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From Topological Strings to Integrable Hierarchies and back

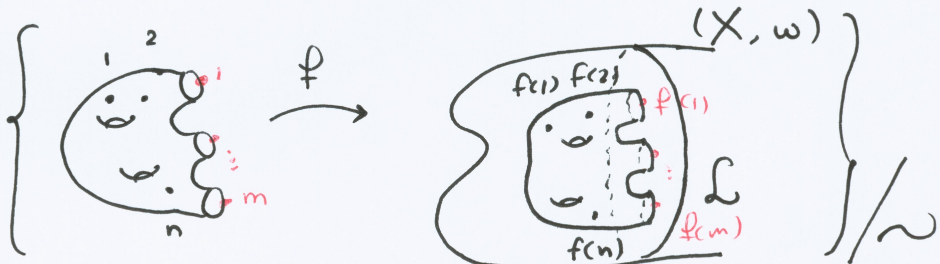
Lecture #4:
* Open GWI
* Foundational Aspects
* Computational Aspects
* Disc Invariants for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$

§1. Open Gromov-Witten Invariants

Physicists tell us that open strings propagate w/ ends constrained on a Lagrangian submanifold, and sweep out Riemann surfaces with boundaries

Mathematically, there should be a moduli space:

$$\overline{M}_{g,n,\ell,m}(X, \mathcal{L}; \beta, \vec{\gamma}) = \left\{ \quad \right\} \Big/ \sim$$



g: genus; n: interior marks; ℓ: boundary circles; m: boundary marks; X: target; $\mathcal{L}$: Lagrangian submanifold of X; $\beta \in H_2(X, \mathcal{L})$; $\gamma_i \in H^1(\mathcal{L})$

* Necessary condition for this moduli space to be non-empty: $\partial\beta = \sum \gamma_i$

Would like:

* proper, orientable, moduli space
* virtual fundamental class

$\not\Rightarrow$ define G.W.-like invariants.

Obs [AKV]: open invariants are not intrinsic to the geometry of $(X, \mathcal{L})$
they depend on extra parameter, in physics called framing
Mathematically, this will correspond to some lifting of an S^1-action to a bundle.

Two types of work are being done in math in this area:

- <u>FOUNDATIONAL</u>: define such invariants rigorously

  [J. Salomon] - thesis

  [Pandharipande-Salomon-Walcher] - the quintic threefold

- <u>COMPUTATIONAL</u>:

  [Katz-Liu]: $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

  [Graber-Zaslow]: $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$

These two aspects are almost disjoint!!

§2. Few words on Salomon's foundational work

Constructs a moduli space $\bar{\mathcal{M}}$ for maps from a fixed (conformal structure) pointed, bordered Riemann surface. The following problems arise:

- $\bar{\mathcal{M}}$ is non-orientable (typically when $\mathcal{L}$ isn't)
- Boundary in $\mathbb{R}$-codimension 1, making insertions for G.W.I.'s not necessarily well defined at the level of cohomology

Solution:

- with some mild assumptions, there is a <u>relative</u> orientation

$$\bar{\mathcal{M}} \xrightarrow{\Pi ev_i} L^k$$

boundary points evaluation maps.

- If $\exists$ $\phi : X \to X$ anti-symplectic involution ($\phi^* \omega = -\omega$) such that $\mathcal{L} = \mathrm{Fix}(\phi)$, then can decompose $\bar{\mathcal{M}}$:

$$\bar{\mathcal{M}} = \mathcal{M} \amalg \mathcal{M}^{\text{bad corners}} \amalg \mathcal{M}^{\text{harmless corners}}$$

And can define an orientation reversing involution

$$\tilde{\phi} : \mathcal{M}^{\text{bad corners}} \longrightarrow \mathcal{M}^{\text{bad corners}}$$

and:

* the quotient space $\hat{\mathcal{M}} = \bar{\mathcal{M}}/\tilde{\phi}$ is oriented

** ϕ-invariant cohomology classes descend to $\hat{\mathcal{M}}$,

*** and have support away from $\partial\hat{\mathcal{M}}$.

$\Rightarrow$ $\int_{\hat{\mathcal{M}}} \prod ev_i^*(\alpha_i)$ are well defined, and are the candidates to be the mathematical open GWI's.

