$. Then there is a $T \in \Gamma_\chi$ such that $j' = Tj$, and we define $\chi(j, j') := \chi(T)$.

Note that a consequence of this definition is that

$$c_\chi[j'] = \chi(j, j')c_\chi[j] \tag{11}$$

holds for $c_\chi \in \mathcal{C}_\chi$ and χ-essential indices j and j' in the same orbit. We now come to the crucial part of our method. In accordance with the procedure outlined in Proposition 12, let us consider a χ-symmetric subproblem of the linear equation (5):

$$P_\chi Ac = A(P_\chi c) = P_\chi b.$$

This equation is written as

$$\sum_{j \in J} A[i, j](P_\chi c)[j] = (P_\chi b)[i]\,.$$

Because of the symmetry, we have $(P_\chi c)[j] = 0$ for all j which do not belong to a χ-essential orbit, and hence we can drop the summation over these indices. The remaining indices are split into the χ-essential orbits, and if we consider a selection J'_χ from these orbits, we obtain the double sum

$$\sum_{j \in J'_\chi} \sum_{j' \in \mathcal{O}_\chi(j)} A[i,j'](P_\chi c)[j'] = (P_\chi b)[i]\,.$$

By using (11) applied to $P_\chi c \in \mathcal{C}_\chi$, we obtain

$$\sum_{j \in J'_\chi} \Big(\sum_{j' \in \mathcal{O}_\chi(j)} \chi(j,j') A[i,j'] \Big) (P_\chi c)[j] = (P_\chi b)[i]\,. \tag{12}$$

Formula (12) generalizes a result in [2] concerning a finite difference discretization and the symmetry group $\Gamma = Z_2 \times D_4$. By also reducing the index i to a selection I'_χ from χ-essential orbits in I, and by taking into account the representation, see Lemma 17, of the symmetry column spaces $\mathcal{C}_\chi$ and $\mathcal{B}_\chi$, we easily see that (12) characterizes the reduction of the linear equation (5) to the symmetry spaces $\mathcal{C}_\chi$ and $\mathcal{B}_\chi$. Since the matrix A is nonsingular by assumption, it follows that the matrix of the symmetry reduced linear system (12) is also square and nonsingular. We summarize the above discussion in the following

Theorem 19 *Let the collocation matrix defined in (6) be nonsingular, and let I'_χ, J'_χ be selections from the χ-essential orbits in I, J, respectively. Then the matrix*

$$A_\chi[i,j] := \sum_{j' \in \mathcal{O}_\chi(j)} \chi(j,j') A[i,j']\,, \quad i \in I'_\chi,\ j \in J'_\chi$$

is square and nonsingular, and the solution c_χ to the linear equations

$$\sum_{j \in J'_\chi} A_\chi[i,j] c_\chi[j] = (P_\chi b)[i]\,, \quad i \in I'_\chi \tag{13}$$

can be extended by symmetry in a unique way to a column $c_\chi \in \mathcal{C}_\chi$ as in the second part of the proof of Lemma 17. Furthermore we have $c_\chi = P_\chi c$, where c denotes the solution of (5). In particular, in the presence of a decomposition of the identity $\{\gamma_\chi\}_{\chi \in \mathcal{X}}$, c.f., Definition 10, we obtain $\sum_{\chi \in \mathcal{X}} \#\chi\, \gamma_\chi c_\chi = c$.

For the reader's convenience, let us summarize our method in the following outline.

Method 20 (Character Decomposition) 1. We assume that an operator equation $\mathcal{L}f = g$ with data g and unknown f is given, and that a collocation method has been fixed by choosing the basis functions $\{\phi_j\}_{j \in J}$ and the collocation points $\{s_i\}_{i \in I}$. Moreover we assume that a group Γ commutes with $\mathcal{L}$ and acts on the basis functions and collocation points by permuting the indices in I and J.

2. Let $A[i,j] := (\mathcal{L}\phi_j)(s_i)$ be the system matrix of the discretization. We choose a selection J' from the Γ-orbits of J. The entries of the system matrix need to be computed, but only for indices $i \in I$ and $j' \in J'$, since, if $j = Tj'$, then $A[i,j] = A[T^{-1}i, j']$. This gives a reduction factor in the computation of the matrix A approximately equal to $1/(\#\Gamma)$.

