## If Sherlock Holmes Were a Mathematician...

#### Elin Smith

Colorado State University

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Elin Smith If Sherlock Holmes Were a Mathematician...

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'Data! Data! Data!' he cried impatiently. 'I can't make bricks without clay.'

- (Sherlock Holmes) Sir Arthur Conan Doyle The Adventure of the Copper Beeches

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#### Visualizing Data

An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

#### Finding a Better Basis: PCA

The Singular Value Decomposition

Principal Component Analysis

Properties

Closest Rank d Approximation to a Matrix X

# Applying PCA to Visualize Data of a Rotating Object Projection into $\mathbb{R}^3$

### Using PCA for Illumination Spaces

Illumination Spaces

A Minimal Energy Point Configuration Problem

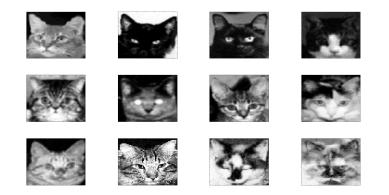
**Global Minimizers** 

An Alternative Setting

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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

#### Data Set 1: 99 Images of Cat Faces



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#### Visualizing Data

Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis



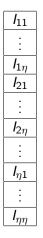
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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

#### Creating a vector from a matrix:

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<i>I</i> <sub>21</sub>	<i>I</i> <sub>22</sub>	• • •	$I_{2\eta}$
÷	÷		÷
$I_{\eta 1}$	$I_{\eta 2}$		$I_{\eta\eta}$



 $\mathbf{x}^{(i)}$ 

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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

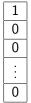
Data matrix X:

where each  $\mathbf{x}^{(i)} \in \mathbb{R}^m$ , for *m* equal to the number of pixels per image.

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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

Standard Spanning Set for Data Set X:





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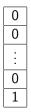


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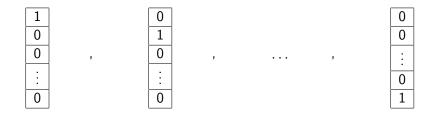
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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

Standard Spanning Set for Data Set X:



A better set and a lower dimension?

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#### Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object

An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

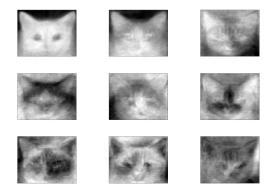
plying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces

Using Principal Component Analysis, we get an ordered basis of 'eigenpictures':

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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

Using Principal Component Analysis, we get an ordered basis of 'eigenpictures':



Only need 34 of these basis 'vectors' to capture most of the energy of the data set.

An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

We can use this basis to compare data sets: Data Set 2: 99 Images of Dog faces



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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

#### How much of a given dog face looks like a cat?



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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

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#### How much of a given dog face looks like a cat?





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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

#### How much of a given dog face looks like a cat?







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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis





How much do Chris and Renzo look like a cat?

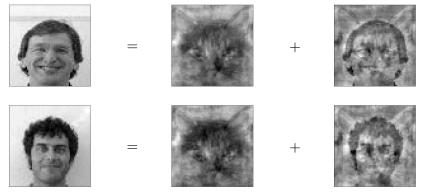
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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis





How much do Chris and Renzo look like a cat?



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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis





How much do Jorge and William look like a dog?

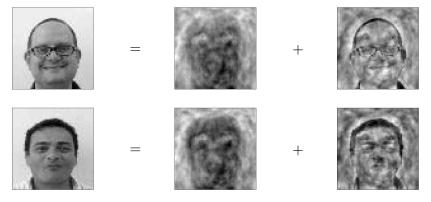
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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis





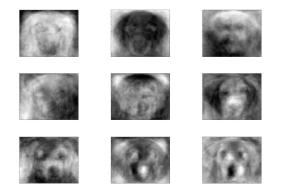
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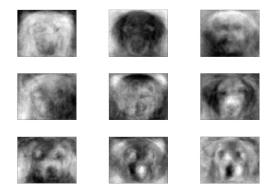
#### Classification? Basis of eigenpictures for dogs:



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An Example with Cats and Dogs Constructing a Data Set of Images A Better Basis

#### Classification? Basis of eigenpictures for dogs:



What is PCA and how do we find such a basis?

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**Theorem** (Singular Value Decomposition) Let  $X \in \mathbb{R}^{m \times n}$ , and let  $\ell = \min\{m, n\}$ . Then there exist real orthogonal matrices U and V such that

 $X = U \Sigma V^{T},$ 

where U has size  $m \times m$ , V has size  $n \times n$ , and  $\Sigma$  is an  $m \times n$  diagonal matrix.

The entries of  $\Sigma = \text{diag} \left( \sigma^{(1)}, \ldots, \sigma^{(\ell)} \right)$  are ordered by  $\sigma^{(1)} \ge \cdots \ge \sigma^{(\ell)} \ge 0$ .