§ 3. Computing open GWI's : $[\mathbb{C}^3/\mathbb{Z}_3]$

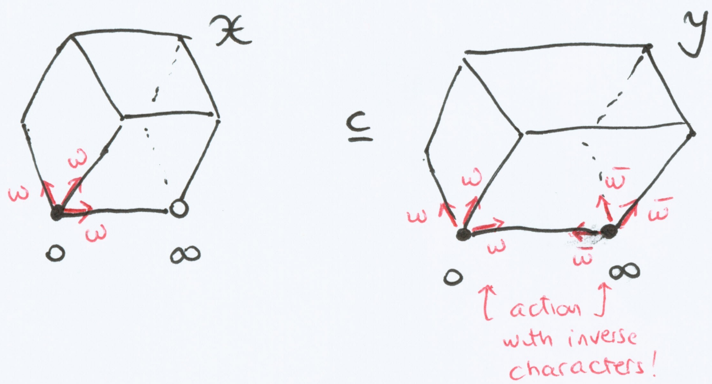
(more or less joint work with P. Johnson, H.H. Tseng)

Premise: as usual, the physicists beat us to it:

[ABKM] compute disc + annulus invariants for $[\mathbb{C}^3/\mathbb{Z}_3]$

In fact, this project for me was born from a visit of V. Bouchard.

Geometric Setup: view $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$ as an open chart of the global quotient $\mathcal{Y} = [\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)/\mathbb{Z}_3]$



Strategy: use localization. Even if I don't understand $\mathcal{M}$, if (when) such a moduli space exists and if V.F.C. doesn't behave pathologically $\Rightarrow$ A-B localization theorem will hold!

- we understand fixed loci F_i
- there is a natural <u>guess</u> for $1^{vir}_{|F_i}$

$\Rightarrow$ can <u>DEFINE</u> invariants via localization and actually compute them.

There is an anti-holomorphic involution on $\mathcal{Y} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$$\sigma : \mathcal{Y} \longrightarrow \mathcal{Y}$$
$$(z, u, v) \mapsto \left(\frac{1}{\bar{z}}, \bar{v}\bar{z}, \bar{u}\bar{z}\right)$$

descending to the quotient:

$$\sigma : \mathcal{Y} \longrightarrow \mathcal{Y}.$$

$\mathcal{L} := \mathrm{Fix}(\sigma) = \{(e^{i\theta}, u, \bar{u}e^{-i\theta})\}$ is a Lagrangian with topology

$$\mathcal{L} \cong S^1 \times \mathbb{R}^2$$

One can define an S^1 action on $\mathcal{Y}$:

* descending to the quotient $\mathcal{Y}$
* compatible with the involution σ: $\sigma(t.x) = \bar{t}.\sigma(x)$
  $(t \in S^1, x \in \mathcal{Y})$

Such action has weights:

$$\left(\frac{1}{3}, -\frac{a}{3}, -\frac{1}{3} + \frac{a}{3}\right) \text{ at } 0$$

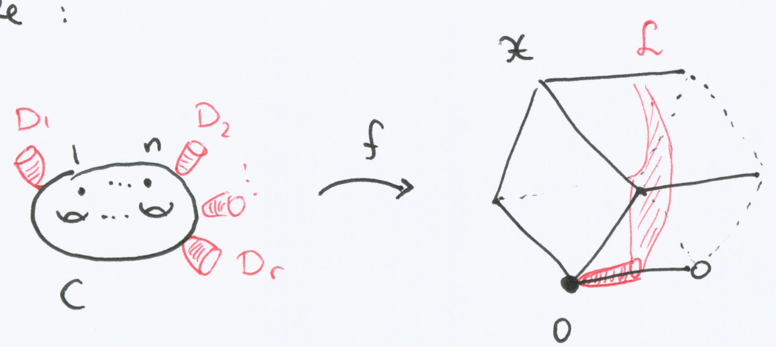
$$\left(-\frac{1}{3}, -\frac{a}{3} + \frac{1}{3}, \frac{a}{3}\right) \text{ at } \infty$$

<u>Rmk</u>: a is a free parameter our invariants will depend on, corresponding to the physicists' framing.