3. Find a decomposition of the identity $\sum_{\chi\in\mathcal{X}} \gamma_\chi \, \chi = \tau$. It is important to use characters χ which have $\#\chi$ as large as possible in order to obtain a small reduction factor as will be discussed below.

4. For each χ appearing in the above decomposition perform the following steps:

    (a) Choose selections $\mathcal{I}'_\chi$ and $\mathcal{J}'_\chi$ from the χ-essential orbits of I and J, respectively.

    (b) Construct the matrix A_χ as defined in Theorem 19.

    (c) Determine $b_\chi[i] := P_\chi g[i]$ for $i \in \mathcal{I}'_\chi$.

    (d) Solve the equation $A_\chi c_\chi = b_\chi$ for c_χ. Note that this determines $c_\chi[j]$ only for $j \in \mathcal{J}'_\chi$.

    (e) Calculate the remaining co-ordinates $c_\chi[j]$ of c_χ by using the χ-symmetry as indicated in Lemma 16 and (11).

5. Reconstruct the solution from

$$f := \sum_{j\in J} \sum_{\chi\in\mathcal{X}} (\#\chi \, \gamma_\chi) \, c_\chi[j] \, \phi_j \, .$$

Let us now examine the reduction in the computational complexity which can be effected by exploiting the above symmetry decompositions. We denote by d_χ the dimension of the symmetry reduced matrix A_χ, and we denote by d the dimension of the full matrix A. As we have seen above, d_χ equals the number of χ-essential orbits in either the index set I or J. Since d_χ indicates the dimension of the χ-symmetric column space $\mathcal{C}_\chi$ or $\mathcal{B}_\chi$, and since the dimension of the image of a linear projection equals its trace, we immediately obtain the formula

$$d_\chi = \mathrm{trace}(P_\chi) = \frac{1}{\#\chi} \sum_{T\in\Gamma_\chi} \chi(T) \, \mathrm{trace}(T)$$

where the trace is taken over the operators viewed as actions on the column space $\mathcal{C}$ or $\mathcal{B}$. Since T may be viewed as a permutation matrix on $\mathcal{C}$, it is easy to see that $\mathrm{trace}(T)$ gives the number of fixed indices of T acting as a permutation on J. An analogous remark holds for $\mathcal{B}$ and I. Since the identity is always an element of Γ_χ, we immediately obtain

$$d_\chi = \frac{d}{\#\chi} + \sum_{\substack{T\in\Gamma_\chi \\ T\neq 1}} \chi(T) \, \{\#\text{ of indices in } J \text{ fixed by } T\} \, .$$

If the discretization is not very coarse with respect to the symmetry group, then a fixed index of $T \neq 1$ will be the exception. Hence, for practical purposes, it is reasonable to take the approximation

$$d_\chi \approx \frac{d}{\#\chi}\,. \tag{14}$$

The system matrix of a collocation method for boundary integral equations usually is numerically treated as a full matrix, and hence the computational complexity of solving (5) can be assumed to be Cd^3 floating point operations, where C is a constant depending on the choice of method. In contrast, the number of floating point operations required to solve one of the subproblems (13) is $C\alpha_\chi d_\chi^3$ where $\alpha_\chi \in \{1, 4\}$ depends on whether the character χ is real or complex, whether the compiler treats real and complex arithmetic differently, and whether the original problem was already complex or not. A typical situation would be that the original equation (5) is taken over the real numbers, and that the decomposition of the identity is taken in such a way that the complex characters occur in conjugate pairs, say $\chi = \bar{\mu}$, see Section 5 for details. In this case, if $c_\chi[j]$ is a solution of (13) with respect to χ, it is immediately seen that the complex conjugate $\bar{c}_\chi[j]$ is a solution of (13) with respect to μ and hence obtained at no computational cost. For the sake of simplicity in our complexity analysis, we therefore make the following choices: $\alpha_\chi = 1$ for a real character χ, $\alpha_\chi = 2$ for a complex character χ, and we use the approximation (14). Furthermore, we neglect the overhead of generating the subproblems (13). The reader can easily modify these choices according to a specific situation in order to obtain more accurate estimates of the computational complexity. In the presence of a decomposition of the identity, we now obtain the approximate computational cost for solving (13) via symmetry reduction to be

$$C \sum_{\substack{\chi \in \mathcal{X} \\ \gamma_\chi \neq 0}} \frac{\alpha_\chi}{d_\chi^3}\,.$$

