A look into the mirror (I)

an overview of Mirror Symmetry

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Topics in Algebraic Geometry Seminar

Outline

1. Physics: the big black box
2. Math
   - Calabi-Yau threefolds
   - The A-model
   - The B-model
   - The Mirror Map
3. Mirror conjecture

A slogan

Mirror Symmetry is a correspondence between pairs of (families of) Calabi-Yau threefolds

$$X \longleftrightarrow \check{X}$$

that interchanges complex and symplectic geometry.

Mirror Symmetry is motivated by physics.

Strings

A physical theory should satisfy some natural axioms that give it the structure of a SCFT.

SUSY is a required feauture of a SCFT. It eliminates in a very natural way a lot of the difficulties arising in constructing a string theory.

A mathematical realization of a SCFT is given by a sigma model, a construction depending upon the choice of:

- a Calabi-Yau threefold X;
- a complexified Kahler class ω.

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Moduli of SCFT

Moduli space of SCFT

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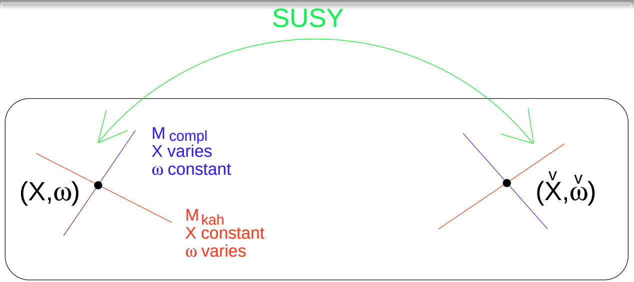

Moduli space of SCFT

SUSY suggests the existence of an involution on the moduli space of SCFT such that:

$$H^q(X, \Lambda^p T_X) \cong H^q(\check{X}, \Lambda^p \Omega_{\check{X}})$$
$$H^q(X, \Lambda^p \Omega_X) \cong H^q(\check{X}, \Lambda^p T_{\check{X}})$$

Moduli of SCFT

In particular, looking at $p = q = 1$

$$T_{M_{compl}} = H^1(X, T_X) \cong H^1(\check{X}, \Omega_{\check{X}}) = T_{M_{kah}}$$

$$T_{M_{kah}} = H^1(X, \Omega_X) \cong H^1(\check{X}, T_{\check{X}}) = T_{M_{compl}}$$

we obtain an identification of tangent spaces, and hence local isomorphisms between complex and kahler moduli spaces of the mirror pair. Such isomorphisms are called the Mirror Maps.

Yukawa Couplings

Physics hands us two trilinear forms called Yukawa couplings:

- A-model YC: $\left(T_{M_{kah}}\right)^3 \to \mathbb{C}$;
- B-model YC: $\left(T_{M_{compl}}\right)^3 \to \mathbb{C}$.

Mirror symmetry postulates that such functions should get identified via the mirror maps!

This is how mirror symmetry makes enumerative predictions about rational curves in CY threefolds.

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Definition

A CY threefold X is a projective threefold (possibly with mild singularities) such that:

- $K_X \cong \mathcal{O}_X$.
- $H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$.

Hodge diamond

Combining the above definition with Serre duality and $h^{p,q} = h^{q,p}$ we obtain that the Hodge diamond of a CY threefold is:

$$
\begin{array}{lccccccc}
b_6 : & & & & 1 & & & \\
b_5 : & & & 0 & & 0 & & \\
b_4 : & & 0 & & h^{1,1} & & 0 & \\
b_3 : & 1 & & h^{2,1} & & h^{2,1} & & 1 \\
b_2 : & & 0 & & h^{1,1} & & 0 & \\
b_1 : & & & 0 & & 0 & & \\
b_0 : & & & & 1 & & &
\end{array}
$$

Kahler forms

A kahler form ω is a closed $(1,1)$ (real) form such that ω^3 is non-degenerate.

The kahler cone

$$\mathcal{K}(X)$$

is the space of all possible kahler forms. It is an open subset of $H^{1,1}(X,\mathbb{R})$.