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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X

#### Given a matrix X, we define the **covariance matrix** C to be

 $C = XX^T$ .

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It can be shown that

$$XX^{T}\mathbf{u}^{(i)} = \left(\sigma^{(i)}\right)^{2}\mathbf{u}^{(i)}.$$

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The left singular vectors of X are eigenvectors of the covariance matrix with eigenvalues equal to the singular values squared.

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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X

#### Let rank(X) = r. Then column i of X is given by

$$\mathbf{x}^{(i)} = \sum_{j=1}^{r} \sigma^{(j)} \mathbf{v}_{i}^{(j)} \mathbf{u}^{(j)}.$$

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$$\mathbf{x}^{(i)} = \sum_{j=1}^{r} \sigma^{(j)} \mathbf{v}_{i}^{(j)} \mathbf{u}^{(j)}.$$

The set of the first r left singular vectors form a basis for the column space of the matrix X.

Visualizing Data	The Singular Value Decomposition
Finding a Better Basis: PCA	Principal Component Analysis
Applying PCA to Visualize Data of a Rotating Object	Properties
Using PCA for Illumination Spaces	Closest Rank $d$ Approximation to a Matrix $X$

#### Principal Component Analysis:

# Given a data set $\left( \begin{array}{c|c} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)} \end{array} \right)$ ,

find the optimal ordered orthonormal basis  $\mathcal{B}$  for X.

Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Object Using PCA for Illumination Spaces

What does it mean to be optimal?

Suppose our data set X is n-dimensional.

Let  $\mathcal{B} = \{\phi^{(1)}, \dots, \phi^{(n)}\}$  be an ordered orthonormal basis for X.

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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Object Using PCA for Illumination Spaces

What does it mean to be optimal?

Suppose our data set X is *n*-dimensional. Let  $\mathcal{B} = \{\phi^{(1)}, \dots, \phi^{(n)}\}$  be an ordered orthonormal basis for X.

Any point  $\mathbf{x} \in X$  can be expressed in terms of this basis:  $\mathbf{x} = \alpha_1 \phi^{(1)} + \dots + \alpha_n \phi^{(n)}$ .

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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X

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The *d*-term truncation of this expansion for **x** is given by  $\mathbf{x}_d = \alpha_1 \phi^{(1)} + \cdots + \alpha_d \phi^{(d)}$ .

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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Object Using PCA for Illumination Spaces

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The mean squared truncation error is given by  $\epsilon = \left\langle \|\mathbf{x} - \mathbf{x}_d\|^2 \right\rangle$ .

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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Object Using PCA for Illumination Spaces

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The mean squared truncation error is given by  $\epsilon = \left\langle \|\mathbf{x} - \mathbf{x}_d\|^2 \right\rangle$ .

We define  $\mathcal{B}$  to be an optimal basis if, among all orthonormal bases,  $\mathcal{B}$  minimizes the mean squared truncation error,  $\epsilon$ .

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Visualizing Data	The Singular Value Decomposition
Finding a Better Basis: PCA	Principal Component Analysis
Applying PCA to Visualize Data of a Rotating Object	Properties
Using PCA for Illumination Spaces	<b>Closest Rank</b> $d$ <b>Approximation to a Matrix</b> $X$

#### Properties of the Principal Component Basis:

The basis can be computed by finding the left singular vectors of X.

Visualizing Data	The Singular Value Decomposition
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Any truncation of  ${\mathcal B}$  captures more statistical variance than any other basis of the same dimension.

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Properties of the Principal Component Basis:

The basis can be computed by finding the left singular vectors of X. (Hence the term eigenpictures!)

Any truncation of  ${\mathcal B}$  captures more statistical variance than any other basis of the same dimension.

A reconstruction of X using any d-term truncation of  $\mathcal{B}$  gives the best rank d approximation of X.

Visualizing Data	The Singular Value Decomposition
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Visualizing Data	The Singular Value Decomposition
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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X







Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X









The Singular Value Decomposition Principal Component Analysis Properties Closest Rank d Approximation to a Matrix X











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The Singular Value Decomposition Principal Component Analysis Properties Closest Rank d Approximation to a Matrix X











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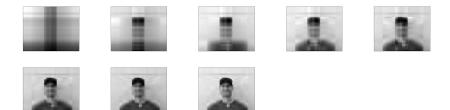
The Singular Value Decomposition Principal Component Analysis Properties Closest Rank d Approximation to a Matrix X



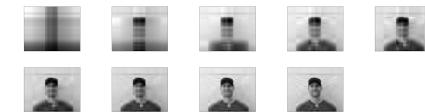
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The Singular Value Decomposition Principal Component Analysis Properties Closest Rank d Approximation to a Matrix X