By using the symmetry reduction and the approximation (14), we approximate the *reduction factor* of the computational complexity by

$$\rho := \sum_{\substack{\chi \in \mathcal{X} \\ \gamma_\chi \neq 0}} \frac{\alpha_\chi}{\#\chi^3}\,. \tag{15}$$

The reduction factor gives the fraction of the computational cost of solving (5) if symmetry is exploited as opposed to not exploiting symmetry. In other words: If we measure the computational cost of solving equation (5) without exploiting symmetry as one unit of work, then the computational cost of solving equation (5) via exploiting symmetry is approximately ρ units of work. It is clear that the aim is to choose a decomposition of the identity in such a way that this factor is as small as possible. This essentially means that we are looking for decompositions where the minimal order of the groups involved is as large as possible. This motivates the choices discussed in the next section.

Let us again stress that Galerkin methods may be treated in an analogous fashion. The only difference is that the evaluation at a collocation point is replaced by an inner product (integral) with a basis function, and hence the two index sets I and J coincide. Thus the action of the group on these two sets is really the same. This did not necessarily hold for collocation methods.

5 Examples of Decompositions of the Identity

Let us first note that a decomposition of the identity $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$ can be viewed as a solution to a linear system of equations. We obtain a matrix $D[T,\chi] := \chi(T)$ indexed over $T \in \Gamma$ and $\chi \in \mathcal{X}$, and a decomposition of the identity is merely a solution column γ to the equation $D\gamma = \tau$ where τ denotes the first standard unit vector. We will display a decomposition of the identity $\{\gamma_{\chi_\nu}\}_{\nu=1,\ldots,N}$ with respect to a group $\Gamma = \{T_1, \ldots, T_k\}$ of order k as indicated in Table 1.

	γ_1	$\ldots$	γ_N
T_1	$\chi_1(T_1)$	$\ldots$	$\chi_N(T_1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
T_k	$\chi_1(T_k)$	$\ldots$	$\chi_N(T_k)$

Table 1: Generic Decomposition Table

The matrix inside the table is again the matrix D considered above. The simplest example is a cyclic group $\Gamma = \{1, F\}$ and may be viewed as an isometry group generated by a reflection. Its standard decomposition of the identity is given in Table 2.

	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1
F	1	-1

Table 2: Decomposition Table for a Cyclic Group of Order Two

This table reflects the familiar decomposition of an arbitrary function into an odd and an even part. The reduction factor as approximated according to (15) is $\rho = \frac{1}{4}$.

More generally, we consider a cyclic group $\Gamma = \{1, R, R^2, \ldots, R^{k-1}\}$ of order k. This may be viewed as the group of rotations over a regular k-gon. If ω is a primitive k-th root of unity, then a standard decomposition of the identity for Γ is given in Table 3.

	$\frac{1}{k}$	$\frac{1}{k}$	$\frac{1}{k}$	$\dots$	$\frac{1}{k}$
1	1	1	1	$\dots$	1
R	1	ω	ω^2	$\dots$	ω^{k-1}
R^2	1	ω^2	ω^4	$\dots$	$\omega^{2(k-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
R^{k-1}	1	ω^{k-1}	$\omega^{2(k-1)}$	$\dots$	$\omega^{(k-1)(k-1)}$

Table 3: Decomposition Table for a Cyclic Group of Order k

The reduction factor as approximated according to (15) is

$$\rho = \begin{cases} \frac{2k-2}{k^3} + \frac{1}{k^3} = \frac{2}{k^2} - \frac{1}{k^3} & \text{for } k \text{ odd,} \\ \frac{2k-4}{k^3} + \frac{2}{k^3} = \frac{2}{k^2} - \frac{2}{k^3} & \text{for } k \text{ even.} \end{cases}$$

