APPENDIX A

Appendix

Our intent in this appendix is to provide a linkage to the two previous papers the authors have written on restricted orbit equivalence. We say that an r-size is a size function m as defined in this current work, in Section 2.2. (The "r" denotes "rearrangement".) We will see that the notion of m-equivalence developed in [**20**] (where m is a p-size, where "p" denotes "permutation") is subsumed by our work here. On the other hand, we will not quite be able to show this for the work in [**36**]. What we will see though is that a slight strengthening in the definition of the equivalence relation associated with a 1-size, (size as defined in [**36**]) will make it possible to describe the equivalence relation as a restricted orbit equivalence in the sense we describe here. As we will see, this change will have no effect on the examples described in [**36**], and the m-f.d. systems for the original equivalences is unchanged by this strengthening. Whether some equivalence classes of arrangements are possibly changed for some 1-size, we do not know. As we now consider the definition in [**36**] to have been a very preliminary, perhaps unrefined, attempt to axiomatize the notion of restricted orbit equivalence, we have not pursued this issue further. Our main interest here is to bring the examples, in particular examples 3 and 4 (referred to as m_ψ and m_ϕ) under the umbrella of our work here.

We break this work into two sections. First we will handle the notion of a size, here called a 1-size (for 1-dimensional)) used in [**36**] for actions of $\mathbb{Z}$. This will be our most detailed section. Then we will discuss the notion of size first put forward in [**20**], here called p-sizes, for actions of $\mathbb{Z}^d$ In so doing we will also consider a preliminary axiomatization we have given for discrete amenable group actions, as it lies somewhere between that of [**20**] and what we have discussed here.

What we do here is rather technical, and is perhaps not of great interest to the general reader, we will not provide a great deal of motivation. We also will assume that an interested reader has a copies of [**20**] and [**36**] in hand to refer to.

A.1. 1-sizes

The notion of a 1-size, as set up in [**36**] begins at the level of permutations π of intervals of integers $(i, i+1, \ldots, j)$, lifts from here to bijections of $\mathbb{Z}$ via the definition

$$m(f) = \liminf_{\substack{i \to -\infty \\ j \to \infty}} m(\pi_{f,(i,j)})$$

where $\pi_{f,(i,j)}$ is the "push-together" of $f|_{(i,j)}$, i.e. the permutation of (i,j) that reorders points exactly as $f|_{(i,j)}$ does. In [**36**] a bijection of $\mathbb{Z}$ is indicated by a function f. To any such bijection f there corresponds a unique bijection fixing 0 that reorders points exactly as f does:

$$h(n) = f(n) - f(0).$$

(as we are in $\mathbb{Z}$ we use additive notation). Axiom ii) (page 7 of [**36**]) requires that m be "stationary", i.e. as a calculation on a permutation depend only on the number of terms and the way they are reordered, not on where the block (i,j) is placed. Thus the calculation $m(f)$ is translation invariant (regarding f as an element of $\mathbb{Z}^{\mathbb{Z}}$), $m(h) = m(f)$. This then implies, using our notation, that

$$m(h) = m(S^{\alpha}(h)) \text{ as } S^{\alpha}(h)(n) = \sigma(f)(n) - \sigma(f)(0).$$

The value of the 1-size distance between a pair of arrangments $(m(\alpha_1, \alpha_2))$ (called orderings on $\mathbb{Z}$) is now set to be the almost-surely constant value of $m(h_x^{\alpha_1,\alpha_2})$.

Notice that we could have written this as

$$m(\alpha_1, \alpha_2) = \int m(h_x^{\alpha_1,\alpha_2})\, d\mu.$$

Written this way, we see this can be viewed as the integral with respect to various invariant measures of the Borel function $m(h)$ on $\mathcal{G}$.

Two arrangements are defined to be m-equivalent (written $\alpha_1 \overset{m}{\sim} \alpha_2$) in [**36**] if there are full-group elements ϕ_i and $m(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0$. The given m is not a metric (soon we will see how easily it can be made one) but taking it to be one, what we are doing is taking the m-closure of the set of all $\alpha_1\phi$ in the full-group.

This approach is most different from our current one in that it evaluates a distance between arbitrary arrangements rather than just rearrangement pairs (α, ϕ). This is best understood by the introduction of the *collapsing* of one arrangement onto another, which converts a pair of arrangments into a rearrangment. This has a serious fault, that the collapsing of a pair $\alpha, \alpha\phi$ will produce a new pair $\alpha, \alpha\phi'$ where ϕ and

ϕ' are not necessarily close in L^1. This leads to a serious difficulty in obtaining Axiom 3 for the r-size we construct. This is more than just technical, as Axiom 3 plays a pivotal role in all of our development. It did as well in the development of [**36**], in particular in trying to develop the m-f.d. notion. This is the motivation for the consideration there of *dividing* rearrangements, and the restrictive definition of an m-joining, requiring the two processes to be covered by a third in which they sit linked by a *bounded rearrangement*. Lying behind this is the fact that if (α, ϕ) is in fact bounded then collapsings (α, ϕ') can be chosen such that ϕ and ϕ' are L^1-close (Corollary 4.5 of [**36**]).

This problem that a collapsing of a rearrangement is not necessarily a small L^1-change is also hidden in the interplay between the definition of $m(\alpha_1, \alpha_2)$ via collapsing of blocks and the requirement that m satisfy an approximate triangle inequality. For these to coexist for the same notion of m forces severe restrictions, and is in some sense the reason we can show all of our examples are equivalent to r-sizes, and also the reason that α-equivalence [**7**] comes from a p-size but not a 1-size.

We introduce two ideas to provide the linkage we wish to demonstrate between 1-sizes and r-sizes.

DEFINITION A.1.1. *We say a rearrangement (α, ϕ) is bounded if the function $f_x^{\alpha,\phi} = \alpha(x, \phi(x))$ is bounded μ-a.s. We say a rearrangement (α, ϕ) is blocked and bounded (abbreviated B&B) over a subset F if the return-time $r_F(x) = \min(n > 0 : T_n^\alpha(x) \in F)$ is bounded, and ϕ acts as a permutation of each return-time block*

$$(x, T_1^\alpha(x), \ldots, T_{r_F(x)-1}^\alpha(x)), x \in F.$$

LEMMA A.1.2. *Those ϕ with (α, ϕ) bounded form a subgroup of the full group and Theorem 5.2 tells us that those ϕ with (α, ϕ) B&B over some set are $\|\cdot, \cdot\|_s^\alpha$-dense in the full group.*

DEFINITION A.1.3. *Axiom v) of a 1-size tells us that for any $\varepsilon > 0$ there are δ and N so that if (α, ϕ) is B& B over some set F with*

1. *$r_F(x) \geq N$ for all $x \in F$ and*
2. *$m(\pi_{x,(0,r_F(x)-1)}) < \delta$ then*

$m(\alpha, \alpha\phi) < \varepsilon$. Any (α, ϕ) blocked and bounded over a set F satisfying (1.) and (2.) we will call m, ε-B& B.

DEFINITION A.1.4. *We say a sequence of rearrangements (α, ϕ_i) is nicely-blocked over F_1 if each of the rearrangements $(\alpha\phi_i, \phi_i^{-1}\phi_{i+1})$ is*

blocked and bounded over a set F_i where $F_{i+1} \subseteq F_i$. Moreover on each return time block

$$(x, T_1^\alpha(x), \ldots, T^\alpha_{r_{F_i}(x)-1}(x)), x \in F_i$$

the next full group element ϕ_i acts as a fixed power of T^α. That is to say, for each $x \in F_i$ there is a value $j(x)$ and for all x' in the return-time block over x, $\phi_{i+1}(x') = T^\alpha_{j(x)}(x')$.

We say the sequence of rearrangements is m-nicely blocked over F_i if for all $i \leq j$, $(\alpha\phi_i, \phi_i^{-1}\phi_j)$ is B&B over F_i and is in fact $m, 2^{-i} - 2^{-j}$-B&B.

LEMMA A.1.5. *If (α, ϕ_i) is nicely blocked over the sequence of sets F_i and $\mu(F_i) \underset{i}{\to} 0$ then $\alpha\phi_i$ is converging in L^1 to an arrangement β. Furthermore the sequence (β, ϕ_i^{-1}) is also nicely blocked over the sequence of sets $\phi_i(F_i)$ and $\beta\phi_i^{-1} \underset{i}{\to} \alpha$ in L^1. We get no symmetry like this for m-nicely blocked sequences as m is only assumed symmetric for arrangements and not for permutations in that knowing $m(\pi)$ is small does not a-priori imply that $m(\pi^{-1})$ is small.*

DEFINITION A.1.6. *For m a 1-size we say $\alpha_1 \overset{m}{\sim} \alpha_2$ nicely if there is an m-nicely blocked sequence of rearrangements ϕ_i with*

$$m(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0.$$

This relation cannot be assumed either symmetric or transitive.

DEFINITION A.1.7. *We say a 1-size m is a 1^+-size if for any $\varepsilon > 0$ there is a δ and for any rearrangement (α, ϕ) with $m(\alpha, \alpha\phi) < \delta$, and any $\varepsilon_1 > 0$ there is a bounded ϕ' with*

1. $\mu(\{x : \phi(x) \neq \phi'(x)\}) < \varepsilon_1$ *and*
2. $m(\alpha, \alpha\phi') < \varepsilon$.

THEOREM A.1.8. *All of the examples of 1-sizes discussed in* [**36**] *are 1^+-sizes.*

PROOF. Note: In these arguments we will assume the reader has access to [**36**]. The two examples (m_0 and m_∞) and two classes of examples (m_ψ and m_ϕ) all have the common feature that $m(\alpha_1, \alpha_2)$ can be defined without reference to collapsing. For m_0 this is done in Lemma 2.6 of [**36**], for m_ψ is Lemma 2.10 of [**36**], for m_ϕ in Lemma 2.14 of [**36**], and notice for m_∞, if $m_\infty(\alpha_1, \alpha_2) < 1$ then $\alpha_1 = \alpha_2$. Modifications of their proofs give the arguments we seek.