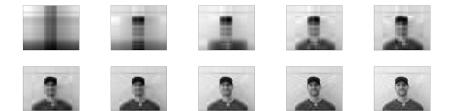


Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X



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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Other Spaces Using PCA for Illumination Spaces



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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X



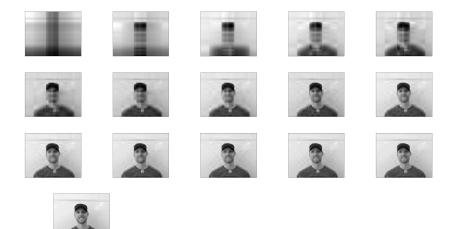
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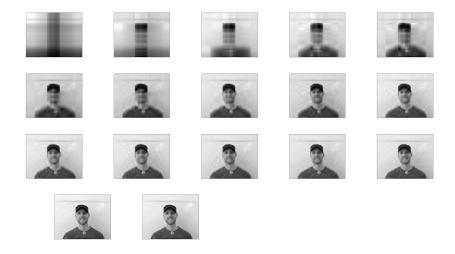
Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Closest Rank d Approximation to a Matrix X



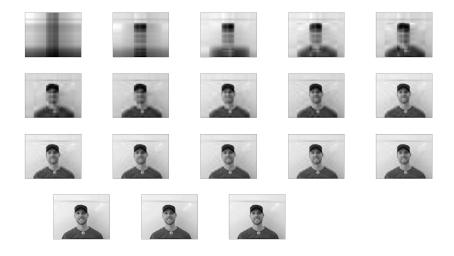
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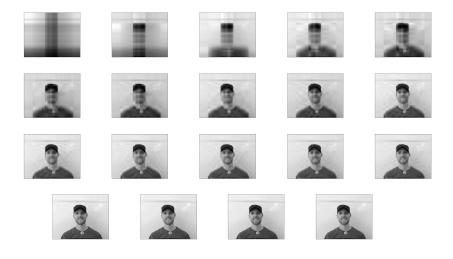




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Visualizing Data Finding a Better Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces Other Interpreties Using PCA for Illumination Spaces



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Projection into  $\mathbb{R}^3$ 

### Data Set 3: Images of a Rotating Object





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$$X = U\Sigma V^{T}$$
$$\hat{U} = \left( \mathbf{u}^{(1)} \middle| \mathbf{u}^{(2)} \middle| \mathbf{u}^{(3)} \right)$$

Projection into  $\mathbb{R}^3$ 

**Projection:** 

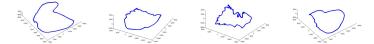
$$\begin{aligned} \pi \colon \mathbb{R}^{262,144 \times 1500} & \to & \mathbb{R}^{3 \times 1500} \\ X & \mapsto & \hat{U}^{\mathsf{T}} X. \end{aligned}$$

#### Projection into $\mathbb{R}^3$

### Volleyball: Red, Green, Blue, and Gray



### Jack O' Lantern: Red, Green, Blue, and Gray



3.0

Projection into  $\mathbb{R}^3$ 

# **Volleyball Eigenpictures 1, 2, and 3:** Red Filter







### Green Filter







### Blue Filter



# Grayscale











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If Sherlock Holmes Were a Mathematician...

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Projection into  $\mathbb{R}^3$ 

# Jack o' Lantern Eigenpictures 1, 2, and 3: Red Filter







### Green Filter







### Blue Filter



# Grayscale









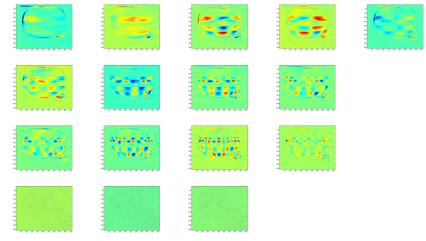
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Projection into  $\mathbb{R}^3$ 

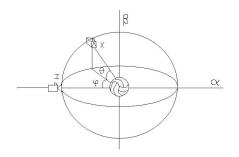
## Jack o' Lantern Eigenpictures 1-13,1000-1002: Grayscale



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Illumination Spaces A Minimal Energy Point Configuration Problem Global Minimizers An Alternative Setting

Data Set 4: Capturing the Illumination Space of an Object



To each point of the sphere, we attach a vector space called the illumination space.

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### Consider all possible illuminations of a person.



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Consider all possible illuminations of a person.



The vector space which captures the majority of the energy of this set is called the **illumination space**.

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# **Determining Optimal Camera Distribution:**

Fix a number of camera locations n and fix the camera resolution. What is the camera location distribution that optimizes capture of variance in illumination space?

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# **Determining Optimal Camera Distribution:**

Fix a number of camera locations n and fix the camera resolution. What is the camera location distribution that optimizes capture of variance in illumination space?

That is, we wish to find an optimal distribution of points with respect to some measure.