Definition 21 We consider the case that the group Γ factors into a direct product $\Gamma = \Gamma_1 \times \Gamma_2$. Given a character χ_1 on Γ_1 and a character χ_2 on Γ_2, one gets a character $\chi_1 \otimes \chi_2$ on Γ, defined by

$$(\chi_1 \otimes \chi_2)(T_1, T_2) = \chi_1(T_1)\chi_2(T_2)\,.$$

We call $\chi_1 \otimes \chi_2$ the *tensor product* of the two characters χ_1 and χ_2. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are two sets of characters on Γ_1 and Γ_2 respectively, we denote by

$$\mathcal{X}_1 \otimes \mathcal{X}_2 := \{\chi_1 \otimes \chi_2 : \chi_1 \in \mathcal{X}_1,\ \chi_2 \in \mathcal{X}_2\}\,.$$

These definitions immediately extend to more than two factors in the obvious way.

The proof of the next lemma is straightforward, being obtained by simply multiplying the given decompositions.

Lemma 22 *Let $\Gamma = \prod_{m=1}^{M} \Gamma_m$ be a direct product factorization. We assume that on each factor group Γ_m, there is a decomposition of the identity given: $\{\gamma_{\chi_m}\}_{\chi_m \in \mathcal{X}_m}$. Then*

$$\{\gamma_{\chi_1}\gamma_{\chi_2}\cdots\gamma_{\chi_M}\}_{\chi_1\otimes\chi_2\otimes\cdots\otimes\chi_M \,\in\, \mathcal{X}_1\otimes\mathcal{X}_2\otimes\cdots\otimes\mathcal{X}_M}$$

is a decomposition of the identity on Γ.

Corollary 23 *It is well known that any finite abelian group can be written as a direct product of cyclic groups. Since we have a standard decomposition for cyclic groups, see Table 3, we immediately obtain a standard decomposition of the identity for any finite abelian group. Furthermore, from Lemma 22 it follows that this decomposition involves $\#\Gamma$ number of characters on Γ. Hence the reduction factor as approximated according to (15) is*

$$\rho = \frac{2}{(\#\Gamma)^2} \, .$$

The finite groups of rigid motions of 2 and 3 dimensions over the real numbers $\mathbf{R}$ have been classified, and in $\mathbf{R}^2$ the only groups are the cyclic groups (generated by a rotation) and the dihedral groups (generated by a rotation and a reflection). We consider the latter groups next: Suppose Γ is a dihedral group of order $2k$. Then Γ is generated by two elements R and F, with $R^k = F^2 = 1$, and $RF = FR^{k-1}$. Every element of Γ can be uniquely written as $F^\epsilon R^i$, where $\epsilon \in \{0,1\}$ and $0 \leq i \leq k-1$. If ω is a primitive k-th root of unity, then a standard decomposition of the identity for Γ is given in Table 4.

	$\frac{1}{2k}$	$\frac{1}{2k}$	$\frac{1}{k}$	$\frac{1}{k}$	$\dots$	$\frac{1}{k}$
1	1	1	1	1	$\dots$	1
R	1	1	ω	ω^2	$\dots$	ω^{k-1}
R^2	1	1	ω^2	ω^4	$\dots$	$\omega^{2(k-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
R^{k-1}	1	1	ω^{k-1}	$\omega^{2(k-1)}$	$\dots$	$\omega^{(k-1)(k-1)}$
F	1	-1	0	0	$\dots$	0
FR	1	-1	0	0	$\dots$	0
FR^2	1	-1	0	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
FR^{k-1}	1	-1	0	0	$\dots$	0

Table 4: Decomposition Table for the Dihedral Group of Order $2k$

The reduction factor as approximated according to (15) is

$$\rho = \begin{cases} \frac{2}{(2k)^3} + \frac{2k-2}{k^3} = \frac{2}{k^2} - \frac{1.75}{k^3} & \text{for } k \text{ odd,} \\ \frac{2}{(2k)^3} + \frac{2k-3}{k^3} = \frac{2}{k^2} - \frac{2.75}{k^3} & \text{for } k \text{ even.} \end{cases}$$

This is only slightly better than the reduction factor for the cyclic group in Table 3. In other words, the fact that we have an additional symmetry with respect to a reflection is not fully exploited. However, in the important case $k =$

4, a better decomposition of the identity can be obtained. This decomposition was found in [1], and in our terminology and approach, it takes the following form indicated in Table 5.