We really only need consider m_ψ and m_ϕ. Suppose we consider $\alpha = \alpha_1$ and $\alpha\hat{\phi} = \alpha_2$. Theorem 4.0.27 gives a method to modify $\hat{\phi}$ by less than any preassigned amount in L^1 to a $\hat{\phi}'$ so that $(\alpha, \hat{\phi}')$ is bounded. We show that given any $\varepsilon > 0$ there is a δ so that if the original $m(\alpha, \alpha\hat{\phi}) < \delta$ then the new value $m(\alpha, \alpha\hat{\phi}') < \varepsilon$. When we refer to a *tower block* or block of the tower here we mean a consecutive sequence of orbit points that begins at the base of the tower and moves up to the top of the tower.

For m_ψ, consider the construction in Theorem 4.0.27 in which the rearrangement $(\alpha, \hat{\phi})$ is modified to a bounded rearrangement $(\alpha, \hat{\phi}')$ by cutting the orbit into tower blocks by a Rokhlin tower, and taking those points thrown out of a block by $\hat{\phi}$ and mapping them to those points whose preimages are outside the block. We describe now why $m(\alpha, \alpha\hat{\phi}')$ must be small if $m(\alpha, \alpha\hat{\phi})$ is. In fact one can use the same set A of Lemma 2.10 of [**36**] to show this. Consider an interval (u, v) as a block of points in the orbit of some point x. This segment of orbit will be covered by a sequence of Rokhlin tower blocks, with two perhaps partial blocks at the end, which we write as (u, u_1) and (v_1, v). The cardinality of the set $f_x^{\alpha,\alpha\hat{\phi}'}((u, v))\triangle(u, v)$ is twice the cardinality of the set $B'(u, v)$ consisting of those points in (u, v) whose image under $f_x^{\alpha,\hat{\phi}'}$ is not in (u, v). Now the only points that can possibly be in $B'(u, v)$ are those in (u, u_1) or (v_1, v). Let $B(u, v)$ be those points in (u, v) that are thrown out by the original map $f_x^{\alpha,\hat{\phi}}$. Notice that if $j \in B'(u, v)$ but $j \notin B(u, v)$ it must be the case that j originally was thrown out of the Rokhlin tower block containing $T_j^\alpha(x)$, but to a point closer to x on the orbit, and when $\hat{\phi}'$ was constructed, it was moved to a point further out away from x but inside the Rohklin tower block containing $T_j^\alpha(x)$. This means j lies outside (u_1, v_1) but its image under $f_x^{\alpha,\hat{\phi}}$ lies inside (u_1, v_1). This implies that one of the two calculations

$$\#(f_x^{\alpha,\hat{\phi}}((u, v))\triangle(u, v)) \quad \text{or} \quad \#(f_x^{\alpha,\hat{\phi}}((u_1, v_1))\triangle(u_1, v_1))$$

is at least half the size of

$$\#(f^{\alpha,\hat{\phi}'}((u, v))\triangle(u, v)).$$

As $\psi(n)/\psi(2n)$ is required to be bounded away from zero in n, we obtain the result.

For m_ϕ, work from the characterization of m_ϕ in Lemma 2.14 of [**36**]. Note that we will represent full-group elements here by $\hat{\phi}$ and $\hat{\phi}'$ to avoid misunderstanding with the use of ϕ as a parameter of m. To

begin, fix $1/4 > \varepsilon > 0$ and select a value B_0 so that

$$\mu(\{x : |\alpha(x, \hat{\phi}(x))| < B_0\}) > 1 - \varepsilon/10$$

Next choose a value B_1 so that for any $|i| \geq B_1$,

$$\frac{\phi(i + B_0)}{\phi(i)} < 1 - \varepsilon/10 \quad \text{and} \quad \frac{\phi(\frac{\varepsilon}{10} i)}{\phi(i)} > 1 - \varepsilon/10.$$

Next choose a value B_2 so that

$$\mu(\{x : |\alpha(x, \hat{\phi}(x))| < B_2\}) > 1 - \frac{\varepsilon}{20 B_1}.$$

Thus:

$$\mu(\{x : f_x^{\alpha,\phi}((-B_1, B_1)) \subseteq (-B_2 - B_1, B_2 + B_1)\}) > 1 - \varepsilon/10.$$

Choose N so large that $N\varepsilon/20 > B_2 + B_1$.

In Theorem 4.0.27 be sure the Rokhlin tower has height $H \geq N$ and covers all but $\varepsilon/10$ in measure of X. Apply the construction of Theorem 4.0.27 to this tower. That is, map those points x in the tower whose image under $\hat{\phi}$ throws the point out of the tower block containing x to those points in this tower block that do not have $\hat{\phi}$-preimages in it. On points outside the tower, we replace $\hat{\phi}$ with the identity, to create the new full-group element $\hat{\phi}'$.

It is a direct calculation from our choices for B_0, B_1, B_2, N, and H that $\mu(\{x : \hat{\phi}(x) \neq \phi(x)\}) < \varepsilon$. To see that $m_\phi(\alpha, \alpha\hat{\phi}')$ is still small, we assume $m_\phi(\alpha, \alpha\hat{\phi}) < \delta < 1/4$ and let A be the set given by Lemma 2.14 of [**36**].

Remove from A all points that are

1. outside the tower,

2. within $\varepsilon H/5$ of either end of a tower block, or for which

3. $h_x^{\alpha,\alpha\hat{\phi}}|_{(-B_1,B_1)} \neq h_x^{\alpha,\alpha\hat{\phi}'}|_{(-B_1,B_1)}$.

Call the remaining set A'. It follows from our estimates that

$$\mu(A') > \mu(A) - \varepsilon.$$

We wish to show that this set now can be used in Lemma 2.14 of [**36**] to show that $m_\phi(\alpha, \alpha\hat{\phi}') < 3\delta + \varepsilon$. We do this by showing that for any $x \in A'$ and any x' in the orbit of x, that

$$\Big|\frac{\phi(|\alpha(x,x')|)}{\phi(|\alpha(\hat{\phi}'(x),\hat{\phi}'(x'))|)}-1\Big|<\delta+\varepsilon/3.$$

Note: if $|a-1|<d<1/2$ then $|a-\frac{1}{a}|<3d$ and if $|a-\frac{1}{a}|<d$ then $|a-1|<d$.

To begin, we only need consider those x' with $\hat{\phi}(x')\neq\hat{\phi}'(x')$ as otherwise we already have the estimate. Hence we can assume $|\alpha(x,x')|>B_1$. Suppose x' is such a point. There are two possibilities:

1. $\hat{\phi}'(x')=x'$ or

2. x' lies in a tower block, but is thrown out of it by $\hat{\phi}$, and hence $x''=\hat{\phi}(x')$ lies in this tower block, but $\hat{\phi}^{-1}(x'')$ does not.

To understand (1.) notice

$$\alpha(\hat{\phi}'(x),\hat{\phi}'(x'))=\alpha(x,x')+\alpha(\hat{\phi}'(x),x).$$

As $|\alpha(\hat{\phi}'(x),x)|\leq B_0$ and $|\alpha(x,x')|>B_1$, we conclude:

$$\Big|\frac{\phi(|\alpha(\hat{\phi}'(x),\hat{\phi}'(x'))|)}{\phi(|\alpha(x,x')|)}-1\Big|<\varepsilon/10.$$

To understand (2.) we describe the case when all three points x,x' and x'' lie in the same tower block. When they do not, the estimates improve. Once more there are three possibilities:

1. Both $|\alpha(x,x')|$ and $|\alpha(\hat{\phi}(x),x'')|>\varepsilon H/10$, or

2. $|\alpha(x,x')|\leq$ both $|\alpha(\hat{\phi}(x),x'')|$ and $\varepsilon H/10$ or

3. $|\alpha(\hat{\phi}(x),x'')|\leq$ both $|\alpha(x,x'))|$ and $\varepsilon H/10$.

In case (1.) we will have that $\frac{|\alpha(x,x')|}{|\alpha(\hat{\phi}(x),x'')|}$ must lie between $\varepsilon/10$ and $10/\varepsilon$ and both numerator and denomerator are larger than B_1 and we conclude that

$$\Big|\frac{\phi(|\alpha(x,x')|)}{\phi(|\alpha(\hat{\phi}'(x),\hat{\phi}'(x')|)|}-1\Big|<\varepsilon/10.$$

Cases (2.) and (3.) will be handled in a parallel fashion. For (2.), we will have $|\alpha(\hat{\phi}(x),\hat{\phi}(x'))|>\varepsilon H/10$ but $|\alpha(\hat{\phi}(x),x'')|\leq H$ and so:

$$1 \geq \frac{|\alpha(x,x')|}{|\alpha(\hat{\phi}(x),x'')|} \geq \frac{\varepsilon}{10}\frac{|\alpha(x,x')|}{|\alpha(\hat{\phi}(x),\hat{\phi}(x')|}.$$

Thus

$$\begin{aligned}\frac{\phi(|\alpha(x,x')|)}{\phi(\frac{10}{\varepsilon}|\alpha(\hat{\phi}(x),\hat{\phi}(x'))|)} &= \frac{\phi(|\alpha(x,x')|)}{\phi(|\alpha(\hat{\phi}(x),\hat{\phi}(x')|))}\frac{\phi(|\alpha(\hat{\phi}(x),\hat{\phi}(x'))|)}{\phi(\frac{10}{\varepsilon}|\alpha(\hat{\phi}(x),\hat{\phi}(x'))|)}\\ &> (1-\delta)(1-\varepsilon/10) > 1-\delta-\varepsilon/10.\end{aligned}$$

