Lecture 2

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$G(k, n)$: Projective Space's Big Brothers

Let us consider an n-dimensional vector space V, and choose once and for all a basis $e_1, \ldots, e_n$. For a fixed $k \leq n$ we call **Grassmannian** the moduli space of linear subspaces of V of dimension k. We denote this space by $G(k, n)$.

We now work towards the understanding of the following

Fact: $G(k, n)$ *is a smooth compact (in fact projective) manifold of dimension* $k(n - k)$.

G(2,3): a Motivating Example

As usual we start from a (relatively) simple case to gather intuition and motivation for the general theory. We already know that $G(2, 3) \cong \mathbb{P}^2$, because to any plane in $\mathbb{R}^3$ we can associate in a canonical and unique way the perpendicular line through the origin. However, we are going to ignore this fact and find charts for $G(2, 3)$ in terms of the planes that $G(2, 3)$ parameterizes.

Look at figure 1!

Here we have chosen the two planes $x = 0$ and $y = 0$ as screens. Most every plane in $\mathbb{R}^3$ will intersect each of these two planes in a line. Then we further intersect with the vertical lines $y = 1$ and $x = 1$, and we find two points. A general plane defines these two points, and conversely, given two points on those vertical lines uniquely determines a plane in $\mathbb{R}^3$. But the only free coordinate for those two points is the z coordinate, therefore we have an $\mathbb{R}^2$ worth of generic planes, or, if we prefer, a map:

$$\phi_{x,y} : \quad \begin{array}{ccc} \mathbb{R}^2 & \to & G(2, 3) \\ (z_1, z_2) & \mapsto & \text{plane through} \\ & & 0, (1, 0, z_1), (0, 1, z_2) \end{array}$$

The planes that are not in the image of this chart are those that contain the z axis. There is a projective line worth of them: since one dimension is "taken" by the z axis, all we need to identify one of these planes is the perpendicular direction, therefore reducing our problem to parameterizing lines through the origin in the horizontal plane.

At the end of the day we have recovered what we already knew:

$$G(2, 3) = \mathbb{R}^2 \sqcup \mathbb{P}^1 = \mathbb{P}^2$$
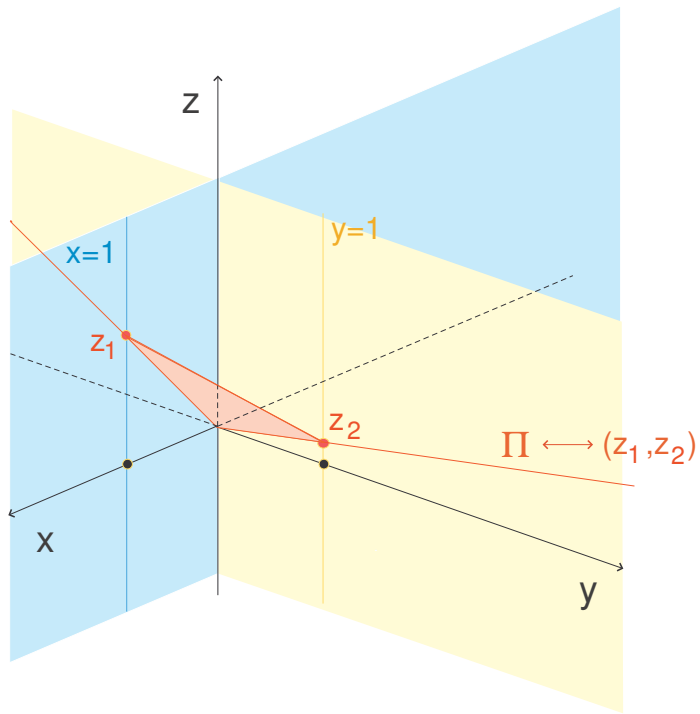
Figure 1: A natural chart for $G(2,3)$.

Problem 1. *Find out the transformation between these charts for $G(2,3)$ and the charts given by associating to a plane its perpendicular line and then using the standard charts for $\mathbb{P}^2$.*

G(k,n) is a Manifold

Let us now try to generalize what we have done. We will do it in two ways.

Geometric Approach

What did we do in the previous section?

1. We chose k linear subspaces $L_1, \ldots, L_k$ of V of dimension $n - k + 1$. Each of them intersects a general k-subspace of $\mathbb{R}^n$ in line. Let us call these lines $\ell_1, \ldots, \ell_k$.

2. Inside each of the L_i, we chose a hyperplane H_i not through the origin. At this point the intersection

$$H_i \cap \ell_i = P_i$$

   is one point in an $n - k$ dimensional linear space.