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Visualizing Data	Illumination Spaces
Finding a Better Basis: PCA	A Minimal Energy Point Configuration Problem
Applying PCA to Visualize Data of a Rotating Object	Global Minimizers
Using PCA for Illumination Spaces	An Alternative Setting

Consider the problem of finding an optimal distribution of points on the sphere based on a potential energy function.

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Consider the problem of finding an optimal distribution of points on the sphere based on a potential energy function.

Let f be a decreasing and continuous function on  $[0, \infty)$ , and let C be a finite set of points on  $S^{n-1}$ . The **potential energy of** C is defined to be

$$\sum_{x,y\in C, x\neq y} f(|x-y|^2).$$

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**Goal:** Minimize the potential energy of *C*.

Visualizing Data Finding a Etter Basis: PCA Applying PCA to Visualize Data of a Rotating Object Using PCA for Illumination Spaces A A Minimal Energy Point Configuration Problem Global Minimizers An Alternative Setting

One way to find an optimal distribution of points is to start with a random configuration and let the points push away from each other.

(Loading movie...)

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One way to find an optimal distribution of points is to start with a random configuration and let the points push away from each other.

(Loading movie...)

Frequently, this will result in a point configuration which gives a local minimum of our potential energy function.

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Under what conditions can we guarantee a global minimum of our potential function?

A sufficient condition for a local energy minimizer to be a global minimizer for potential energy was obtained by Cohn and Kumar.

## Theorem (Cohn, Kumar 2007):

For any completely monotonic potential function, sharp configurations are global potential energy minimizers.

Illumination Spaces A Minimal Energy Point Configuration Problem Global Minimizers An Alternative Setting

## Examples of sharp configurations:



http://en.wikipedia.org/wiki/Platonic\_solid

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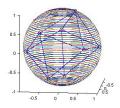
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Illumination Spaces A Minimal Energy Point Configuration Problem Global Minimizers An Alternative Setting

## Examples of sharp configurations:



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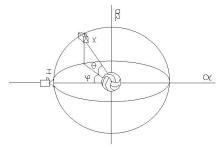


Square Antiprism

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#### Return to



We wish to let points push away from each other on the sphere, this time driven by 'closeness' of illumination spaces.

Visualizing Data	Illumination Spaces
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# Given two subspaces $\mathcal{L}$ and $\mathcal{M}$ of $\mathbb{R}^n$ , with $\dim(\mathcal{L}) = \ell \leq \dim(\mathcal{M}) = m$ , what is the distance between $\mathcal{L}$ and $\mathcal{M}$ ?

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Given two subspaces  $\mathcal{L}$  and  $\mathcal{M}$  of  $\mathbb{R}^n$ , with  $\dim(\mathcal{L}) = \ell \leq \dim(\mathcal{M}) = m$ , what is the distance between  $\mathcal{L}$  and  $\mathcal{M}$ ?

Let  $Q_{\mathcal{L}}$  and  $Q_{\mathcal{M}}$  be orthonormal bases for  $\mathcal{L}$  and  $\mathcal{M}$ , respectively. The principal angles  $\theta_1, \ldots, \theta_\ell$  between  $\mathcal{L}$  and  $\mathcal{M}$  are given by

 $\cos(\theta_i) = \sigma_i.$ 

where the  $\sigma_i$  are the singular values of  $Q_{\mathcal{M}}^T Q_{\mathcal{L}}$ .

Some example metrics on the Grassmannian:

$$\checkmark \sqrt{\sum_{i=1}^{\ell} \theta_i^2}$$

• 
$$\cos^{-1}\left(\prod_{i=1}^{\ell}\cos(\theta_i)\right)$$

• 
$$\sqrt{\sum_{i=1}^{\ell} \sin(\theta_i)^2}$$
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Visualizing Data	Illumination Spaces
Finding a Better Basis: PCA	A Minimal Energy Point Configuration Problem
Applying PCA to Visualize Data of a Rotating Object	Global Minimizers
Using PCA for Illumination Spaces	An Alternative Setting

- We have a way of using a potential energy function to push points away from each other.
- We can quickly calculate principal angles between any two illumination spaces.
- We can therefore efficiently determine the distance between any two illumination spaces.
- Goal: Generalize the theorem of Cohn and Kumar's to this setting to determine if a given configuration yields a global minimum.

Visualizing Data Illumination Spaces Finding a Better Basis: PCA A Minimal Energy Point Configuration Problem Global Minimizers Using PCA for Illumination Spaces An Alternative Setting

Variations in illumination spaces:









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Elin Smith If Sherlock Holmes Were a Mathematician...

### Variations in illumination spaces:



Bjørn Rørslett www.naturfotograf.com

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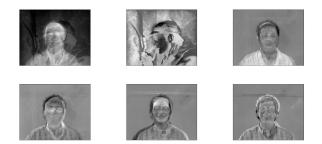






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