G-Hodge Integrals, Gerby Localization and $\mathcal{GW}(\ [\mathbb{C}^3/\mathbb{Z}_3]\ )$

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Roadmap for the talk

GW Theory	Orbifold *GW* Theory
↕ *localization*	↕ *localization*
Hodge Integrals	*G*-Hodge Integrals
↕ *relations*	↕ *relations*
F-P technique	Gerby localization

Application (Cadman-C, 2007):

gerby localization produces recursions that determine any genus 0 equivariant Gromov-Witten invariant of $[\mathbb{C}^3/\mathbb{Z}_3]$.

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Outline

1. Motivation and History
   - Hodge Integrals
   - *G*-Hodge integrals
2. $[\mathbb{C}^3/\mathbb{Z}_3]$
3. Gerby Localization

The Hodge bundle:

The *Hodge bundle*

$$\mathbb{E} \to \overline{\mathcal{M}_g}$$

is a rank g vector bundle, whose fiber over a curve C is:

- the holomorphic differential 1-forms on C (if C is smooth).
- the global sections of the relative dualizing sheaf (K_C if C smooth).
- the dual to $H^1(C, \mathcal{O}_C)$.

The *i-th Hodge class* is

$$\lambda_i := c_i(\mathbb{E}).$$

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What's nice about λ classes?

1. They are tautological (Mumford).
2. They "split nicely" when restriced to the boundary.
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Hodge Integrals

Hodge Integrals are top intersection numbers of λ classes. When organized in generating function, they exhibit a surprising amount of structure.

Example:(Faber-Pandharipande/Bertram-C-Todorov)

$$\mathcal{F}(u) := \sum_{g=1}^{\infty} \left(\int_{\overline{H}_g} \lambda_g \lambda_{g-1} \right) \frac{u^{2g-1}}{(2g-1)!} = \frac{1}{2} \tan\left(\frac{u}{2}\right)$$

(used to show that dim $R^{g-2}\mathcal{M}_g$=1)

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A one slide sidetrack

Theorem (C, 2007)

$$\mathcal{P}(u; x_1, \ldots) := \sum_{g=1}^{\infty} \left(\int_{\overline{H}_g} \lambda_g \lambda_{g-i} \psi_1^{i_1} \ldots \psi_n^{i_n} \right) \frac{u^{2g-1}}{(2g-1)!} x_1^{i_1} \ldots x_n^{i_n} = \\ = \frac{1}{2} \left(\cos\left(\frac{u}{2}\right)\right)^{-2\sum x_i} \tan\left(\frac{u}{2}\right)$$

Hodge Integrals and GW Theory

Let X be a space with a torus action and isolated fixed points. Then Gromov-Witten invariants of X can be computed via localization and expressed in terms of Hodge integrals:

- induce a torus action on $\overline{M}_{g,n}(X, \beta)$;
- fixed loci parameterize maps where all high genus components are collapsed;

$$\Rightarrow F \cong (comb.\ mess) \prod \overline{M}_{g_i,n_i}$$

- the virtual fundamental class restricted to a fixed locus is a polynomial in λ classes;
- the normal bundle to a fixed locus is a rational function in ψ classes.

Hodge Integrals and open GW Theory

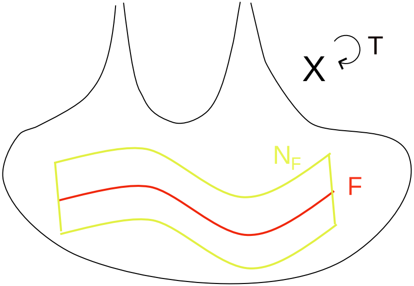
Let X be a non-compact space with a torus action with compact fixed locus F. GW invariants for X can be defined via localization as intersection numbers on

$$\overline{M}_{g,n}(F,\beta)$$

"corrected" by a class

$$e(Ob),$$

(obstruction bundle "coming" from N_F). This again causes the appearance of Hodge Integrals in these expressions.



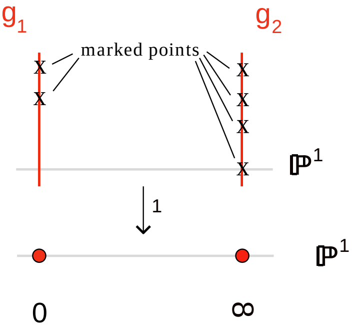
Faber-Pandharipande's technique

Faber and Pandharipande study Hodge integrals through the equivariant GW theory of $\mathbb{P}^1$.