	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
1	1	1	1	1	1	1
R	1	1	−1	−1	0	0
R^2	1	1	1	1	−1	−1
R^3	1	1	−1	−1	0	0
F	1	−1	1	−1	1	−1
FR	1	−1	−1	1	0	0
FR^2	1	−1	1	−1	−1	1
FR^3	1	−1	−1	1	0	0

Table 5: Decomposition Table for the Dihedral Group of Order 8

The reduction factor as approximated according to (15) is

$$\rho = \frac{4}{8^3} + \frac{2}{4^3} \approx 0.039\,.$$

For $k = 4$, the standard decomposition of Table 4 would give a reduction factor of approximately 0.082, i.e., about twice the computational work. The advantage of the decomposition of Table 5 stems mainly from the fact that all characters have real values.

Let us now turn to our main point of interest, namely to the finite groups of rigid motions in $\mathbf{R}^3$. Generally such groups include both reflections and rotations. Often it is the case that the antipodal map $-$Identity leaves the body Ω under consideration invariant, i.e., $-1 \in \Gamma$; in this case the group factors as

$$\Gamma = \Gamma_0 \times \{\pm 1\}$$

where Γ_0 is the group of rotations leaving Ω invariant. If this happens, since $\{\pm 1\}$ is cyclic of order two, the problem of finding a decomposition of the identity for Γ is reduced to that for Γ_0, see Lemma 22. For this reason we will concentrate on the finite groups of rotations of $\mathbf{R}^3$. Suppose Γ is a finite subgroup of the group $\mathbf{SO}(3)$ of 3×3 orthogonal matrices with determinant 1. Then Γ is isomorphic to one of the following groups, see, e.g., [6]:

1. A cyclic group of order k (the group of rotations of a cone over a regular k-gon).

2. A dihedral group of order $2k$ (the group of rotations of a prism over a regular k-gon).

3. A_4, the alternating group of even permutations on 4 symbols (which is the rotational symmetry group of a regular tetrahedron; any rotation gives an even permutation of the 4 vertices).

4. S_4, the symmetric group of all permutations on 4 symbols (which is the rotational symmetry group of the cube; any rotation of the cube permutes the four inner diagonals).

5. A_5, the alternating group of even permutations on 5 symbols (which is the rotational symmetry group of the regular icosahedron).

We remark that S_4 is also the group of the octahedron (which is the dual regular figure to the cube) and A_5 is also the group of the dodecahedron (which is the dual of the icosahedron). Let us also note that if Ω is indeed the cube, then it has antipodal symmetry, and the remarks above apply: the group of its rigid motions is $S_4 \times \{\pm 1\}$.

Cases 1) and 2) have already been discussed above. The group A_4 permits the decomposition shown in Table 6 where ω denotes a primitive cube root of unity.

	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1	1	1	1	1	1	1
(12)(34)	1	1	1	1	-1	-1
(13)(24)	1	1	1	-1	1	-1
(14)(23)	1	1	1	-1	-1	1
(123)	1	ω	ω^2	0	0	0
(132)	1	ω^2	ω	0	0	0
(124)	1	ω^2	ω	0	0	0
(142)	1	ω	ω^2	0	0	0
(134)	1	ω	ω^2	0	0	0
(143)	1	ω^2	ω	0	0	0
(234)	1	ω^2	ω	0	0	0
(243)	1	ω	ω^2	0	0	0

Table 6: Decomposition Table for the Alternating Group A_4

The reduction factor as approximated according to (15) is

$$\rho = \frac{5}{12^3} + \frac{3}{4^3} \approx 0.05 \,.$$

We conclude our examples with S_4, see Table 7, the rotational symmetry group of the cube, which is extremely important in numerical applications. Again ω denotes a primitive cube root of unity.

The reduction factor as approximated according to (15) is

$$\rho = \frac{2}{24^3} + \frac{4}{12^3} + \frac{6}{8^3} \approx 0.014\,.$$

Let us recall that the full group of rigid motions of the cube is given by $S_4 \times \{\pm 1\}$. By using Lemma 22, the decomposition table 7 immediately induces a decomposition table for $S_4 \times \{\pm 1\}$. The corresponding reduction factor is obtained by dividing the above ρ by 4.

