19) An arrangement of \( n \) coins in rows such that every coin touches exactly two coins in the row below, with the coins in each row forming a single contiguous block, and \( k \) coins in the bottom row is called a \( (n, k) \)-block fountain. For example, the following depicts a \((33, 12)\)-block fountain:

![Diagram of a (33, 12)-block fountain]

a) For a given \( k \), what is the maximum \( n \), such that there is an \((n, k)\)-block fountain?

b) Let \( f(k) \) denote the number of block fountains with a bottom row of length \( k \). Show that this satisfies the recursion

\[
f(j) = \sum_{j=1}^{k} (k - j)f(j) + 1.
\]

c) Show that the generating function \( F(t) = \sum f(n) t^n \) satisfies the functional equation:

\[
F(t) = \frac{t}{(1 - t)^2} (F(t) - 1) + \frac{t}{1 - t} + 1.
\]

**Hint:** Split the sum, from the \((k - j)\) factor, into two sums and use that

\[
\frac{1}{(1 - t)^{k+1}} = \sum_{n} \binom{n + k}{n} t^n.
\]

d) Find a closed form expression for \( F(t) \) and from that determine a formula for \( f(n) \). To verify your result you can use that the first few values of \( f(n) \) are 1, 1, 2, 5, 13, 34, 89. Does this remind you of something?

20) a) Show that the number of different mountain ranges you can draw with \( n \) upstrokes and \( n \) downstrokes is given by the Catalan number \( C_{n+1} \):

![Diagram of mountain ranges]

..._1..._1..._2.....

\[
\frac{1}{(1 - t)^{k+1}} = \sum_{n} \binom{n + k}{n} t^n.
\]
b) The pirate Blackbeard Catalan lets prisoners walk the plank in a particularly suspenseful way: The prisoner is placed on the plank edge on the boat and Blackbeard randomly draws balls from a bowl with \( n \) red and \( n \) blue balls. If he draws a red ball the prisoner must take a step forward. If he draws a blue ball, the prisoner may take a step back. (All steps are equal size.) If the prisoner ever steps back beyond the start of the plank he is pardoned. If he always stays on the plank he will have to walk to the end into the sea. What is the probability that the prisoner is pardoned?

21) Show that there are \( C_{n+1} \) rooted trees (fixing a particular “root” vertex) with \( n \) edges when distinguishing left and right branches:

\[
\begin{array}{c}
\cdots \\
1 \\
\vdots \\
2 \\
\vdots \\
5 \\
\vdots \\
\ddots
\end{array}
\]

22) Show that (ignoring symmetries) there are \( C_n \) different ways to divide a polygon with \( n + 1 \) sides into triangular regions by inserting diagonals that do not intersect in the interior.

23) Determine the number of ways to color the squares of a \( 1 \times n \) chessboard using red, yellow and green, where the number of red squares is even and there is at least one green square.

23\) (This is harder than you may think! See R.Guy, *The Second Strong Law of Small Numbers*, Mathematics Magazine, Vol. 63, No. 1 (Feb., 1990), pp. 3-20, Example 72.) Consider the number of different ways to fold a strip of \( n \) stamps, disregarding the distinction of front and back:

How many different possibilities are there to fold a strip of five stamps?