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Complexified kahler moduli space

The complexified kahler moduli space of X is

$$M_{kah} := H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}) + i\mathcal{K}(X).$$

A basis $\{C_\beta\}$ of $H_2(X,\mathbb{Z})$ gives coordinates (called kahler parameters) on M_{kah},

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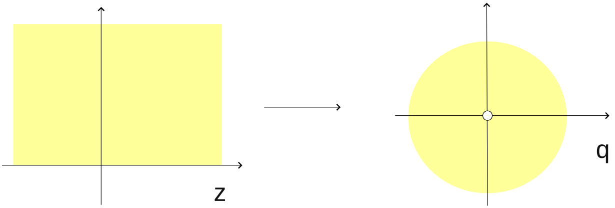
The one-dimensional case

If $Pic(X) = \mathbb{Z} =< H >$, then

$$M_{kah} = \mathbb{R}/\mathbb{Z} + i\mathbb{R}_{>0}$$

is equivalent to the punctured disk Δ^* via the exponential coordinates

$$q = e^{2\pi i z}$$



In general

For higher Picard number, a framing is a choice of a basis for $H^2(X, \mathbb{Z})$, that identifies a simplicial cone in $\overline{\mathcal{K}(X)}$.

An exponential transformation from the kahler parameters identifies the corresponding portion in M_{kah} with a punctured polydisc.

The Yukawa coupling

For $D_1, D_2, D_3 \in H^2(X, \mathbb{Z})$, define:

$$< D_1, D_2, D_3 >:= D_1 \cdot D_2 \cdot D_3 + \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} < D_1, D_2, D_3 >_{\beta}^{g=0} q^{\beta},$$

where

$$< D_1, D_2, D_3 >_{\beta}^{g=0} = \int_{[\overline{M}_{0,3}(X,\beta)]^{vir}} ev_1^*(D_1) \cdot ev_2^*(D_2) \cdot ev_3^*(D_3)$$

is a three pointed Gromov-Witten invariant for X.

Note: from the above formula we can extract, after correcting for multiple cover contributions, the (virtual) number of rational curves on the threefold in any given homology class.

Deformation spaces

Idea: the moduli space of complex structures is too complicated, so we study it locally.

A deformation space for X is the data illustrated in the following universal property diagram:

$$
\begin{array}{ccccc}
\mathcal{X} & & \longrightarrow & & \mathcal{U} \\
& \nwarrow & & \nearrow & \\
\downarrow & & X & & \downarrow \\
& & | & & \\
S & & \longrightarrow & & Def(X) \\
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Facts and observations

1. The tangent space to $Def(X)$ at x_0 is classically identified with $H^1(X, T_X)$.
2. For a CY threefold, the choice of a global non-vainshing holomorphic 3-form gives an isomorphism

$$H^1(X, T_X) \cong H^1(X, \Lambda^2\Omega_X) = H^{2,1}(X)$$

($\Rightarrow$ symmetry in the Hodge diamond of a mirror pair)
3. Bogomolov-Tian-Todorov theorem: for a CY threefold, the deformation problem is unobstructed. (i.e. any infinitesimal deformation can be integrated).
4. A family $\mathcal{X} \to S$ induces a map $T_{S,s_0} \to T_{Def(X)}$ called the Kodaira-Spencer morphism. If we assume it to be an isomorphism, we can work on the tangent space of a concrete family rather than on $T_{Def(X)}$.

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Hodge Bundle

Given a family of CY threefolds $\pi : \mathcal{X} \to S$ we can define the Hodge bundle to be

$$\mathbb{E} := R^3\pi_*(\mathbb{C}) \otimes \mathcal{O}_S.$$

What is going on:

$$\begin{array}{ccc} H^3(\mathcal{X}_s, \mathbb{C}) & \to & \mathbb{E} \\ \downarrow & & \downarrow \\ s & \to & S \end{array}$$

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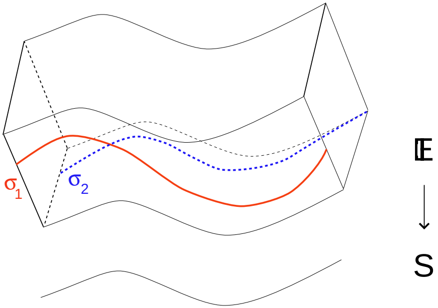