For (3.) just do the same argument as for (2.) but from the point of view of $\hat{\phi}^{-1}$, i.e. replace x with $\hat{\phi}(x)$ and x' with x'' and $\hat{\phi}$ with $\hat{\phi}^{-1}$ throughout. This completes the calculation and the proof. □

Having seen that the examples of [**36**] are 1^+-sizes, we now want to explain the link between nice m-equivalences and 1^+-sizes. To begin we now remind the reader in more detail of the construction of collapsings. This is carried out on pages 47–48 of [**36**]. By a *collapsing* $\overline{\alpha}_2$ of α_2 on α_1 one means the construction of a full group element ϕ ($\overline{\alpha}_2 = \alpha_1\phi$) so that (α_1,ϕ) is blocked and bounded over a set F and for each $x \in F$,

$$\pi_{f_x^{\alpha_1,\alpha_2},(0,r_F(x)-1)} = \begin{cases} \pi_{f_x^{\alpha_1,\overline{\alpha}_2},(0,r_F(x)-1)} & \text{or} \\ \text{id} & \end{cases}.$$

That is to say, $\overline{\alpha}_2$ is obtained by *pushing together* or *collapsing* the image points $f_x^{\alpha_1,\alpha_2}((0,\ldots,r_F(x)-1))$ into a consecutive block. It is convenient to allow the collapsing to act as the identity on some of these blocks. Such a collapsing is called an $\hat{\varepsilon},K$-collapsing if (in our vocabulary)

$$\mu(\{x : h_x^{\alpha_1,\alpha_2}|_{(-K,K)} = h_x^{\alpha_1,\overline{\alpha}_2}|_{(-K,K)}) > 1-\hat{\varepsilon}.$$

Note: Having $\hat{\varepsilon}$ in the definition in [**36**] will give an $\hat{\varepsilon}$ in this definition. Having an $\hat{\varepsilon}$ in this definition will give $\sqrt{\hat{\varepsilon}}$ in the [**36**] definition.

DEFINITION A.1.9. *In* [**36**] *there was also given the notion of an* $\varepsilon,\hat{\varepsilon},K$*-collapsing where one also asked that*

$$m(\alpha_1,\overline{\alpha}_2) < \varepsilon.$$

We give here a stronger definition for this notion than the one actually used in [**36**]. *Notice that an* $\hat{\varepsilon},K$*-collapsing is a B&B rearrangement. If it is in fact an* m,ε*-B&B arrangement we call it an* $\varepsilon,\hat{\varepsilon},K$*-collapsing (See Definition A.1.2) This in particular forces* $m(\alpha_1,\alpha_1\phi) < \varepsilon$.

LEMMA A.1.10. *(3.2 of* [**36**]*) Given any $\varepsilon > 0$ there is a δ so that if $m(\alpha_1, \alpha_2) < \delta$ then for all $\hat{\varepsilon}, K$ there exists an $\varepsilon, \hat{\varepsilon}, K$-collapsing $\overline{\alpha}_2$ of α_2 on α_1.*

PROOF. Although [**36**] only claims to obtain its definition of an $\varepsilon, \hat{\varepsilon}, K$-collapsing, it does so by obtaining a collapsing satisfying our Definition A.1.9. □

Collapsings behaves particularly nicely with respect to bounded rearrangements (or more generally what are called dividing reorderings in [**36**]). We recall a few simple facts about rearrangments of $\mathbb{Z}$. Recall that $\alpha\phi = \alpha\phi'$ iff $\phi' = T_j^\alpha \circ \phi$ (we are assuming ergodicity universally.) Next recall that for (α, ϕ) a bounded rearrangement,

$$\int \alpha(x, \phi(x))\, d\mu$$

is always an integer, which we call $J(\alpha, \phi)$ (the average translation induced by ϕ.) Thus we can replace ϕ by $\phi' = T^\alpha_{-J(\alpha,\phi)}\phi$ to obtain a rearrangement (α, ϕ') with $J(\alpha, \phi') = 0$ and $\alpha\phi = \alpha\phi'$. As the calculation of m for a rearrangment is $m(\alpha, \alpha\phi)$, this change is invisible to m, and hence we can always assume our rearrangments are such that $J(\alpha, \phi) = 0$. Also notice that for a collapsing $\alpha_1\phi$ of any α_2 on α_1 we always have $J(\alpha, \phi) = 0$. The next lemma explains the value of this.

LEMMA A.1.11. *If (α, ϕ) is a bounded rearrangement with $J(\alpha, \phi) = 0$ then for the collapsing $\alpha\phi'$ of $\alpha\phi$ onto α over a set F,*

$$\mu(\{x : \phi(x) \neq \phi'(x)\}) \leq 2\mu(F)\|\alpha(x, \phi(x))\|_\infty.$$

PROOF. See Lemma 4.3 and Corollaries 4.4 and 4.5 of [**36**]. □

We now need a little trick which we will discuss in more detail later under the notion of a *convergence criterion.* For now we need just a very special case.

LEMMA A.1.12. *Suppose $(\alpha, \phi_i), i = 1, \ldots, N$ is a finite sequence of blocked and bounded rearrangements which for all $i \leq j$, $(\alpha\phi_i, \phi_i^{-1}\phi_j)$ is $m, 2^{-i} - 2^{-j}$-B&B. Assume further that they are nice in that the base sets F_i over which they are blocked are nested and on any return-time block of F_i the rearrangement $(\alpha\phi_i, {\phi_i}^{-1}\phi_{i+1})$ acts by a constant translation. (This is saying that the finite list is the beginning of a potentially m nicely blocked sequence.)*

There is then a δ_N depending on this finite list so that for any bounded rearrangement (α, ϕ_{N+1}) with $J(\alpha, \phi_{N+1}) = 0$ and

$$m(\alpha\phi_N, \alpha\phi_{N+1}) < \delta$$

we will be able to construct a new list (α, ϕ_i'), $i = 1, \ldots, N+1$ *of blocked and bounded rearrangements, satisfying all the conditions of the original list but with* N *replaced by* $N+1$ *and with, for all* $i = 1, \ldots, N+1$,

$$\mu(\{x : \phi_i(x) \neq \phi_i'(x)\}) \leq 2^{-N-9}.$$

PROOF. Notice that if δ is small enough, we can obtain $\alpha\phi_{N+1}'$ as a collapsing of $\alpha\phi_{N+1}$ on $\alpha\phi_N$ with all its conditions, and its base set F_{N+1} as small as we like. Set $\psi = \phi_{N+1}^{-1}\phi_{N+1}'$ and define $\phi_i'' = \phi_i\psi$ for all i. We can assume $\mu(\{x : \psi(x) \neq x\}) \leq 2^{-N-10}$. The rearrangements ϕ_i'' are B&B with base sets $F_i'' = \psi^{-1}(F_i)$. The base set F_{N+1} may not lie in the various F_i'', $i \leq N$. This is really all we lack. For each $x \in F_{N+1}$ consider the return-time block for F_N'' containing it, and the return-time blocks just preceding and following this one. Be sure that $\mu(F_{N+1})$ is so small that the set all points in these blocks has measure less than 2^{-N-11}. Modify each ϕ_i' to ϕ_i'', $i \leq N$ by making it the identity on these blocks. Now modify the sets F_i'', $i \leq N$ on these blocks so that all the new return-time blocks are still long enough for the collapsings on them to still be $m, 2^{-i} - 2^{-j}$-B&B, so that they are nested and all contain the new point in F_{N+1}. □

The following result now follows rather directly from this corollary. It is complex to state, but the basic picture is easy to understand.

THEOREM A.1.13. *For each* N *and finite list of bounded rearrangements* $(\alpha, \phi_1), (\alpha, \phi_2), \ldots, (\alpha, \phi_N)$ *with* $J(\alpha, \phi_i) = 0$ *there is a value*

$$\delta_N((\alpha, \phi_1), \ldots, (\alpha, \phi_N))$$

so that if (α, ϕ_i) *is an infinite sequence of bounded rearrangements all with J=0 satisfying*

$$m(\alpha\phi_i, \alpha\phi_{i+1}) < \delta_i((\alpha, \phi_1), \ldots, (\alpha, \phi_i)),$$

then there is an m*-nicely blocked and bounded sequence of rearrangements* (α, ϕ_i') *with*

$$\mu(\{x : \phi_i'(x) \neq \phi_i(x)\}) \leq 2^{-7}$$

for all i.

This will mean that there is a β *with* $m(\alpha\phi_i', \beta) \underset{i}{\to} 0$ *and hence that* $\alpha \overset{m}{\sim} \beta$. *Furthermore we will see that there is a* ψ *in the full-group with*

$$\|\alpha\phi_i, \beta\psi\|_1 \underset{i}{\to} 0$$

which is to say the $\alpha\phi_i$ *are converging to an arrangement* $\beta\psi$ *that is* m*-equivalent to* α.

PROOF. We will construct inductively using Lemma A.1.12. At stage N of the induction we will have constructed a sequence of rearrangements (α, ϕ_i^N), $i = 1, \ldots, N$. These will satisfy the hypotheses of Lemma A.1.12. The value δ_N will be the value δ_N of this Lemma. Supposing

$$m(\alpha\phi_N, \alpha\phi_{N+1}) < \delta_N,$$

let $\phi_{N+1}\phi_N^{-1}\phi_N$ be the "ϕ_N" of the lemma, as $(\alpha\phi_N, \phi_N^{-1}\phi_{N+1})$ and $(\alpha\phi'_N, {\phi'_N}^{-1}\phi_{N+1}{\phi'_N}^{-1}\phi_N)$ are identical in distribution. Lemma A.1.12 now tells us we can modify each of the ϕ_i^N to the new ϕ_i^{N+1} and add on the new term ϕ_{N+1}^{N+1}.