They compute via localization some known intersection numbers on

$$\overline{M}_{g,n}(\mathbb{P}^1, 1)$$

and extract relations among Hodge integrals from the fixed loci contributions.



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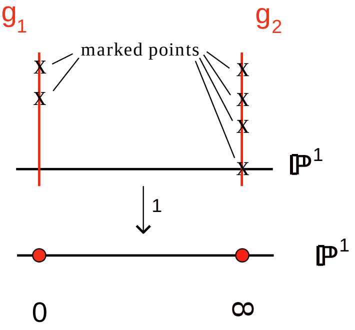
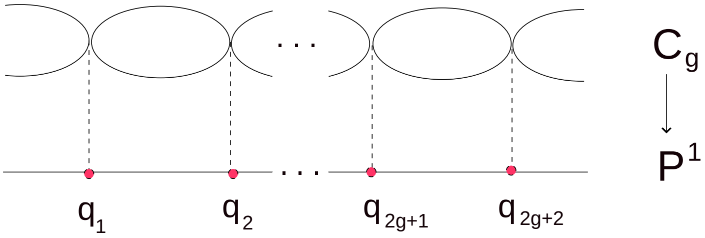


G actions on curves

Let G be a finite group. Consider the locus of curves in C in $\overline{\mathcal{M}_g}$ that admit a G action such that the quotient is a rational curve.



Example: the hyperelliptic locus is such a locus for the group $G = \mathbb{Z}_2$.

The splitting of the Hodge bundle

The G action on the curves induces a G action on the 1-forms. The Hodge bundle then splits according to the irreducible representations of G

$$\mathbb{E} = \mathbb{E}_{\rho_1} \oplus \ldots \oplus \mathbb{E}_{\rho_n},$$

We now define:

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Why we like G-λ classes

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G-Mumford relations:

$$c(\mathbb{E}_\rho \oplus \mathbb{E}^\vee_{\rho^\vee}) = 1$$

Top intersection numbers of G-λ classes are called G-Hodge integrals.

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Twisted stable maps to *BG*

The locus in $\overline{\mathcal{M}_g}$ of curves admitting a G action can be interpreted as a moduli space of genus 0 twisted stable maps to the classifying stack BG.

$$\text{Admissible } G \text{ covers} \cong \overline{\mathcal{M}}_{0,n}(BG, 0)$$

Idea: think of the genus g curve C as a principal G bundle over an orbifold rational curve. This gives a map to BG. The marked points keep track of the orbifold points on the base, or, if you prefer, of the branch points for the ramified cover.

$[\mathbb{C}^3/\mathbb{Z}_3]$

By $[\mathbb{C}^3/\mathbb{Z}_3]$ we mean the stack quotient of $\mathbb{C}^3$ by the action:

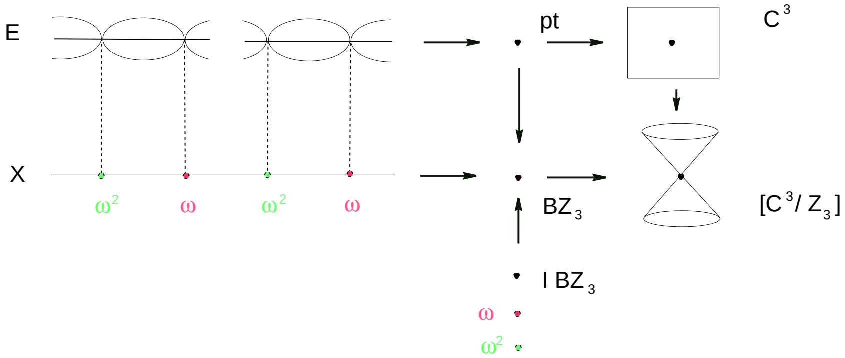
$$\omega \mapsto \begin{bmatrix} \omega & & \\ & \omega & \\ & & \omega \end{bmatrix}$$

This orbifold is very fascinating to both mathematicians and physicists, because of its role in mirror symmetry. Aganagic-Bouchard-Klemm predicted all of its Gromov-Witten invariants (physicists are always one step ahead...) Verifying mathematically the physics predictions and the CRC is a quest that has only been fulfilled in genus 0.