3. The coordinates of the P_i's inside the H_i's define the chart to the Grassmannian. Since we are free to choose k points inside $n - k$ dimensional linear spaces we see that the dimension of $G(k, n)$ is $k(n - k)$.

4. What are the k-subspaces that are not in the image of this chart? Those that intersect any of the subspaces L_i in more than just a line!

Algebraic Approach

If you prefer linear algebra here's another approach. To give a k-subspace of V you can simply give k linearly independent vectors $v_1, \ldots, v_k \in V$, or, if you prefer, a $(k \times n)$ matrix of maximal rank:

$$A = \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \ldots & v_{kn} \end{bmatrix}$$

Of course there is a lot of redundancy in this description, because we can choose to arbitrarily change the basis for our k-subspace. This corresponds to multiplying A on the left by a matrix in $GL(k)$. Now let us choose k columns, for example the first k, just so we don't get a headache with general notation.

If the determinant of the $k \times k$ minor is different from zero, then there is a unique matrix in $\Upsilon \in GL(k)$ (namely the inverse of that minor) such that

$$\Upsilon A = \begin{bmatrix} 1 & 0 & \ldots & 0 & v_{1(k+1)} & \ldots & v_{1n} \\ 0 & 1 & \ldots & 0 & v_{1(k+1)} & \ldots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & v_{k(k+1)} & \ldots & v_{kn} \end{bmatrix}$$

We see immediately that the remaining coefficients give local coordinates for the chart, and that they correspond to a copy of $\mathbb{R}^{k(n-k)}$.

What's the Complement of a Chart?

Algebraically, the k-subspaces that are not parameterized in the chart above are those such that the determinant of the chosen $k \times k$ minor vanishes.

For the particular chart above, this corresponds to k-subspaces that intersect the space $\{x_1 = x_2 = \ldots = x_k = 0\}$ in more than just a point. By distinguishing all possible cases, you should now tackle the following

Problem 2 (Challenge).

$$G(k,n) = \mathbb{R}^{k(n-k)} \sqcup G(1,k) \times G(k-1,n-k) \sqcup G(2,k) \times G(k-2,n-k) \sqcup \ldots \sqcup G(k-1,k) \times G(1,n-k)$$

G(k,n) is Projective

We still need to veryfy that $G(k,n)$ is compact. We do so by showing that $G(k,n)$ is a closed subvariety of an appropriate projective space.

Our approach is again to try and reproduce what we did with projective space when we constructed homogeneous coordinates for $\mathbb{P}^n$. As usual, rather than going crazy with indices, let's work with a specific example: $G(2,4)$.

Start from a point $P \in G(2,4)$, that we think as a 2×4 matrix A.

$$A = \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \end{bmatrix}$$

We want to associate to this matrix a well chosen set of numbers. The problem is, as usual, that A is not the only matrix that represents P. Any matrix ΥA also represents the same point in $G(2,4)$, provided that Υ is an invertible 2×2 matrix. Therefore we try and choose as coordinates numbers that will be as unperturbed as possible by the multiplication by Υ.

Take the determinants of all 2×2 minors of the matrix A:

$$(u_0, \ldots, u_5)(A) = \left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \ldots, \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix} \right)$$

What happens when you multiply on the left by Υ?

$$(u_0, \ldots, u_5)(\Upsilon A) = \left(det(\Upsilon) \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \ldots, det(\Upsilon) \begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix} \right)$$

Yes! All numbers are multiplied by the same constant $det(\Upsilon)$! This means that this construction associates to a point in $G(2,4)$ a proportionality class of sextuples - i.e., a point in $\mathbb{P}^5$.

Problem 3. *Show that in general this construction defines a(n injective) map:*

$$H : G(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$$

We know however that $G(2,4)$ is 4 dimensional, so the image of the previous map cannot be all of $\mathbb{P}^5$.

Problem 4. *Show that the points in the image of H must satisfy the homogeneous quadratic equation:*

$$u_0 u_5 + u_1 u_4 + u_2 u_3 = 0.$$

Problem 5. *Generalize all of this for $G(k,n)$*

Moduli Spaces of Points on $\mathbb{P}^1_{\mathbb{C}}$

$M_{0,n}$

A family of moduli spaces of a completely different flavor parameterizes configuration of labelled points on the complex projective line (a.k.a. the Riemann Sphere). To make the problem more interesting than just taking product spaces, we introdce the following rules:

1. We parameterize configurations of n labelled points $(P_1, \ldots, P_n)$ on $\mathbb{P}^1_{\mathbb{C}}$.

2. No two points are allowed to coincide.

3. We introduce the following equivalence relation:

$$(P_1, \ldots, P_n) \sim (Q_1, \ldots, Q_n)$$

   if there exists an automorphism φ of $\mathbb{P}^1_{\mathbb{C}}$ such that

$$\varphi(P_i) = Q_i$$

For reasons that will become apparent later, this moduli space is denoted $\mathbf{M_{0,n}}$.