S^1-invariant curve in $\mathcal{L} \subseteq \mathcal{X}$ ($\subseteq \mathcal{Y}$):

$$C = \{(e^{i\theta}, 0, 0)\}$$

Therefore, the fixed loci in the moduli space of open maps will look like :

$F = \Big\{$  $\Big\} / \sim$

$$f_{|C} : C \longrightarrow \underline{0}$$

$$f_{|D_i} : D_i \longrightarrow y$$

$$z_i \longmapsto (z_i^{\,k}, 0, 0)$$

$$\Rightarrow \quad F \cong \overline{\mathcal{M}}_{0,n+r}(B\mathbb{Z}_3, 0) \times \prod B\mathbb{Z}_k$$

sloppy : I am not sure this is a trivial gerbe, but for the purpose of $\cap$ theory it doesn't matter

$1^{vir}_{|F}$ has two parts :

- a part coming from $\pi_* f^*(T_{\mathbb{P}^1/\mathbb{Z}_3})$
- a part coming from $R^1\pi_* f^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_3)$

Remark : In fact, the obstruction theory is naturally expressed in terms of (push-pull of) sheaves of holomorphic functions with values in the above bundles, but with boundary values in totally Real sub-bundles of $\text{bundle}_{|L}$. Via $\mathbb{C}_x$-doubling these weights can be expressed in terms of $\mathbb{C}^*$-weights of $\mathbb{C}_x$-bundles over all of $\mathbb{P}^1/\mathbb{Z}_3$, as said above.

After much combinatorial pain:

$$\langle\ \rangle = \left(\int_{[\bar{M}_{0,n}(B\mathbb{Z}_3, 0)]} \frac{e^{eq}\left(\mathbb{E}(+\tfrac{a}{3}) \oplus \mathbb{E}(\tfrac{+1-a}{3}) \oplus \mathbb{E}(-\tfrac{1}{3})\right)}{1-\psi} \right) \cdot \prod \left\{ \begin{matrix} \text{pure} \\ \text{weight} \\ \text{factors} \end{matrix} \right\}$$

pull push of the bundles restricted to D_i's.

And therefore the open theory can be computed by knowing:

- closed theory w/ one descendant insertion
- Combinatorial weights associated to discs.

§4. Explicit Asnwer for disc invariants

Recall Givental's J-function:

$$J(z; \tau_1, \tau_\omega, \tau_{\bar{\omega}} = 0; s_1, s_2, s_3) =$$

$$= e^{\tau_1/z} \left[z\phi_1 + \tau_\omega \phi_\omega + \sum \frac{\phi^\alpha}{n!} \left\langle \phi_\omega^n, \frac{\phi_\alpha}{z-\psi} \right\rangle \right]$$

Define a combinatorial 'disc' function:

$$D(d,a) = \frac{1}{\lfloor \frac{d}{3} \rfloor !} \cdot \frac{1}{d^{3\langle d \rangle}} \cdot \frac{\Gamma\left(\frac{d}{3} + \langle \frac{d}{3} \rangle - \frac{da}{3}\right)}{\Gamma\left(1 - \langle \frac{d}{3} \rangle - \frac{da}{3}\right)} \phi^{w^d}$$

Then:

$$F_0^{disk}(x,y,a) := \sum_{n,d} \langle \phi_\omega^n ; a \rangle_{0,d}^{disk} \frac{x^n}{n!} \frac{y^d}{d!} =$$

$$= \sum_d \frac{y^d}{d!} \left[J \cdot D(d,a) \right] \to \text{specializing}$$

$\boxed{S_i \to \text{weights at } 0}$

$z \to 1$