Orbifold GW theory

Orbifold GW invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$ can be interpreted in terms of $\mathbb{Z}_3$-Hodge integrals on moduli spaces of Galois $\mathbb{Z}_3$-admissible covers. This is best illustrated in a picture.

Orbifold GW theory



$\mathbb{Z}_3$-Hodge Integrals

For the orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$, the obstruction bundle consists of three copies of $\mathbb{E}_\omega^\vee$, the ω-subrepresentation of the dual of the Hodge bundle on the cover curve.

Remarks:

1. Monodromy condition. We denote by (n_1, n_2) the genus 0, (degree 0) invariant corresponding to n_1 ω-insertions and n_2 ω^2-insertions. For such an invariant to be non-zero the condition

$$n_1 - n_2 \equiv 0 \quad (3)$$

must be satisfied.

2. The obstruction bundle exceeds the dimension of the moduli space precisely by the number of ω^2 points. Invariants of type $(3k+3, 0)$ are defined non-equivariantly, whereas all other invariants are polynomials in the equivariant parameters.

The genus 0 Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$

Genus of the cover curves:

g=1	*g=2*	*g=3*	*g=4*	*g=5*
			(6,0)	
(3,0)		(4,1)		(5,2)
	(2,2)		(3,3)	
(0,3)		(1,4)		(2,5)
			(0,6)	

number of ω insertions

number of ω^2 insertions

The genus 0 Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$

The 3 pointed invariants, defined over a zero dimensional moduli space, are easily computed by hand and considered initial conditions. All other invariants are polynomials in λ^ω classes and in the three equivariant parameters t_1, t_2, t_3.

All the invariants are computed recursively in terms of the three pointed invariants. Relations are developed via:

- WDVV;
- gerby localization.

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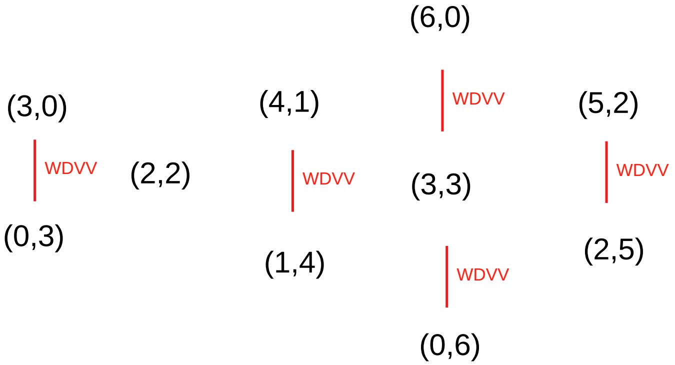
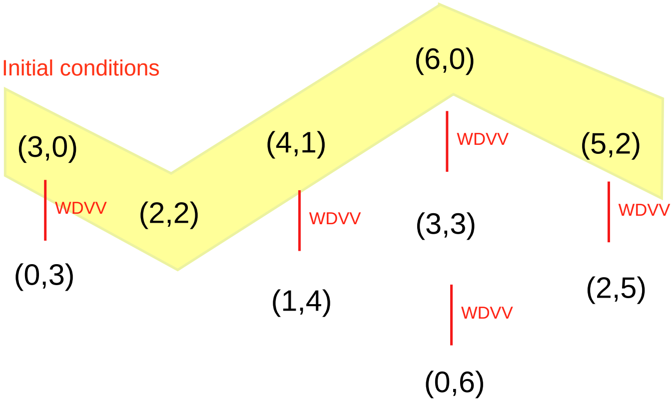
Strategy: WDVV



Strategy: gerby localization



Gerbes

Intuition: a G-gerbe over a space X is, roughly speaking, a BG bundle over X.

We focus on $\mathbb{Z}_3$-gerbes over $\mathbb{P}^1$. A concrete way to describe them is:

$$G_k \cong [\{\mathcal{O}_{\mathbb{P}^1}(k) \setminus Z\}/\mathbb{C}^*],$$

where $\mathbb{C}^*$ acts on the fibers of the line bundle via:

$$(\alpha, v) \mapsto \alpha^3 v$$

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Maps to the trivial gerbe

For X a rational twisted curve, a map:

$$X \xrightarrow{d=1} G_0 \cong \mathbb{P}^1 \times B\mathbb{Z}_3$$

is equivalent to a $\mathbb{Z}_3$ admissible cover of X, where one (and only one!) of the twigs of X is a parameterized $\mathbb{P}^1$.