Problem 6. *In case you are not familiar with automorphisms of $\mathbb{P}_1$, try to convince yourself that an automorphism*

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1$$

can be expressed in any of these three equivalent ways:

- *a pair of homogeneous polynomials of degree 1.*

$$(y_0 : y_1) = (ax_0 + bx_1 : cx_0 + dx_1)$$

- *a Mobius transformation (on one chart):*

$$y = \frac{ax + b}{cx + d}$$

- *a 2×2 invertible (!) matrix:*

$$\left[\begin{array}{c} y_0 \\ y_1 \end{array} \right] = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \left[\begin{array}{c} x_0 \\ x_1 \end{array} \right]$$

Conclude the following:

1. *There is a 3 dimensional group of automorphisms of $\mathbb{P}^1$.*

2. *There is a unique automorphism that sends any three (distinct) points P_1, P_2, P_3 to any other three points Q_1, Q_2, Q_3.*

Let us now explore some of these moduli spaces:

$M_{0,3}$: by 2. above this moduli space is just one point, because any triple of points can be moved via a unique automorphism to the configuration $0, 1, \infty$.

$M_{0,4}$: $\mathbb{P}_1 \smallsetminus \{0, 1, \infty\}$. Spend your automorphism to send the first three points to $0, 1, \infty$, then the fourth point is free to roam on the sphere, except on those three points.

$M_{0,5}$: $M_{0,4} \times M_{0,4} \smallsetminus \Delta$. Here Δ means "the diagonal" in the product space.

$M_{0,n}$: $M_{0,4}^{n-3} \smallsetminus$ all diagonals. Here by diagonal I mean any subpace of the product space where at least two coordinates are the same.

We have a very good understanding of these moduli spaces. Unfortunately, they are very much not compact. The quest for a good compactification leads us to yet another family of moduli spaces, which is by far more interesting!

$\overline{M}_{0,n}$

The winning idea here is very simple and elegant. We enlarge the class of objects that we wish to parameterize to include some "degenerate" objects. By doing so we construct a compact space that contains $M_{0,n}$ as a dense open set.

Heuristically, when two points (or more) want to come together...we don't allow them to. At the moment in which they would crash together, we make them "jump" on a new sphere attached to the old one at the point of collision. Let us be a little more formal now.

Definition 1. *A **tree of projective lines** is a connected curve with the following properties:*

1. *Each irreducible component is isomorphic to $\mathbb{P}^1$.*

2. *The points of intersection of the components are ordinary double points.*

3. *There are no closed circuits, i.e., if a node is removed then the curve becomes disconnected.*

Each irreducible component is called a twig. We draw a marked tree as in Figure 2, where each line represents a twig.
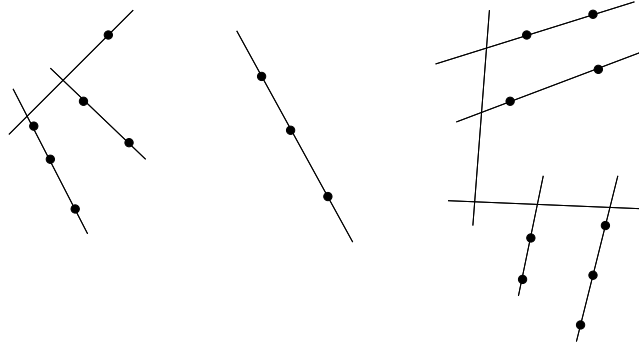

Figure 2: stable marked trees.

Definition 2. *A marked tree is **stable** if every twig has at least three special points (marks or nodes).*

Problem 7. *This stability condition is equivalent to the existence of no non-trivial automorphisms of the tree that fix all of the marks.*

Problem 8. *Show that if we define $\overline{M}_{0,4}$ to be the moduli space of isomorphism classes of four pointed stable trees, we obtain $\overline{M}_{0,4} \cong \mathbb{P}^1$.*

Fact: The moduli space $\overline{M}_{0,n}$ of n-pointed rational stable curves compactifies $M_{0,n}$.