6 Criteria for Direct Sum Decompositions

In the introduction we mentioned that our decompositions of the identity do not necessarily yield a direct sum decomposition of the underlying function space $\mathcal{F}$ into symmetry spaces. A counterexample is given below. However, it is possible to characterize some cases where the direct sum is obtained. Assume that Γ acts linearly on a vector space V. The associated twisted Reynolds operators P_χ are then defined on V, as are the subspaces V_χ of χ-symmetric vectors, and again P_χ is a projection onto V_χ. Examples are the column spaces $\mathcal{B}$, $\mathcal{C}$ and the function space $\mathcal{F}$ which we have introduced above.

Proposition 24 *Let $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$ be a decomposition of the identity, and let us assume that $\gamma_\chi = 1/(\#\chi)$ for all $\chi \in \mathcal{X}$. Then*

$$V = \bigoplus_{\chi\in\mathcal{X}} V_\chi\,.$$

Proof. Suppose first that V is finite dimensional. As in Lemma 11, we have $\sum_{\chi\in\mathcal{X}} P_\chi = I$ where I denotes the identity operator on V. This implies $V = \sum_{\chi\in\mathcal{X}} V_\chi$. By applying the trace to the first formula we obtain

$$\sum_{\chi\in\mathcal{X}} \operatorname{trace} P_\chi = \operatorname{trace} I\,.$$

Note that $\operatorname{trace} P_\chi = \dim V_\chi$ and $\operatorname{trace} I = \dim V$. The conclusion follows immediately from dimension considerations.

Even if V is infinite dimensional, we still have $V = \sum_{\chi\in\mathcal{X}} V_\chi$. It is sufficient to prove the following: if $v_\chi \in V_\chi$ is such that $\sum_{\chi\in\mathcal{X}} v_\chi = 0$, then each $v_\chi = 0$. To show this, let W be a Γ-invariant finite dimensional linear subspace of V containing all the v_χ (e.g., W is the span of the union of the orbits of all v_χ). From the first part of the proof, we know that W splits into a direct sum, forcing each v_χ to be zero. **QED**

	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
1	1	1	1	1	1	1	1	1	1	1
(12)(34)	1	1	1	1	1	1	-1	-1	-1	-1
(13)(24)	1	1	1	1	-1	-1	1	1	-1	-1
(14)(23)	1	1	1	1	-1	-1	-1	-1	1	1
(123)	1	1	ω	ω^2	0	0	0	0	0	0
(132)	1	1	ω^2	ω	0	0	0	0	0	0
(124)	1	1	ω^2	ω	0	0	0	0	0	0
(142)	1	1	ω	ω^2	0	0	0	0	0	0
(134)	1	1	ω	ω^2	0	0	0	0	0	0
(143)	1	1	ω^2	ω	0	0	0	0	0	0
(234)	1	1	ω^2	ω	0	0	0	0	0	0
(243)	1	1	ω	ω^2	0	0	0	0	0	0
(1324)	1	-1	0	0	1	-1	0	0	0	0
(1423)	1	-1	0	0	1	-1	0	0	0	0
(1234)	1	-1	0	0	0	0	1	-1	0	0
(1432)	1	-1	0	0	0	0	1	-1	0	0
(1243)	1	-1	0	0	0	0	0	0	1	-1
(1342)	1	-1	0	0	0	0	0	0	1	-1
(12)	1	-1	0	0	-1	1	0	0	0	0
(34)	1	-1	0	0	-1	1	0	0	0	0
(13)	1	-1	0	0	0	0	-1	1	0	0
(24)	1	-1	0	0	0	0	-1	1	0	0
(14)	1	-1	0	0	0	0	0	0	-1	1
(23)	1	-1	0	0	0	0	0	0	-1	1

Table 7: Decomposition Table for the Symmetric Group S_4

	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
1	1	1	1	1	1	1	1	1
R	1	1	-1	-1	0	0	0	0
R^2	1	1	1	1	-1	-1	-1	-1
R^3	1	1	-1	-1	0	0	0	0
F	1	-1	1	-1	1	-1	0	0
FR	1	-1	-1	1	0	0	-1	1
FR^2	1	-1	1	-1	-1	1	0	0
FR^3	1	-1	-1	1	0	0	1	-1

Table 8: A Counterexample

It is possible to come up with decompositions of the identity for various groups which do not satisfy the hypotheses of the previous lemma. An example, see Table 8, is obtained by complicating Table 5.