Obtaining relations

Let us write the number 0 in a more interesting way:

$$\int_{(3k+3,0)\to G_0} e(R^1_\omega \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) ev_1^*(\infty) = 0$$

Localizing this integral gives relations between $\mathbb{Z}_3$-Hodge integrals (with descendants).

Remarks

1. in genus 0 removing descendants is but a technical/combinatorial issue. In higher genus this is currently the obstacle preventing us from succeeding with this approach.
2. although it is possible to concoct many auxiliary integrals producing relations, it is really quite hard to produce independent relations.
3. a clever choice for the auxiliary integrals and for the linearizations of the bundles is instrumental to keeping the combinatorial complexity of the recursions manageable.
4. despite our most creative efforts, auxiliary integrals on maps to the trivial gerbe do NOT produce sufficiently many relations to determine the invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$!!

Maps to non-trivial gerbes

We turn our attention to maps to non-trivial gerbes. For example:

$$\int_{\mathcal{G}_1(3k+1,0)} e\left(R^1\pi_* f^*\left(\mathcal{O}_{G_1}\left(-\frac{2}{3}\right)\oplus\mathcal{O}_{G_1}\left(-\frac{2}{3}\right)\oplus\mathcal{O}_{G_1}\left(-\frac{2}{3}\right)\right)\right)=0$$

Determining the invariants

To prove that invariants are reconstructed, we proceed by induction on "three genera at a time". Invariants (with at most 2 ω^2 insertions) for genera $g, g+1$ and $g+2$ are determined by 4 monomials in G-λ classes. The two relations above plus two more give an invertible 4×4 matrix of principal parts. The invariants can therefore be computed in terms of invariants of genus $< g$.

Some Numbers:

(9,0): $\frac{1}{9}$

(6,0): $-\frac{1}{27}$

(7,1): $-\frac{5}{27}t$

(3,0): $\frac{1}{3}$

(4,1): $\frac{1}{9}t$

(5,2): $\frac{7}{27}t^2$

(6,3): $\frac{398}{243}t^3$

(2,2): $-\frac{1}{3}t^2$

(3,3): $-\frac{8}{27}t^3$

(4,4): $-\frac{98}{81}t^4$

(0,3): $\frac{1}{3}t^3$

(1,4): $\frac{2}{9}t^4$

(2,5): $\frac{19}{27}t^5$

(3,6): $\frac{1274}{243}t^6$

(0,6): $-\frac{10}{27}t^6$

(1,7): $-\frac{179}{81}t^7$

(0,9): $\frac{686}{81}t^9$

Generating functions

Our recursions are neatly packaged in terms of generating functions for $\mathbb{Z}_3$ Hodge integrals with one descendand insertion. Let:

$$\mathcal{L}^{\omega}(x, y; u, v) := \sum_{m,n,i,j} \int_{(m,n)} \lambda_{top}\lambda_{top-i}\lambda_{top-j}\psi_{\omega}^{i+j-n}\frac{x^m}{m!}\frac{y^n}{n!}u^i v^j$$

and define $\mathcal{L}^{\bar{\omega}}$ analogously. Then the two (families of) relations written earlier are equivalent to the following PDE's on these generating functions.

PDE's

Relation 1:

$$2\mathcal{L}_x^{\omega}(-x,0;1,0) = 3\mathcal{L}_y^{\bar{\omega}}(-x,0;1,0)\mathcal{L}_{xx}^{\omega}(-x,0;0,0)$$

Relation 2:

$$\mathcal{L}_y^{\bar{\omega}}\left(-x,0;\frac{2}{3},\frac{2}{3}\right) - \mathcal{L}_y^{\bar{\omega}}\left(-x,0;\frac{2}{3},0\right) =$$

$$= \frac{1}{9}\mathcal{L}_x^{\omega}\left(-x,0;\frac{2}{3},\frac{2}{3}\right)\mathcal{L}_x^{\omega}\left(-x,0;\frac{2}{3},0\right)$$