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From Topological Strings to Integrable Hierarchies & back.

Lecture # 2:

* Tautological classes on $\overline{M}_{g,n}$
* Admissible G-covers
* G-Hodge classes
* Application: computing $\int_{H_g} \lambda_g \lambda_{g-1}$
* C.R.C. á la Bryan-Graber: $[\mathbb{C}^2/\mathbb{Z}_2]$

§1. Tautological Classes on $\overline{M}_{g,n}$

These are Chow classes on $\overline{M}_{g,n}$ constructed in a 'natural' geometric way. We focus on two families of such classes.

Ⓐ ψ classes / Gravitational Descendants

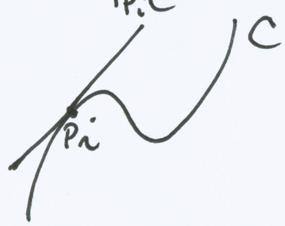
$$\begin{tikzcd} \mathbb{L}_i = \mathbb{T}_i^* \arrow[d] & \\ \overline{M}_{g,n} \arrow[r, bend right, "\sigma_i"'] & \overline{M}_{g,n+1} \arrow[l, "\pi"'] \end{tikzcd}$$

The i-th cotangent line bundle on $\overline{M}_{g,n}$:

$$\mathbb{L}_i := \sigma_i^*(\omega_\pi)$$

Fiber over a point $[(C, p_1, \dots p_n)]$ is $T^*_{p_i}C$

$T_{p_i}C$: fiber of $\mathbb{T}_i$; $T^*_{p_i}C$: fiber of $\mathbb{L}_i$



Def: $\psi_i := c_1(\mathbb{L}_i)$

Importance:

① In $g=0$ they are boundary class $\Rightarrow$ give rise to T.R.R.

② If $D = \left[\text{X}_{p_1 = p_2} \right] \Rightarrow N_{D/\overline{M}_{g,n}} \cong \mathbb{T}_1 \otimes \mathbb{T}_2$

(B) Hodge Classes

$$\begin{array}{c} \mathbb{E}_g \\ \downarrow \\ \overline{M}_g \xleftarrow{\pi} \overline{M}_{g,1} \end{array}$$

The Hodge bundle $\mathbb{E}_g$ is a rank g vector bundle:

$$\mathbb{E}_g := \pi_* \omega_\pi$$

Fiber over $[C] \in M_g$:

- $H^0(C, K_C)$: global sections of the canonical bundle;
- $\Omega^1(C)$: global holomorphic 1-forms on C;
- $H^1(C, \mathcal{O}_C)^*$: by Serre duality.

Nice properties of $\mathbb{E}_g$:

① $\mathbb{E}_g|_{[\underset{g_1\ g_2}{\times}]} \cong \mathbb{E}_{g_1} \oplus \mathbb{E}_{g_2}$

② $\mathbb{E}_g|_{[\alpha^{g-1}]} \cong \mathbb{E}_{g-1} \oplus \mathcal{O}$

③ (Mumford's Relations): $\mathbb{E} \oplus \mathbb{E}^\vee$ is a flat bundle

($\Rightarrow$ all Chern classes ($\neq c_0$) vanish)

Def: $\lambda_i := c_i(\mathbb{E}_g)$ – ith Hodge class.

Remark: properties above translate to combinatorial relations:

① $\Rightarrow$ $\lambda_g \lambda_{g-1}|_{\overline{M}_g - M_g} \equiv 0$

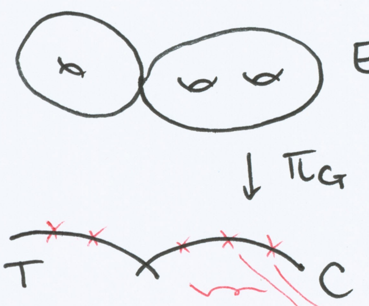
② $\Rightarrow$ $\lambda_g|_{[\alpha]} \equiv 0$

③ $\Rightarrow$ $\lambda_g^2 = 0 \qquad g \neq 0$

$$\lambda_2 = \frac{\lambda_1^2}{2}$$

§2. Admissible G-covers

I want to describe a moduli space for:



E^{2G}

$\downarrow \pi_G$ $\qquad$ $\Downarrow$ EQUIVALENTLY

$$T \cup C \xrightarrow{\ f\ } \cdot\, BG = [pt/G]$$

$$\uparrow$$

$$[g_1]\cdot\ IBG = \coprod_{[g]\in G} BC(g)$$

$\xrightarrow{ev_i}$

$[g_r]\cdot$

specify ramification profile over branch points.

- nodes correspond to nodes
- Kissing condition: matching ramification profile at the shadows of the nodes.

Such a creature is called an <u>admissible G-cover</u>

- Fixing ramification profiles is equivalent to specifying image of the (refined) evaluation maps to IBG.