From Lemma A.1.12 we know that for each i,

$$\mu(\{x : \phi_i^N(x) \neq \phi_i^{N+1}(x)\}) \leq 2^{-N-9}$$

implying the full-group elements ϕ_i^N converge in N to the full-group element we call ϕ'_i with

$$\mu(\{x : \phi_i(x) \neq \phi'_i(x)\}) \leq \mu(\{x : \phi_i^i(x) \neq \phi_i(x)\}) + 2^{-i-8}.$$

Our inductive construction makes

$$\mu(\{x : \phi_i^i(x)\phi_i^{-1}(x) \neq \phi_{i-1}^{i-1}(x)\phi_{i-1}^{-1}(x)\}) \leq 2^{-i-9}$$

and so summing backward to $i = 1$ we see that for all i

$$\mu(\{x : \phi'_i(x) \neq \phi_i(x)\}) \leq 2^{-7}.$$

(We will refine this estimate a bit later.)

That the limit elements ϕ'_i form an m-nicely blocked and bounded sequence of rearrangments is clear from the inductive construction.

Notice that for an m-nicely blocked sequence of rearrangements (α, ϕ_i) the $\alpha\phi_i$ converge in m to some β.

We saw above that

$$\mu(\{x : \phi_i^i(x)\phi_i^{-1}(x) \neq \phi_{i-1}^{i-1}(x)\phi_{i-1}^{-1}(x)\}) \leq 2^{-i-9}$$

and so

$$\mu(\{\{x : \phi'_i(x)\phi_i^{-1}(x) \neq \phi'_{i-1}(x)\phi_{i-1}^{-1}(x)\}) \leq 2^{-i-6}$$

which is summable. This tells us that the ${\phi'_i}^{-1}\phi_i$ are L^1-Cauchy in the full group, hence converging to some ψ. Thus $\alpha\phi_i = \alpha\phi'_i{\phi'_i}^{-1}\phi_i$ converges in $\mathcal{A}$ to the arrangement $\beta\psi$ finishing the result. $\square$

The following corollary now indicates the link between 1^+-sizes and m-nicely blocked sequences of rearrangments.

COROLLARY A.1.14. *Suppose m is a 1^+-size and $\alpha \overset{m}{\sim} \beta$. There is then an arrangement γ and full group elements $\hat{\psi}_i$ and $\hat{\psi}'_i$ $i = 0, 1, \ldots$ so that the rearrangments*

$$(\alpha\hat{\psi}_0, \hat{\psi}_0^{-1}\hat{\psi}_i)$$

and

$$(\beta\hat{\psi}'_0, (\hat{\psi}'_0)^{-1}\hat{\psi}'_i)$$

are both m-nicely blocked and bounded and both $m(\alpha\hat{\psi}_i, \gamma)$ and $m(\beta\hat{\psi}'_i, \gamma)$ tend to zero in i.

PROOF. We will begin by constructing sequences ψ_i and ψ'_i with

$$m(\alpha\psi_i, \psi_i^{-1}\psi_{i+1}) < \delta_i((\alpha\psi_0, \psi_0^{-1}\psi_1), \ldots, (\alpha\psi_0, \psi_0^{-1}\psi_i))$$

and

$$m(\beta\psi'_i, {\psi'_i}^{-1}\psi'_{i+1}) < \delta_i((\beta\psi'_0, {\psi'_0}^{-1}\psi'_1), \ldots, (\beta\psi'_0, {\psi_0}^{-1}\psi'_i))$$

and

$$m(\alpha\psi_i, \beta\psi'_i) \underset{i}{\to} 0.$$

Theorem A.1.13 now completes the result by giving $\hat{\psi}_i$ and $\bar{\psi}_i$ that are m-nicely B&B with both

$$\alpha\hat{\psi}_i \to \gamma \quad \text{and} \quad \beta\bar{\psi}_i \to \gamma'$$

and as $m(\alpha\psi_i, \beta\psi'_i) \to 0$, $\gamma = \gamma\psi$ for some ψ. Setting $\hat{\psi}'_i = \bar{\psi}_i\psi$ is all we need do.

To construct ψ_i and ψ'_i we will alternately add terms to each of the sequences ψ_i and ψ'_i. Moreover, at each stage all the full group elements in both lists that we have already constructed will be slightly perturbed by right multiplication by an L^1-small full group element. As we do this to all terms, we do not change the relations among them, and as these perturbations will converge in the full group, the two sequences will converge to sequences in the full-group. The two full-group elements ψ_0 and ψ'_0 are in fact the total perturbations. In this process we will not index over this sequence of perturbations but will always just indicate ψ_i or ψ'_i, noting that they have been perturbed but have kept their names. To begin, let both ψ_0 and ψ'_0 be the identity (they will, as indicated above, change.) ψ_1 is arbitrary, but choose it so that $(\alpha\psi_0, \psi_0^{-1}\psi_1)$ is bounded. Now using Theorem A.1.13 we get a value $\delta_1 = \delta_1(\alpha\psi_0, \psi_0^{-1}\psi_1)$. Choose ψ'_1 so that both

1. $m(\beta\psi'_1, \alpha\psi_1) < \delta_1$ and

2. $(\beta\psi_0', {\psi_0'}^{-1}\psi_1')$ is bounded.

Do this by first obtaining (1.), then perturbing ψ_0' by at most ε_1 in L^1 to obtain (2.) as well.

To construct ψ_2, notice we now can apply Theorem A.1.13 to the β side to get

$$\delta_1' = \delta_1'(\beta\psi_0', {\psi_0'}^{-1}\psi_1').$$

Choose ψ_2 so that both

1. $m(\alpha\psi_2, \beta\psi_1') < \delta_1'$ and

2. $(\alpha\psi_0, \psi_0^{-1}\psi_2)$ is bounded.

Do this by first obtaining(1.), then perturbing both ψ_0 and ψ_1 by right multiplicating by a full-group element that is within some ε_2 in L^1 of the identity to obtain (2.).

To construct ψ_2', apply Theorem A.1.13 to the α side to get

$$\delta_2 = \delta_2((\alpha\phi_0, \phi_0^{-1}\phi_1), (\alpha\phi_0, \phi_0^{-1}\phi_2)).$$

Choose ψ_2' so that both

1. $m(\beta\psi_2', \alpha\psi_2) < \delta_2$ and

2. $(\beta\psi_0', {\psi_0'}^{-1}\psi_2')$ is bounded.

Do this by first obtaining (1.), then perturbing the list ψ_0', ψ_1' by right multiplying by a full-group element that is within some ε_2 in L^1 of the identity to obtain (2.).

To finish we just continue this induction alternating sides and using successively the bounds δ_N of Theorem A.1.13. Making the ε_i summable will force convergence of the perturbations, and hence existence of the ψ_i and ψ_i' as described. □

This rather long effort has now established the role of nicely blocked sequences of rearrangements. The Fundamental Lemma (Lemma 4.8, page 100) of [**36**] actually describes an inductive step of precisely the same form as this result. The sequences of arrangments $(\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\alpha_0', \alpha_1', \ldots \alpha_m')$ constructed there are not directly assumed to be m-nicely blocked and bounded but the mass of structure in the definition of an (n, m)-approximation says we could in fact collapse all the terms and get a sequence that is. This observation is not going to matter

much to us as boundedness is all that will really matter in our discussion of m-f.d. actions. We point this out here just to indicate the parallelism

We proceed to construct an r-size associated with every 1-size which, for 1^+-sizes will have precisely the same equivalence classes, and universally the same finitely determined classes. Both these resulte will be based on the fact that for a sequence of rearrangements $\alpha\phi_i \underset{i}{\to} \beta$ and $\beta\phi_i^{-1} \underset{i}{\to} \alpha$ in L^1 which are bounded in both directions, convergence for the 1-size and the corresponding r-size will be equivalent.

Let m be a 1-size. Our first simple task is to bound m. For a permutation π, define

$$m^1(\phi) = \min(m(\pi), 1).$$

LEMMA A.1.15. *If m is a 1-size, then m^1 as defined above is as well. Moreover, this new 1-size has the same equivalence classes as m.*

PROOF. It is quite simple to check the six conditions of [**36**]. Notice that using m^1 the requirement that $m(\alpha_2, \alpha_3) < 1$ in axiom vi) can be completed omitted, i.e. one has:

iv) given any $\varepsilon > 0$ there is a $\delta > 0$ so that if $m^1(\alpha_1, \alpha_2) < \delta$ then for any α_3,

$$m^1(\alpha_1, \alpha_3) \leq m^1(\alpha_2, \alpha_3) + \varepsilon.$$

To see that the equivalence classes are unchanged, just remember that for $\alpha_1 \overset{m}{\sim} \alpha_2$ means there are full-group elements ϕ_i with

$$m(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0.$$

We observe that $m^1(\alpha, \beta) = \min(m(\alpha, \beta), 1)$. Thus $m(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0$ is equivalent to $m^1(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0$. Thus the two equivalence relations agree. □

Our next step is to replace m^1 with a metric. To do this we must leave behind permutations. We us a standard trick for creating a metric.

For any arrangement α_1 define a function:

$$F_{\alpha_1} : \mathcal{A} \to [0, 1] \text{ by}$$

$$F_{\alpha_1}(\alpha_2) = m^1(\alpha_1, \alpha_2).$$

Set

$$m^2(\alpha_1, \alpha_2) = \|F_{\alpha_1}, F_{\alpha_2}\|_\infty.$$

It is easy to see that m^2 is a metric. The only non-obvious point is that the map $\alpha_1 \to F_{\alpha_1}$ is 1-1. To see this just note that the only zero of this function is at α_1.