In fact, this example does not lead to a direct sum decomposition. This shows that there can be some redundancy in the reduction method, which however does not affect its applicability. In fact, one may be willing to accept some redundancy in order to obtain real-valued characters or bigger subgroups Γ_χ. The reader can easily check that in all the decompositions given in the tables in Section 5, the above criteria are satisfied and therefore lead to direct sum decompositions for the column spaces $\mathcal{C}$ and $\mathcal{B}$ and the function space $\mathcal{F}$.

Let us note that the previous proposition implies that the twisted Reynolds projectors are orthogonal as operators on the vector space V: $P_\chi P_\mu = 0$ for $\chi \neq \mu$. In fact, this orthogonality is a rather formal property which we would now like to discuss.

Definition 25 Two characters χ and μ on Γ are *orthogonal* if

$$\sum_{T\in\Gamma} \chi(T^{-1})\mu(T) = 0\,.$$

We need the following technical

Lemma 26 *Let χ and μ be two characters on Γ which are orthogonal, and let $G \in \Gamma$. Then*

$$\sum_{\substack{T,S\in\Gamma\\ TS=G}} \chi(T)\mu(S) = 0\,.$$

Proof. Let $Q_k \in \Gamma$, $k = 1, \ldots, K$ be such that $Q_k(\Gamma_\chi \cap \Gamma_\mu)$ for $k = 1, \ldots, K$ represents a partition of Γ into the cosets of $\Gamma_\chi \cap \Gamma_\mu$. The proof is now obtained from the following formula:

$$\begin{aligned}
\sum_{\substack{T,S\in\Gamma\\ TS=G}} \chi(T)\mu(S) &= \sum_{T\in\Gamma} \chi(T)\mu(T^{-1}G) \\
&= \sum_{k=1}^{K} \sum_{T\in\, Q_k(\Gamma_\chi\cap\Gamma_\mu)} \chi(T)\mu(T^{-1}G) \\
&= \sum_{k=1}^{K} \sum_{R\in\, \Gamma_\chi\cap\Gamma_\mu} \chi(Q_k R)\mu(R^{-1}Q_k^{-1}G) \\
&= \sum_{\substack{k=1,\ldots,K\\ Q_k\in\Gamma_\chi,\ Q_k^{-1}G\in\Gamma_\mu}} \sum_{R\in\, \Gamma_\chi\cap\Gamma_\mu} \chi(Q_k R)\mu(R^{-1}Q_k^{-1}G) \\
&= \sum_{\substack{k=1,\ldots,K\\ Q_k\in\Gamma_\chi,\ Q_k^{-1}G\in\Gamma_\mu}} \sum_{R\in\, \Gamma_\chi\cap\Gamma_\mu} \chi(Q_k)\chi(R)\mu(R^{-1})\mu(Q_k^{-1}G) \\
&= \sum_{\substack{k=1,\ldots,K\\ Q_k\in\Gamma_\chi,\ Q_k^{-1}G\in\Gamma_\mu}} \chi(Q_k)\mu(Q_k^{-1}G) \sum_{R\in\, \Gamma_\chi\cap\Gamma_\mu} \chi(R)\mu(R^{-1}) \\
&= \sum_{\substack{k=1,\ldots,K\\ Q_k\in\Gamma_\chi,\ Q_k^{-1}G\in\Gamma_\mu}} \chi(Q_k)\mu(Q_k^{-1}G) \sum_{R\in\Gamma} \chi(R)\mu(R^{-1}) = 0\,.
\end{aligned}$$

QED

Now we see that the orthogonality of the characters determines the formal orthogonality of the corresponding twisted Reynolds projectors:

Lemma 27 *Let χ and μ be two characters on Γ which are orthogonal. Then P_χ and P_μ are orthogonal i.e., $P_\chi P_\mu = 0$.*