One of the exciting features of this theory is that all these spaces are related to one another by natural morphisms. Consider the map

$$\pi_i : \overline{M}_{0,n+1} \to \overline{M}_{0,n},$$

defined by forgetting the i-th mark. It is obviously defined if the i-th mark does not belong to a twig with only three special points. If it does belong to such a twig, then our resulting tree is no longer stable. In this case, we must perform **contraction**.

Contraction: We need to consider two cases:

1. The remaining two special points are both nodes. We make the tree again stable by contracting this twig so that the two nodes are now one (see Figure 3).
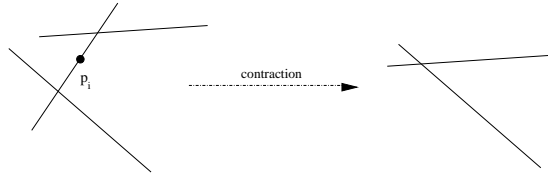
Figure 3: contracting a twig with only two nodes.

2. There is one other mark and one node on the twig in question. We make the tree stable by forgetting the twig and placing the mark where the node used to be (Figure4).
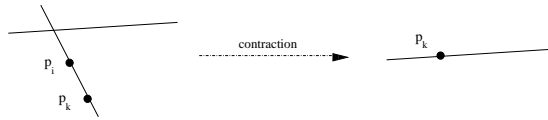

Figure 4: contracting a twig with one node and one mark.

The boundary

The **boundary** is all we have added to make our space compact, i.e. the complement of $M_{0,n}$ in $\overline{M}_{0,n}$. It consists of all nodal stable curves.

Fact: the boundary is a union of irreducible components, corresponding to the different possible ways of arranging the marks on the various twigs.

Problem 9. *Show that the codimension of a boundary component equals the number of nodes in the curves in that component.*

Problem 10. *Notice that the irreducible components of the boundary are isomorphic to products of moduli spaces of rational stable curves with strictly fewer number of points.*
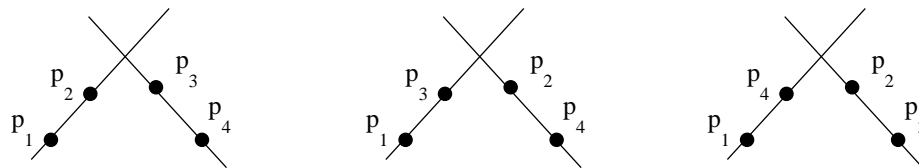

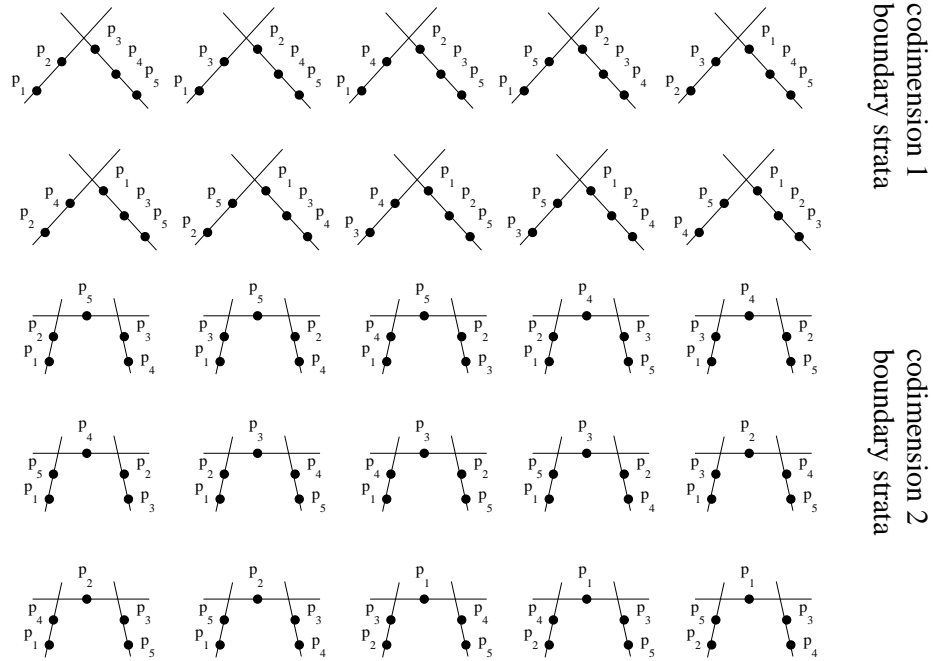Figure 5: irreducible components of the boundary of $\overline{M}_{0,4}$

Figure 6: boundary cycles of $\overline{M}_{0,5}$

In Figures 5 and 6 we draw all boundary strata for $\overline{M}_{0,4}$ and $\overline{M}_{0,5}$.