LEMMA A.1.16. *Given any $\varepsilon > 0$ there is a $\delta > 0$ so that:*

a) *If $m^1(\alpha_1, \alpha_2) < \delta$ then $m^2(\alpha_1, \alpha_2) < \varepsilon$ and*

b) *if $m^2(\alpha_1, \alpha_2) < \delta$ then $m^1(\alpha_1, \alpha_2) < \varepsilon$.*

PROOF. Statement (b) is trivial as

$$m^2(\alpha_1, \alpha_2) = \sup_{\alpha_3}(|m^1(\alpha_1, \alpha_3) - m^1(\alpha_2, \alpha_3)|) \geq m^1(\alpha_1, \alpha_2).$$

For (a), using iv) stated above for m^1, if $m^1(\alpha_1, \alpha_2) < \delta$ then for all α_3,

$$|m(\alpha_1, \alpha_3) - m(\alpha_2, \alpha_3)| < \varepsilon$$

which is what we want. □

LEMMA A.1.17. *For m a 1-size, $\alpha_1 \overset{m}{\sim} \alpha_2$ iff α_2 is in the m^2-closure of the full-group orbit of α_1.*

PROOF. To say $m(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0$, as we just saw in Lemma A.1.16 is equivalent to $m^2(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0$. □

We need to take two more steps in modifying m to become an r-size. At present m^2 satisfies Axioms 1 and 2 of an r-size (Axiom 2, that $(\Gamma, m^2_\alpha) \overset{\mathrm{id}}{\to} (\Gamma, \|\cdot,\cdot\|^\alpha_{\mathrm{w}})$ is uniformly continuous follows from Axiom iv) of a 1-size, Lemma A.1.16 and a bit of thought). As we indicated earlier Axiom 3 is not clearly associated with the conditions of a 1-size and acquiring it will take a little two-step.

DEFINITION A.1.18. *For m a 1-size let*

$$m^3(\alpha_1, \alpha_2) = \inf(m^2(\alpha_1, \alpha_2\phi) + \mu(\{x : \phi(x) \neq x\}) : \phi \in \mathrm{FG}(\mathcal{O})).$$

Note: we use $\mu(\{x : \phi(x) \neq x\})$ rather than $\|\phi, \mathrm{id}\|^\alpha_s$ as it has better equivariance properties. In particular relative to the metric $\mu(\{x : \phi_1(x) \neq \phi_2(x)\})$ the full-group is a Polish metric space and this metric is invariant under both right and left composition.

LEMMA A.1.19. *For m a 1-size, m^3 is a metric and for any arrangement α, the m-orbit closure of $\{\alpha\phi\}$ is the same as the m^3-orbit closure. Moreover m^3 satisfies Axiom 2 of an r-size.*

PROOF. To check the triangle inequality, suppose $m^3(\alpha_1, \alpha_2) = a$ and $m^3(\alpha_2, \alpha_3) = b$. For any $\varepsilon > 0$ there exist ϕ_1 and ϕ_2 with

$$\begin{aligned} a \leq & m^2(\alpha_1, \alpha_2\phi_1) + \mu(\{x : \phi_1(x) \neq x\}) + \varepsilon, \\ b \leq & m^2(\alpha_2, \alpha_2\phi_2) + \mu(\{x : \phi_2(x) \neq x\}) + \varepsilon \\ = & m^2(\alpha_2\phi_1, \alpha_3\phi_2\phi_1) + \mu(\{x : \phi_2\phi_1(x) \neq \phi_1(x)\}) + \varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} m^3(\alpha_1, \alpha_3) \leq & m^2(\alpha_1, \alpha_3\phi_2\phi_1) + \mu(\{x : \phi_2\phi_1(x) \neq x\}) \\ \leq & a + b + 2\varepsilon. \end{aligned}$$

To see that the orbit closures agree, notice that if $m^3(\alpha_1, \alpha_2\phi) < \varepsilon$ then there is a ϕ' with $m^2(\alpha_1, \alpha_2\phi\phi') + \mu(\{x : \phi'(x) \neq x\}) < \varepsilon$ and hence $m^2(\alpha_1, \alpha_2\phi\phi') < \varepsilon$ and the m^2 orbit closure contains the m^3. The other direction is trivial.

As m^2 satisfies Axiom 2 of an r-size, and $\mu(\{x : \phi(x) \neq x\})$ is an r-size, m^3 satisfies Axiom 2 as well. □

DEFINITION A.1.20. *Let*

$$m^4(\alpha, \phi) = \limsup_{\|(\alpha,\phi),(\alpha',\phi')\|_* \to 0} m^3(\alpha', \alpha'\phi').$$

Our next task is to show that m^4 is an r-size.

LEMMA A.1.21. *For m a 1-size and α an arrangement $m^4_\alpha(\phi_1, \phi_2) = m^4(\alpha\phi_1, \phi_1^{-1}\phi_2)$ is a pseudometric on the full group.*

PROOF. This amounts to verifying the triangle inequality, i.e. given α, ϕ_1, and ϕ_2 that

$$m^4(\alpha, \phi_2) \leq m^4(\alpha, \phi_1) + m^4(\alpha\phi_1, \phi_1^{-1}\phi_2).$$

This is a job for a copying lemma.

By Corollary 5.9, for any $\delta > 0$ there is a $\delta_1 > 0$ so that if

$$\|(\alpha, \phi_2), (\alpha', \phi_1')\|_* < \delta_1,$$

then there is a ϕ_1' in the full group of $T^{\alpha'}$ with both

$$\|(\alpha, \phi_1), (\alpha', \phi_1')\|_* < \delta \quad \text{and} \quad \|(\alpha\phi_1, \phi_1^{-1}\phi_2), (\alpha', \alpha'\phi_1'^{-1}\phi_2')\|_* < \delta.$$

Note: This is stretching Corollary 5.9 a bit. To see that this stretch is acceptable notice that as

$$\|(T^{\alpha}, g_{(\alpha,\phi_1)} \vee g_{(\alpha,\phi_2)}), (T^{\alpha'}, g_{(\alpha',\phi_1')} \vee g_{(\alpha',\phi_2')})\|_* \to 0$$

we will have both

$$\|(\alpha, \phi_1), (\alpha', \phi_1')\|_* \to 0 \qquad \text{and}$$
$$\|(\alpha\phi_1, \phi_1^{-1}\phi_2), (\alpha', \alpha'\phi_1'^{-1}\phi_2')\|_* \to 0.$$

Thus use finite partitions that approximate the two possibly infinite partitions $g_{(\alpha,\phi_2)}$ and $g_{(\alpha',\phi_2')}$ as Q and Q_1 in Corollary 5.9.

Continuing, for any $\varepsilon > 0$ we can choose δ and δ_1 small enough that

$$m^4(\alpha, \phi_2) \leq m^3(\alpha', \alpha'\phi_2') + \varepsilon,$$
$$m^3(\alpha', \alpha'\phi_1') \leq m^4(\alpha, \phi_1) + \varepsilon, \text{ and}$$
$$m^3(\alpha'\phi_1', \phi_1'^{-1}\phi_2') \leq m^4(\alpha\phi_1, \phi_1^{-1}\phi_2) + \varepsilon.$$

We conclude

$$m^4(\alpha, \phi_2) \leq m^4(\alpha, \phi_1) + m^4(\alpha\phi_1, \phi_1^{-1}\phi_2) + 3\varepsilon$$

to complete the result. □

THEOREM A.1.22. *For m a 1-size, m^4 is an r-size.*

PROOF. Lemma A.1.20 says we only need to verify Axioms 2 and 3. Lemma A.1.19 tells us m^3 satisfies Axiom 2 and as

$$m^4(\alpha, \phi) \geq m^3(\alpha, \phi)$$

we conclude m^4 does as well. The definition of m^4 is such as to make Axiom 3 direct. □

We will complete our work now by showing that for any 1-size, if $\alpha_1 \overset{m}{\sim} \alpha_2$ by an m-nicely blocked sequence of rearrangements then $\alpha_1 \overset{m^4}{\sim} \alpha_2$.

Suppose (α, ϕ) is B&B over a set F and hence r_F is bounded. For each $x \in F$ let π_x^F be the permutation of $(0, 1, \ldots, r_F(x) - 1)$ with

$$\phi(T_j^{\alpha}(x)) = T^{\alpha}_{\phi_x^F(j)}(x),\ 0 \leq j < r_F(x).$$

This is a finite list of permutations. Add on to this all identity permutations of finite blocks $(0, \ldots, N-1)$ and call the resulting collection of permutations $\Pi(\phi, F)$. This is an infinite set of permutations only because we included all the identities.

LEMMA A.1.23. *Suppose (α, ϕ) is blocked and bounded over a set F, and for some N, $r_F(x) \geq N$ for all $x \in F$. Then for any $\varepsilon > 0$ there is a δ so that for any (α', ϕ') with*

$$\|(\alpha, \phi), (\alpha', \phi')\|_* < \delta$$

there is a ϕ'' with $\mu(\{x : \phi'(x') \neq \phi''(x')\}) < \varepsilon$ and (α', ϕ'') is blocked and bounded over some set F' with

1. $\Pi(\phi'', F') \subseteq \Pi(\phi, F)$ *and*
2. $r_{F'}(x') \geq N$ *for all* $x' \in F'$.

PROOF. As r_F is bounded let $r_F \leq B$ and choose $4M = [10B/\varepsilon]+1$. Choose δ so small that knowing

$$\|(\alpha, \phi), (\alpha', \phi')\|_* < \delta$$

implies that for all but $\varepsilon/10$ in measure of the $x' \in X'$ there is an $x \in X$ with

$$f^{\alpha,\phi}(T_j^{\alpha}(x)) = f^{\alpha',\phi'}(T_j^{\alpha'}(x')), \quad 0 \leq j < M.$$

Call a choice for this point $x(x')$. We can and do choose the point $x(x')$ to depend only on the list of values $f^{\alpha',\phi'}(x'), \quad 0 \leq j < M$. Hence there are only finitely many points of the form $x(x')$. Let Q be the finite partition of X' into sets according to the value $f^{\alpha',\phi'}(x)$ if it is $\leq B$ in absolute value, and the complimentary set. Construct a Rokhlin tower for $T^{\alpha'}$ with base C and height M, covering all but $\varepsilon/10$ of X' with

$$C \perp \bigvee_{j=0}^{M-1} T_{-j}^{\alpha'}(Q).$$

Let $C_0 \subseteq C$ consist of those x' for which there is **no** value $x(x')$. Hence $\mu(C_0) \leq \frac{\varepsilon}{10}\mu(C)$.