A basis $\{\sigma_i\}$ for $H^3(X, \mathbb{Z})$ gives a local frame for $\mathbb{E}$: any local section is

$$\sigma = \sum f_i(s)\sigma_i(s).$$

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$$\nabla_{\frac{\partial}{\partial s_j}} \sigma = \sum \frac{\partial f_i}{\partial s_j} \sigma_i.$$

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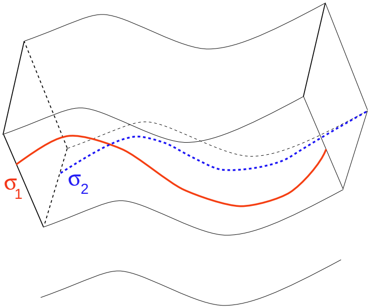

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The Yukawa Coupling

We can now define a cubic form on $T_{Def(X)} \overset{KS}{\cong} T_{S,s}$.

Choose a family of Calabi-Yau forms $\Omega(s)$ (non-vanishing $(3,0)$ forms).

$$\left\langle \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3} \right\rangle := \int_X \Omega \wedge \nabla_{\frac{\partial}{\partial s_1}} \nabla_{\frac{\partial}{\partial s_2}} \nabla_{\frac{\partial}{\partial s_3}} \Omega$$

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Remarks

1. third derivatives are necessary to obtain something non-trivial, by Griffiths transversality.
2. the coupling depends on the choice of $\Omega(s)$. Any two Calabi-Yau families differ by a non-vanishing holomorphic function $f(s)$, and the coupling transforms by multiplication by $f^2(s)$.

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The Mirror Map

Mirror Map "=" a set of canonical coordinates q on $Def(X)$ that we can identify with the q's on (part of) M_{kah} coming from the choice of a framing.

Observation: on the kahler side $q = 0$ corresponded to a degenerate kahler metric. This suggests that we should try and "center" our canonical coordinates somewhere on the "boundary" of the complex moduli space.

Simplification: from now on, let us restrict our attention to the situation of $dim(Def(X)) = 1$ and look very locally around some point. I.e., we consider families

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Periods

For a fixed pair (X, Ω), the period map is

$$P_{X,\Omega}: \quad \begin{array}{ccc} H_3(X,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ \beta & \mapsto & \int_\beta \Omega. \end{array}$$

Local torelli tells us the period map is a local coordinate for the complex moduli space.

Problems:

1. for a family $\mathcal{X} \to \Delta^*$ we can define a period map only on the universal cover $\mathcal{H}$ of the punctured disc.

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Monodromy

$$P(z+1) = P(z) \circ T,$$

where $T : H_3(X, \mathbb{C}) \to H_3(X, \mathbb{C})$ is a linear map called monodromy transformation.

If we were lucky enough to have a basis for $H_3(X, \mathbb{C})$ such that

$$T = \begin{bmatrix} 1 & n & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

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Canonical coordinates

$$w(z) := \frac{\int_{A_1} \Omega(z)}{\int_{A_0} \Omega(z)}$$

and the canonical coordinate (recall $s = e^{2\pi i z}$):

$$q(s) := e^{2\pi i w}$$

Such luck happens only around special points in the boundary of the complex moduli space, called large complex structure limit points.

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GKZ

The periods of a family of CY threefolds are the solutions of a GKZ system of differential equations, called Picard-Fuchs equations.

The technology we have developed this semester allows us to systematically:

1. find the solutions to the Picard-Fuchs equations.
2. identify a family centered around a large complex structure limit point.
3. extract the basis vectors necessary to define canonical coordinates.

The Mirror conjecture

It is possible to correspond:

$\mathcal{X} \to (\Delta^*)^s$ 0 a large CS limit point	$\leftrightarrow$	$\check{X}$
canonical coordinates q	$\leftrightarrow$	a framing on $\overline{\mathcal{K}(\check{X})}$ giving coordinates q for M_{kah}
$(2, 1)$-YC (Quantum Cohomology)	$\leftrightarrow$	$(1, 1)$-YC

Punchline

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Stay tuned!