There is plenty more to be said about the spaces $\overline{M}_{0,n}$, their relationships, and their boundaries, but time is short and we need to get to Kontsevich's proof by next week, so I'll stop here. The book [KV99] is an **excellent** reference for beginners.

Moduli Spaces of Curves

We are going to wrap up the day with a whirlwind tour of higher genus...this will be unbelievably fast and imprecise, but hopefully it will make you want to know more about these topics!

Recall that a projective variety is the zero set of a bunch of homogeneous polynomials in some projective space $\mathbb{P}^n$.

Definition 3. *A **projective curve** is a projective variety of dimension* 1.

Problem 11. *Show that* $\mathbb{P}^1$ *is a projective curve.*

A large class of examples is given by plane curves: zero sets of one homogeneous polynomial in x, y, z, considered as varieties in $\mathbb{P}^2$.

If we work over the complex numbers, then curves are really "surfaces". It is not hard to see that projective curves actually are complex manifolds (well, smooth curves are, singular curves are...almost everywhere). The complex structure forces orientability, and being a closed subset of $\mathbb{P}^n$ tells us that they are compact.

Fact: topologically, projective curves are all "doughnuts". The number of holes, called the **genus**, is the unique discrete topological invariant.

Let's point out, at the cost of being boring, that $\mathbb{P}^1$ is THE smooth curve of genus 0.

Why did I write "THE"? It was not a typo, nor a language mishap. In fact it is true that any smooth curve of genus 0 is isomorphic to $\mathbb{P}^1$. That is why moduli spaces of genus 0 curves are not very interesting by themselves, and we had to "spice them up" by adding mark points!

Let me point out another feature, that I find quite amazing. While we are not too surprised that an algenraic curve is a complex manifold (after all polynomials are holomorphic functions), it is kind of amazing that the converse is true as well, i.e. any compact complex manifold of complex dimension 1 is in fact algebraic. And more is true in general.

Fact. The following mathematical structures on a topological surface of genus g are equivalent:

1. A structure of an algebraic curve.

2. A complex structure.

3. (for $g \geq 2$) A hyperbolic structure (i.e. a metric with constant negative curvature) up to isotopy.

While $\mathbb{P}^1$ is the unique curve of genus 0, for higher genera there are lots and lots of curves (or of complex structures, or of hyperbolic metrics).

In fact there is a $3g-3$ dimensional moduli space of smooth curves of genus g, called $\mathbf{M_g}$. It is homeomorphic to the quotient of a ball by a finite group, called the mapping class group...but this is another story.

Again, M_g is very much non compact, because curves can degenerate. Again, the solution is to allow nodal curves to enter the picture.

Definition 4. *A stable curve of genus g is:*

1. *A connected nodal curve.*

2. *The genus of the curve is g. For nodal curves the genus is counted in the following way:*

    (a) add the genus of all irreducible components.

    (b) add the number of loops created by the irreducible components.

3. *Each component of genus* 0 *must have at least three nodes.*

4. *Each component of genus* 1 *must have at least one node.*

Problem 12. *Make sense of the above definition by drawing some pictures!*

The moduli space of stable curves of genus g, denoted $\overline{\mathbf{M}}_{\mathbf{g}}$ is a compactification of M_g.

Of course, if you want, you can do all of the above with marked points as well. The corresponding moduli space is denoted $\overline{\mathbf{M}}_{\mathbf{g,n}}$ and it is a moduli space of dimension $3g-3+n$.

Final Observation: look at the boundary of $\overline{M}_g$. Its components are isomorphic to products of moduli spaces of marked curves! So, even if you only care about unmarked curves, studying moduli spaces of marked curves is essential in order to understand the boundary.

I have quickly told you the existence of a huge network of interesting geometric spaces connected by a ton of natural maps. I hope I conveyed the fact that we just scratched the very tip of a huge iceberg. Let me leave you with some reading reccomendations, for anyone eager to know a little more. The book [HM98] is certainly a pleasant and innstructive read. The survey by Ravi Vakil (*The moduli space of curves and Gormov-Witten theory* - available on the ArXiv), starts basic and gets steep quickly, but it's very well done. Realistically, the first 10 pages may be accessible to a general audience, but that would still be a lot of good stuff.

References

[HM98] Joe Harris and Ian Morrison. *Moduli of Curves.* Springer, 1998.

[KV99] Joachim Kock and Israel Vainsencher. *Kontsevich's formula for rational plane curves.* IMPA, 1999.