Partition the remainder of C according to $\vee_{j=0}^{M-1} T_{-j}^{\alpha'}(Q)$ which is the same as to say, according to the value $x(x')$. Call these sets $C_1, C_2, \ldots, C_k$. Let x_i be the value $x(C_i), \quad i \geq 1$.

For each x_i let

$$0 \leq j_1^i < j_2^i < \cdots < j_{L(i)}^i < M$$

be those indices with

$$T_{j_t^i}^{\alpha}(x_i) \in F.$$

Any two successive values $j_t^i < j_{t+1}^i$ are at least N and at most B apart.

To start a definition of F', put

$$\bigcup_{i=1}^{K} \bigcup_{t=2}^{l(i)-1} T_{j_t^i}^{\alpha'}(C_i)$$

in F'.

To start a definition of ϕ'', for each $x' \in C_i$ and each block of points

$$(T_{j_t^i}^{\alpha'}(x'), \ldots, T_{j_{t+1}^i - 1}(x')), \quad 2 \leq t < L(i) - 1$$

just set $\phi'' = \phi$. Thus, as this is a block between two occurrence of F' and the functions $f^{\alpha', \phi''}(x')$ and $f^{\alpha, \phi}(x_i)$ agree on $(0, M-1)$, we have

$$\pi^{\phi''}_{T_{j_t^i}^{\alpha'}(x')} = \pi^{\phi}_{T^{\alpha}_{j_t^i(x_i)}}.$$

For all other points x', those in the tower over C_0 or outside of the tower, or below level j_2^i or above level $j_{L(i)-1}^i - 1$ in the tower over C_i, set $\phi''(x') = x'$. Notice this is a full-group element. Moreover, notice that the set on which we just defined $\phi'' = \mathrm{id}$ always occurs in an orbit in blocks at least $2N$-long. In each such block then we can select a list of points to add to F' so that $r_{F'}$ is bounded, but is always $\geq N$. On each such block, the permutation induced by ϕ'' will just be the identity. Both conditions (1.) and (2.) are now evident. □

COROLLARY A.1.24. *Suppose m is a 1-size. Given any $\varepsilon > 0$ there is a $\delta > 0$ so that if (α, ϕ) is bounded, $J(\alpha, \phi) = 0$ and $m(\alpha, \alpha\phi) < \delta$. Then $m^4(\alpha, \phi) < \varepsilon$.*

PROOF. Choose ε_1 so that if $m(\alpha, \phi) < \varepsilon_1$ then $m^3(\alpha, \phi) < \varepsilon$. Choose δ so that if $m(\alpha, \alpha\phi) < \delta$ and bounded with $J(\alpha, \phi) = 0$ then for all δ_1 and K there are m, ε_1-nicely blocked ε, δ_1, K-collapsings $(\alpha\phi_1)$ of $\alpha\phi$ onto α with $\mu(\{x : \phi_1(x) \neq \phi(x)\}) < \delta_1$. Arguing analogously to Lemma A.1.23 for any (α', ϕ') close enough in distribution to (α, ϕ) we will be able to copy the collapsing $\alpha\phi_1$ into (α', ϕ') as an $\varepsilon_1, \delta_1, K$-collapsing with $\mu_1(\{x' : \phi_1'(x') \neq \phi'(x')\}) < 2\delta_1$. In particular the rearrangment (α', ϕ_1') will be m, ε_1-nicely blocked and hence $m(\alpha', \phi_1') < \varepsilon_1$ and $m^3(\alpha', \phi') \leq \varepsilon$. But of course this says $m^4(\alpha, \phi) < \varepsilon$. □

THEOREM A.1.25. *Suppose m is a 1-size. For any α_1 and α_2 for which there are ϕ_i, $m(\alpha_1\phi_i, \alpha_2) \underset{i}{\to} 0$ and all (α_1, ϕ_i) and (α_2, ϕ_i^{-1}) are bounded we will conclude $\alpha_1 \overset{m^4}{\sim} \alpha_1$. In particular, if m is a 1^+-size then whenever $\alpha_1 \overset{m}{\sim} \alpha_2$, we also have $\alpha_1 \overset{m^4}{\sim} \alpha_2$ and any T which is m-f.d. is m^4-f.d.*

PROOF. Corollary A.1.24 tells us that both (α_1, ϕ_i) and (α_2, ϕ_i^{-1}) must be m^4-Cauchy, completing the result. To finish the other observations, notice that any nicely blocked sequence of rearrangements is bounded in both the forward and backward directions. Corollary A.1.14 tells us that we can interpose a γ between α_1 and α_2 that will be m^4-equivalent to both of them. That the notion of m-f.d. implies m^4-f.d. is simply that the joining demanded in the definition of m-f.d. in [**36**] has to be a bounded rearrangment and hence m^4-small. □

We now want to see that if $\alpha_1 \overset{m^4}{\sim} \alpha_2$ then $\alpha_1 \overset{m}{\sim} \alpha_2$. This may seem trivial now as we know that any m^4-Cauchy sequences of rearrangements will be m^3-Cauchy. The problem is that we do not know that just because (α_1, ϕ_i) is m-Cauchy, and $\alpha_1\phi_i \to \alpha_2$ in L^1 that then $m(\alpha_1\phi_i, \alpha_2) \to 0$. The way we handle this issue is much like what we already did for the other direction; we introduce convergence criterion.

DEFINITION A.1.26. *Suppose (X, d) is a metric space and $m : X \times X \to \mathbb{R}^+$. We say a sequence of functions*

$$\delta_n : X^n \to \mathbb{R}^+$$

is a convergence criterion for the function m if for any sequence of values $\{x_n\}$ in X with

$$m(x_i, x_{i+1}) \leq \delta_i(x_1, \ldots, x_i)$$

there is a (necessarily unique) point $x \in X$ with

$$d(x_i, x) \underset{i}{\to} 0.$$

LEMMA A.1.27. *For m a 1-size, and $\langle\alpha\rangle_m$ the m-equivalence class of an arrangement α, we know m^2 is a metric on $\langle\alpha\rangle_m$. Relative to the metric space $(\langle\alpha\rangle_m, m^2)$, there are convergence criteria for both m and m^2.*

PROOF. This construction just abstracts what is done in both Theorem 3.1 and Lemma 4.8 of [**36**]. To construct the convergence criterion for m, select sets $A_1, A_2, \ldots$ and bounds $K_1 < K_2 < \ldots$ inductively with

1. $\mu(A_N) > 1 - 2^{-(N+2)}$ for all N, and for each $x \in A_N$ there is a list of blocks, $(n_{i,N}, m_{i,N})$ with

   $$n_{i,N}(x) \leq -N < 0 < N \leq m_{i,N}(x)$$

   so that

2. $[-N, N] \subseteq f_x^{\alpha_1,\alpha_N}([-n_{i,M}(x), m_{i,n}(x)]) \subseteq [-K_N, K_N]$ and

3. $m(\pi_{f_x^{\alpha_i,\alpha_N},(n_{i,N}(x),m_{i,N}(x))}) < m(\alpha_i, \alpha_N) + 2^{-(N+1)}$.

That $m(\alpha_1, \alpha_N) = m(f_x^{\alpha_1,\alpha_N})$ μ-a.s. and that $f_x^{\alpha_1,\alpha_N}$ is a bijection tell us we can select such bounds K_N and sets A_N.

Choose $\delta_N(\alpha_1, \ldots, \alpha_N)$ so small that if

$$m(\alpha_N, \alpha_N + 1) < \delta_N(\alpha_1, \ldots, \alpha_N)$$

then for all $i \leq N$,

a) $m(\alpha_i, \alpha_{N+1}) < m(\alpha_i, \alpha_N) + 2^{-(N+1)}$ and

b) $\mu(\{x : f_x^{\alpha_N,\alpha_{N+1}}|_{[-K_N,K_N]} = \mathrm{id}\}) > 1 - 2^{-(N+2)}$.

That we can do this is a consequence of Lemma 2.6 concerning m_0 and Axiom v) of [**36**], which requires that all 1-sizes essentially dominate m_0.

Suppose $\alpha_1, \alpha_2, \ldots$ satisfy

$$m(\alpha_{i+1}, \alpha_i) < \delta_i(\alpha_1, \ldots, \alpha_i).$$

Let $K_1, K_2, \ldots$ and $A_1, A_2, \ldots$ be the associated sets and bounds. Let

$$\hat{A}_j = \{x : x \in A_N, N \geq j \text{ and } f_x^{\alpha_N,\alpha_{N+1}}|_{[-K_N,K_N]} = \mathrm{id}, N \geq j\}.$$

Notice that $\mu(\hat{A}_j) \geq 1 - 2^{-j}$. For $x \in \hat{A}_j$, for each $N \geq j$ and $i < N$ we will have values

$$n_{i,N}(x) < -N < 0 < N < m_{i,m}(x),$$

$$[-N, N] \subseteq f_x^{\alpha_i,\alpha_N}([n_{i,N}(x), m_{i,N}(x)]) \subseteq [-K_N, K_N]$$

and further, for all $k \geq N$,

$$f_x^{\alpha_k,\alpha_{k+1}}|_{[-K_k,K_k]} = \mathrm{id}.$$

As the K_k's increase, by composing we see

$$f_x^{\alpha_N,\alpha_{k+1}}|_{[-K_N,K_N]} = \mathrm{id}$$

and so for all $j \geq N$,

$$f_x^{\alpha_i,\alpha_j}|_{[n_{i,N}(x),m_{i,N}(x)]} = f_x^{\alpha_i,\alpha_N}|_{[n_{i,N}(x),m_{i,N}(x)]}.$$

As $[-N,N] \subseteq [n_{i,N}(x), m_{i,N}(x)]$ we conclude that there is a 1-1 function $f_{x,i}$ with

$$f_x^{\alpha_i,\alpha_j} \underset{j}{\to} f_{x,i}.$$

As $[-N,N] \subseteq f_x^{\alpha_i,\alpha_N}([n_{i,N}(x), m_{i,N}(x)])$ we conclude that $f_{x,i}$ is a bijection. Moreover it is calculation that

$$f_x^{\alpha_i,\alpha_j} f_{x,j} = f_{x,i}.$$

Hence defining an arrangement

$$\beta(x, T_j^{\alpha_1}(x)) = f_{x,1}(j),$$

we have

$$\beta(x, T_j^{\alpha_i}(x)) = f_{x,i}(j).$$

It remains only to show that $m(\alpha_i, \beta) \to 0$ which is equivalent to $m^1(\alpha_i, \beta) \to 0$.