Proof. In view of Lemma 26 the proof is straightforward:

$$\begin{aligned}
P_\chi P_\mu &= \frac{1}{\#\chi}\frac{1}{\#\mu}\Big(\sum_{T\in\Gamma}\chi(T)\,T\Big)\Big(\sum_{S\in\Gamma}\mu(S)\,S\Big) \\
&= \frac{1}{\#\chi}\frac{1}{\#\mu}\sum_{T\in\Gamma}\sum_{S\in\Gamma}\chi(T)\mu(S)\,TS \\
&= \frac{1}{\#\chi}\frac{1}{\#\mu}\sum_{G\in\Gamma}\sum_{\substack{T,S\in\Gamma\\ TS=G}}\chi(T)\mu(S)\,G = 0\,.
\end{aligned}$$

QED

Corollary 28 *Let $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$ be a decomposition of the identity, and let us assume that the characters in $\mathcal{X}$ are mutually orthogonal. Then*

$$V = \bigoplus_{\chi\in\mathcal{X}} V_\chi$$

holds.

It is an easy exercise to check that in all the decompositions given in the tables in Section 5, the above criteria are also satisfied and therefore give a different view of the direct sum decompositions for the column spaces $\mathcal{B}$ and $\mathcal{C}$ and the function space $\mathcal{F}$.

It is not a coincidence that both criteria for the direct sum are holding simultaneously. In fact, a more careful scrutiny of the above proofs shows that the following three conditions are equivalent for a given decomposition of the identity $\{\gamma_\chi\}_{\chi\in\mathcal{X}}$:

1. For any linear action of Γ on a vector space V we have $V = \bigoplus_{\chi\in\mathcal{X}} V_\chi$.
2. For each $\chi \in \mathcal{X}$ we have $\gamma_\chi = 1/(\#\chi)$.
3. Any two distinct characters in $\mathcal{X}$ are orthogonal.

References

[1] E. L. Allgower, K. Böhmer, and Z. Mei, *A generalized equibranching lemma with applications to $D_4 \times \mathbf{Z}_2$ symmetric elliptic problems. Part I.* Preprint, University of Marburg, Fed. Rep. Germany, submitted, 1990.

[2] ——, *On a problem decomposition for semi-linear nearly symmetric elliptic problems*, in Parallel Algorithms for PDE's, W. Hackbusch, ed., vol. 31 of Notes on Numerical Fluid Mechanics, Braunschweig, Fed. Rep. Germany, 1991, Vieweg Verlag, pp. 1–17.

[3] E. L. Allgower and K. Georg, *Numerical Continuation Methods: An Introduction*, vol. 13 of Series in Computational Mathematics, Springer Verlag, Berlin, Heidelberg, New York, 1990. pp 388.

[4] E. L. Allgower and S. Gnutzmann, *An algorithm for piecewise linear approximation of implicitly defined two-dimensional surfaces*, SIAM J. Numer. Anal., 24 (1987), pp. 452–469.

[5] K. E. Atkinson, *A survey of boundary integral equation methods for the numerical solution of Laplace's equation in three dimensions*, in Numerical Solution of Integral Equations, M. Goldberg, ed., New York, 1990, Plenum Press.

[6] H. S. M. Coxeter, *Regular Polytopes*, Dover Publ., New York, 1973. Third edition.

[7] M. Dellnitz and B. Werner, *Computational methods for bifurcation problems with symmetries — with special attention to steady state and Hopf bifurcation points*, J. Comput. Appl. Math., 26 (1989), pp. 97–123.

[8] C. C. Douglas and J. Mandel, *The domain reduction method: High way reduction in three dimensions and convergence with inexact solvers*, Philadelphia, 1989, Fourth Copper Mountain Conference on Multigrid Methods, SIAM, pp. 149–160.

[9] ——, *A group theoretic approach to the domain reduction method: The commutative case*. In preparation, 1990.

[10] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, vol. 1, Springer-Verlag, Berlin, Heidelberg, New York, 1985.

[11] M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[12] J.-P. Serre, *Linear Representations of Finite Groups*, vol. 42 of Graduate Texts in Mathematics, Springer Verlag, Berlin, Heidelberg, New York, 1977.

[13] E. Stiefel and A. Fässler, *Gruppentheorethische Methoden und ihre Anwendung*, Teubner, Stuttgart, Fed. Rep. Germany, 1979.

[14] M. J. Todd, *The Computation of Fixed Points and Applications*, vol. 124 of Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, Heidelberg, New York, 1976.

[15] A. Vanderbauwhede, *Local Bifurcation Theory and Symmetry*, Pitman, London, 1982.