To see this just notice that for $x \in \hat{A}_j$, $n \geq j$ and $i < N$

$$\begin{aligned} m(\phi_{f_x^{\alpha_i,\alpha_N},(n_{i,N}(x),m_{i,N}(x))}) &= m(\phi_{f_x^{\alpha_i,\beta},(n_{i,N}(x),m_{i,N}(x))}) \\ &< m(\alpha_i, \alpha_N) + 2^{-(n+1)} < 2^{-i}. \end{aligned}$$

Letting $N \to \infty$ we conclude that $m(f_x^{\alpha_i,\beta}) \leq 2^{-i}$ and we are done. □

As $m^1(\alpha_1, \alpha_2) = m(\alpha_1, \alpha_2)$ once $m(\alpha_1, \alpha_2) \leq 1$ a convergence criterion for m will be one for m^1 as well, and as m^1 and m^2 are uniformly related, there is a convergence criteria for m^2.

LEMMA A.1.28. *For m a 1-size and $\langle\alpha\rangle_m$ the m-equivalence class of α, relative to the metric space $(\langle\alpha\rangle_\alpha, m^3)$ there is a convergence criterion for m^3.*

PROOF. To construct a convergence criterion for m^3 select inductively, for a list $\alpha_1, \ldots, \alpha_N$, full group elements $\phi_1, \ldots, \phi_N$ so that

$$m^3(\alpha_i, \alpha_{i+1}) \leq 2(m^2(\alpha_i\phi_i, \alpha_{i+1}\phi_{i+1})) + \mu(\{x : \phi_i(x) \neq \phi_2(x)\}).$$

Let δ_N be the convergence criterion for m^2 and now define

$$\delta'_N(\alpha_1, \alpha_2, \ldots, \alpha_N) = \min(\delta_N(\alpha_1\phi_1, \alpha_2\phi_2, \ldots, \alpha_N\phi_N), 2^{-N}).$$

Suppose now that α_i satisfies

$$m^3(\alpha_i, \alpha_{i+1}) < \delta'_N(\alpha_1, \ldots, \alpha_N).$$

Then both

$$m^2(\alpha_i\phi_i, \alpha_{i+1}\phi_{i+1}) < \delta_N(\alpha_1\phi_1, \ldots, \alpha_i\phi_i) \quad \text{and}$$

$$\mu(\{x : \phi_i(x) \neq \phi_{i+1}(x)\}) < 2^{-i}.$$

We conclude that there must be a $\beta \in \langle\alpha\rangle_m$ with

$$m^2(\alpha_i\phi_i, \beta) \underset{i}{\to} 0$$

and a ϕ with

$$\mu(\{x : \phi_i(x) \neq \phi(x)\}) \underset{i}{\to} 0.$$

Hence $m^3(\alpha_i\phi, \beta) \underset{i}{\to} 0$ which is the same as $m^3(\alpha_i, \beta\phi) \underset{i}{\to} 0$. □

THEOREM A.1.29. *For m a 1-size, if $\alpha \overset{m^4}{\sim} \beta$ then $\alpha \overset{m}{\sim} \beta$.*

PROOF. The "trick" here is exactly the same as that in Corollary A.1.14. We give a brief sketch. Restrict the discussion to the metric space $(\langle\alpha\rangle_{m^4}, m^4)$. Remember m^4 is the "geodesic" distance, the infimum over sequences of full-group elements leading from one to the other in both directions.

What we construct now are two sequences of full-group elements ψ_i and ψ_i' so that (α, ψ_i) and (β, ψ_i') both satisfy the convergence criterion for m^3 and

$$\|\alpha\psi_i, \beta\psi_i'\|_1 \underset{i}{\to} 0.$$

This of course completes the result as both $\alpha\psi_i$ and $\beta\psi_i'$ will be converging to a common γ to which they are both m^3 and hence m-equivalent.

The construction follows the same lines as that in Corollary A.1.14 by successively pulling in from one side and then the other by full-group elements close enough in $m^4 \leq m^3$ to be sure to obtain the next term of the convergence criterion on the other side at the next step. □

A.2. p-sizes

In a preliminary version of our work on sizes for discrete amenable group actions we put forward an axiomatization that lies between what we have actually set out here, and the axiomatization we used in [**20**]. What we will do in this section is to state that axiomatization, show that a "size" in those terms gives rise to a size in satisfying our axioms here having precisely the same equivalence classes. We then will present the axiomatization used in [**20**] and show that a size in those terms (which we call a p-size) gives rise to a size in terms of our older axiomatization for discrete amenable groups.

Here is that preliminary axiomatization: A size m, as defined in a preliminary version of this work, is a function m from the space of G-rearrangements to $\mathbb{R}^+$ that satisfies the following five axioms:

AXIOM B1. *Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\phi\|_{\text{p.w.}} < \delta$ then $m(\alpha, \phi) < \varepsilon$.*

AXIOM B2. *Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(\alpha, \phi) < \delta$ then $\|\alpha, \alpha\phi\|_{\text{p.w.}} < \varepsilon$.*

AXIOM B3. *Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(\alpha, \phi) < \delta$ then $m(\alpha\phi, \phi^{-1}) < \varepsilon$.*

AXIOM B4. *Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(\alpha\phi, \psi) < \delta$ then $m(\alpha, \phi\psi) < m(\alpha, \phi) + \varepsilon$.*

AXIOM B5. *Given $\varepsilon > 0$ and rearrangement (α, ϕ), there exists $\delta = \delta(\varepsilon, \alpha, \phi) > 0$ such that if $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta$ then $m(\beta, \psi) < m(\alpha, \phi) + \varepsilon$.*

Suppose m is a size function, in the sense that it satisfies the above five (old) axioms. We will construct a new size function m^2 which satisfies the current three axioms, giving the same full group topology, and the same completion. That is to say, the m-Cauchy sequences in the full group are exactly the m^2-Cauchy sequences in the full group. The two steps we take are precisely analogous to those taking m^2 to m^4 in our work on 1-sizes. It is worth pointing out here that Axiom B1 and Theorem 5.2 tell us that we have the analogue of a 1^+-size in that we can always perturb elements of the full-group by a small amount in m to be bounded. Note that the size function m is a conjugacy invariant, meaning that

$$m(\alpha, \phi) = m(\alpha\psi, \psi^{-1}\phi\psi).$$

To each arrangement α and full group element ϕ_0, associate a function

$$k^\alpha_{\phi_0} : \Gamma \to \mathbb{R}^+$$

by letting

$$k^\alpha_{\phi_0}(\phi) = m(\alpha\phi, \phi^{-1}\phi_0).$$

Axioms B1 and B3 imply that $k^\alpha_{\phi_0}$ is uniformly continuous with respect to the L^1 topology on the full group.

For ϕ_1 and ϕ_2 in the full group, define

$$m_\alpha(\phi_1, \phi_2) = m(\alpha\phi_1, \phi_1^{-1}\phi_2),$$

and define

$$m^1(\alpha, \phi) = \|k^\alpha_{\text{id}}, k^\alpha_\phi\|_{\text{sup}}$$

and so

$$m^1_\alpha(\phi_1, \phi_2) = \|k^\alpha_{\phi_1}, k^\alpha_{\phi_2}\|_{\text{sup}}.$$

Clearly, $m^1_\alpha(\cdot, \cdot)$ is a pseudo-metric on the full group. Axioms B1, B3 and B4 imply that m_α and m^1_α give rise to equivalent topologies on

the full group. In particular, any sequence that is m_α-Cauchy is also m^1_α-Cauchy, and vice versa.

Axiom B2 tells us that the identity map on the full-group is uniformly continuous from the m^1 to the L^1-metric.

All that remains is to obtain Axiom 3 of an r-size, i.e. upper semi-continuity with respect to distribution. We do this in exactly the same way as for 1-sizes.

Define

$$m^2 : \{G\text{-rearrangements}\} \to \mathbb{R}^+$$

by letting

$$m^2(\alpha, \phi) = \limsup_{\|(\alpha,\phi),(\beta,\psi)\|_* \to 0} m^1(\beta, \psi).$$

Also define

$$m^2_\alpha(\phi_1, \phi_2) = m^2(\alpha\phi_1, \phi_1^{-1}\phi_2).$$

That m^2 is a pseudometric follows exactly as in Lemma A.1.21 relating m^3 to m^4 for 1-sizes, as the copying lemma (Corollary 5.9) applies in precisely the same way.

LEMMA A.2.1. *m^2_α and m^1_α are equivalent metrics on the full-group. That is to say, given any $\varepsilon > 0$ there is a δ so that*

1. *if $m^1_\alpha(\phi_1, \phi_2) < \delta$ then $m^2_\alpha(\phi_1, \phi_2) < \varepsilon$ and*
2. *if $m^2_\alpha(\phi_1, \phi_2) < \delta$ then $m^1_\alpha(\phi_1, \phi_2) < \varepsilon$*

PROOF. To see this just note that one can rewrite Axiom B5 as

Given any (α, ϕ),

$$\limsup_{\|(\alpha,\phi),(\beta,\psi)\|_* \to 0} m(\beta, \psi) \leq m(\alpha, \phi).$$

As we know that m and m^1 satisfy (1.) and (2.) above we conclude the lemma. □

Notice here that we could have used less than Axiom B5. To conclude that m^2 was equivalent to m all we really needed is the following.

AXIOM B5′. *Given any $\varepsilon > 0$ there exists δ so that for any (α, ϕ) with $m(\alpha, \phi) < \delta$ then*

$$\limsup_{\|(\alpha,\phi),(\beta,\psi)\|_* \to 0} m(\beta, \psi) < \varepsilon.$$

In [**20**], we presented a framework for restricted orbit equivalence for actions of $\mathbb{Z}^d$. A size function was initially defined on permutations of boxes in $\mathbb{Z}^d$. Such a size was required to satisfy six different axioms. This function was then extended to be defined on rearrangements, (α, ϕ), in many ways analogous to what happened for a 1-size. We will avoid all the problems we had in dealing with 1-sizes here in that rather than "collapsing" by "pushing-together" (something that does not make much sense beyond $\mathbb{Z}^1$), we "fill in", analogous to what we did in the proof of Theorem 4.0.27.

Let Π_n be the set of all permutations of the box B_n (in $\mathbb{Z}^d$), and let $\Pi = \cup_n \Pi_n$. According to [**20**], a p-size (p signifying "permutation") is a function $m : \Pi \to [0, 1]$ that satisfies the following axioms.

AXIOM C1. *(Continuity near the identity)*

(a) $m(id) = 0$ *for all identity permutations, and*

(b) *for all $\varepsilon > 0$ there is an N and $\delta > 0$ so that for $n \geq N, \pi \in \Pi_n$, and if*

$$\frac{|\{\vec{v} \in B_n; \pi(\vec{v}) \neq \vec{v}\}|}{|B_n|} < \delta,$$

then $m(\pi) < \varepsilon$.

AXIOM C2. *("Lumpiness" of small permutations)*

For all N_0 and $\varepsilon > 0$, there is an N_1 and $\delta > 0$ so that if $n \geq N_1$ and $\pi \in \Pi_n$ with $m(\pi) < \delta$, then

$$\frac{|\{\vec{v} \in B_n;\ \textit{for some } \vec{v}_0 \in B_{N_0}, \pi(\vec{v} + \vec{v}_0) \neq \pi(\vec{v}) + \vec{v}_0\}|}{|B_n|} < \varepsilon.$$

To make $\pi(\vec{v} + \vec{v}_0)$ well defined we extend π outside B_n as the identity. In such contexts we will always make this choice.

Axioms C1 and C2 are partial converses to one another. Axiom C1 tells us that if a permutation moves very few points, then it must be small. Axiom C2 tells us that if a permutation is small it must move most points in large rigid lumps.

AXIOM C3. *(Continuity of inverses near zero)*
For all $\varepsilon > 0$ there is an N and $\delta > 0$ so that for $n \geq N$ and $\pi \in \Pi_n$ with $m(\pi) < \delta$, then $m(\pi^{-1}) < \varepsilon$.

AXIOM C4. *(Continuity of composition near zero)*
For all $\varepsilon > 0$ there is an N and δ so that for $n \geq N$ and $\pi_1, \pi_2 \in \Pi_n$, if $m(\pi_2) < \delta$, then

$$m(\pi_2\pi_1) < m(\pi_1) + \varepsilon.$$

Axioms C3 and C4 simply tell us the kind of uniform continuity we will want of the group operations within each Π_n. There are two other natural operations within Π and that is the concatenation of permutations on disjoint blocks, and the rearranging of blocks left invariant by a permutation. One of these builds a new permutation on a larger block from smaller ones, the other changes a given permutation to a new one. Our last two axioms concern continuity of such operations.

AXIOM C5. *(Continuity of concatenation near zero)*
For all $\varepsilon > 0$ there is an N_0 and $\delta > 0$ so that if $n_1, n_2, \ldots, n_k > N_0$ and if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{Z}^d$ are such that

(a) *$A_i = B_{n_i} + \vec{v}_i$ are disjoint and for some n;*

(b) *$\cup_{i=1}^k A_i \subseteq B_n$ and $\dfrac{|\cup_{i=1}^k A_i|}{|B_n|} > 1 - \delta$:*

(c) *$\pi \in \Pi_n$ is such that for all i, $\pi(A_i) = A_i$ and setting $\pi_i \in \Pi_{n_i}$ to be*
$\pi_i(\vec{v}) = \pi(\vec{v} + \vec{v}_i) - \vec{v}_i$ we have

(d) *$m(\pi_i) < \delta$ for all i;*
we then must have that $m(\pi) < \varepsilon$.

The picture of what Axiom C5 requires is that if B_n is almost completely covered by translates of various B_{n_i} each of which is fixed by π and if when π is viewed as a permutation of each translate it is small, then π itself must be small.

AXIOM C6. *(Continuity of rearranging large lumps near zero)*
For all $\varepsilon > 0$ there is an N and $\delta > 0$ so that if $N_1 \geq N$ and $K_1, K_2, \ldots, K_s \subseteq B_n$ and

(a) *the $K_i = B_{N_1} + \vec{v}_i$ are and*
$$\frac{|\cup_{i=1}^s K_i|}{|B_n|} > 1 - \delta,$$

(b) *there exist $\vec{u}_i \in B_n$, $i = 1, \ldots, s$ so that $K_i + \vec{u}_i = K_i'$ are also disjoint and $\subseteq B_n$,*

(c) *the permutations $\pi, \pi' \in \Pi_n$ are such that $\pi(K_i) = K_i$ and $\pi'(K_i') = K_i'$ and for $\vec{v} \in K_i$*
$$\pi'(\vec{v} + \vec{u}_i) = \pi(\vec{v}) + \vec{u}_i,$$
(i.e. π acts on K_i exactly as π' acts on K_i') and,

(d) *$m(\pi) < \delta$.*

we then must also have $m(\pi') < \varepsilon$.

This last axiom says that smallness of a permutation is a local property. More precisely, if you have a small permutation (in m) that maps a lot of sets K_i to themselves, where the K_i are all translates of a large box B_{N_1}, and if you simply rearrange where these boxes K_i happen to lie in B_n then the permutation you obtain will still be small.

Suppose now that m is a p-size as defined by these six axioms. Suppose α is an arrangement and $\phi \in FG(\mathcal{O})$. Let $x \in X$ and let $f_x^{\alpha,\phi}$ be the associated bijection of $\mathbb{Z}^d$. Recall that for μ-a.e. $x \in X$ for any $\varepsilon > 0$ there is an $N(x, \varepsilon)$ so that for all $n \geq N$ the permutation $\pi_{x,n}^{\alpha,\phi}$ agrees with $f_{x,n}^{\alpha,\phi}$ all but ε of the "time", i.e. on all but a subset of B_n of density at most ε.

Define:

$$m_{n,x}(\alpha, \phi) = \min_{\pi \in \Pi_n} \max\Big\{m(\pi), \frac{|\{\vec{v} \in B_n; \pi_{x,n}^{\alpha,\phi}(\vec{v}) \neq \pi(\vec{v})\}|}{|B_n|}\Big\} \text{ and}$$

$$m_x(\alpha, \phi) = \liminf_{n \to \infty} m_{n,x}(\alpha, \phi).$$

Finally define the p-size m on rearrangements by

$$m(\alpha, \phi) = \operatorname*{ess\,inf}_{x \in X} m_x(\alpha, \phi).$$

THEOREM A.2.2. *If m is a p-size as defined above, then m satisfies Axioms B1–B4 and B5′ and hence the corresponding m^2 is an r-size giving the same equivalence classes.*

PROOF. As the axiomatization given in Axioms B1–B5 was derived from the work in [**20**] we can just about read off everything we want directly. Axiom B1 is precisely the conclusion of Lemma 2.3 of [**20**]. Axiom B2 is simply Lemma 2.4 of [**20**] reworded a bit using the blocks B_n as a Følner sequence. Axioms B3 and B4 are precisely the conclusions of Lemmas 2.5 and 2.6 of [**20**].

To see that m satisfies Axiom B5′ takes a bit of thought. It is essentially embodied in Axiom C5. To see this first notice that although m is defined as a $\liminf$ in n, Theorem 2.1 of [**20**] tells us that if this $\liminf_n$ is small then the $\hat{m}(\alpha, \phi) = \limsup_n$ is also small. More precisely, for any δ_1 there is a δ so that if $m(\alpha, \phi) < \delta$ then $\hat{m}(\alpha, \phi) < \delta_1$. This means once n is large enough, for all but $\varepsilon/10$ of the $x \in X$, there is a permutation $\pi(x)$ of B_n so that

1. $m(\pi(x)) < \delta_1$ and
2. $\#(\{\vec{v} \in B_n : \pi(x)(\vec{v}) \neq f_x^{\alpha,\phi}(\vec{v}\}) < \varepsilon \# B_n/10$.

Suppose a permutation π of a large box B_N agrees with some π' on all but a fraction δ_1 of B_N, and that we obtain π' by covering all but δ_1 of the B_N by disjoint translates of B_n and permuting the points in each by a permutation of m-size less then δ_1 and applying the identity to the rest of the points. Combining Axioms C1 and C5, if δ_1 is small enough, we will have $m(\pi) < \varepsilon/2$.

Combining the ergodic theorem, and the strong Rokhlin lemma and a bit of thought one sees that if (β, ψ) is close in distribution to (α, ϕ) then there will be large blocks B_N in the orbit of a.e. x' where $\pi_{x',N}^{\beta,\psi}$ will be as described as π is described above. We conclude that $m_x(\beta, \psi) < \varepsilon$ and the result. $\square$