<u>Abramovich-Corti-Vistoli</u>:

$$\left\{\begin{array}{l}\text{Stack of Admissible}\\ G\text{-covers}^{\text{of }\mathbb{P}^1} + \text{assigned}\\ \text{monodromy types}\end{array}\right\} \overset{\text{EQUIV.}}{\longleftrightarrow} \left\{\begin{array}{l}\text{Component of } \overline{M}_{0,n}(BG,0)\\ \text{cut out by the class}\\ \quad \prod_i ev_i^*([1_{g_i}])\end{array}\right\}$$

§3. Tautological classes on Admissible Covers

(A) ψ classes

ψ classes are defined similarly. Only caveat, there are 2 types:

- stacky ψ classes
- coarse ψ classes

[Ionel]: $\boxed{\text{stacky } \psi_i = \frac{1}{|G_{p_i}|} \text{ coarse } \psi_i}$

(B) Hodge classes :

$$\begin{tikzcd} f^*\mathbb{E}_g \arrow[r] \arrow[d] & \mathbb{E}_g \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(BG) \arrow[r, "f"] & \overline{M}_g \end{tikzcd}$$

$$\left[\begin{matrix} E^{\circlearrowleft G} & \\ \downarrow & \\ C & \to BG \end{matrix}\right] \mapsto [E]$$

A 'rank' g Hodge bundle is obtained by pull back from $\overline{M}_g$.

$f^*\mathbb{E}_g := \mathbb{E}$ inherits a G-action from the action on the curves.

$$\mathbb{E} = \bigoplus_{\rho \text{ irreps of } G} \mathbb{E}^{\rho} \leftarrow \text{possibly many copies of } \rho$$

Def: $\lambda_i^{\rho} := c_i(\mathbb{E}^{\rho})$ - G-Hodge classes.

Top intersection numbers of G-Hodge classes are called G-Hodge integrals.

Remark: G-Hodge classes enjoy combinatorial properties (vanishing, splitting along the boundary) similar to ordinary Hodge classes.

§4. Application: computing $\tan\left(\frac{x}{2}\right)$ without GRR

Thm (Faber–Pandharipande)

$$H(x) := \sum_{g=1}^{\infty} \left(\int_{H_g} \lambda_g \lambda_{g-1} \right) \frac{x^{2g-1}}{(2g-1)!} = \frac{1}{2} \tan\left(\frac{x}{2}\right)$$

Proved in '90s using G.R.R.

COMBINATORIAL PROOF (Bertram, C–, Todorov):

① Setting up the 'right' generating functions

$$F(x) := \sum_{g=1}^{\infty} \left(\int_{H_g} \lambda_g \lambda_{g-1} \right) \frac{x^{2g+1}}{(2g+1)!} \qquad F_i(x) := \sum_{g=i}^{\infty} \left(\int_{\bar{H}_g} \lambda_g \lambda_{g-i} \psi^{i-1} \right) \frac{x^{2g+1}}{(2g+1)!} \quad \left(F_0(x) = \frac{x}{2} \right)$$

$$G(x) := F'(x) \qquad G_i(x) := F_i'(x) \qquad G_0 = \frac{1}{2}$$

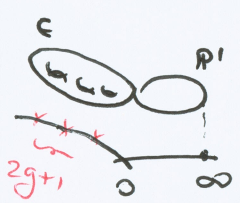
$$H(x) := F''(x) \qquad H_i(x) := F_i''(x)$$

② Auxiliary Localization Computations

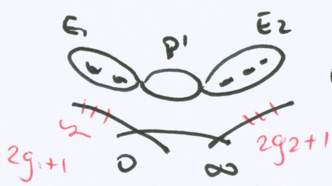
Ⓐ $$\int_{M_{0,\underbrace{2g+2}_{\text{twisted points}}}(\mathbb{P}^1 \times B\mathbb{Z}_2, 1)} \lambda_g \lambda_{g-2}\, ev_1^*(0)\, ev_2^*(0)\, ev_3^*(\infty) = 0 \text{ by dimension reasons.}$$

Compute 0 by localization:

Fixed loci:



$$\longmapsto \int_{\bar{H}_g} \frac{\lambda_g \lambda_{g-2}}{1-\psi} = \int_{\bar{H}_g} \lambda_g \lambda_{g-2} \psi$$



$$\longmapsto \underset{\substack{\text{STACKY} \\ \text{GLUING}}}{2} \binom{2g-1}{2g_2} \int_{H_{g_1}} \lambda_{g_1} \lambda_{g_1-1} \int_{H_{g_2}} \lambda_{g_2} \lambda_{g_2-1}$$

In terms of generating functions:

$$G_2' = 2\,G\,G' \Rightarrow G_2 = G^2 = \frac{1}{2}\cdot\frac{2^2}{2!}\,G^2$$

(B) Repeat the trick:

$$\int \lambda_g \lambda_{g-3} = 0 \Rightarrow G_3 = \frac{1}{2}\,\frac{2^3}{3!}\,G^3$$

And in general:

$$\int \lambda_g \lambda_{g-i} = 0 \Rightarrow G_i = \frac{1}{2}\,\frac{2^i}{i!}\,G^i$$

And:

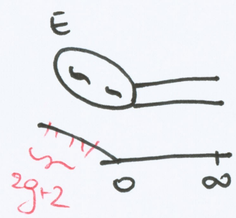
$$\boxed{\sum_0^{\infty} G_i = \frac{1}{2}\,e^{2G}} \quad (1)$$

(C) $$\int_{\bar{M}_{0,2g+2}(\mathbb{P}^1\times B\mathbb{Z}_2,1)} e^{eq}\left(R^1\pi_* f^*(\mathcal{O}\oplus\mathcal{O}(-1))\right) = 0$$

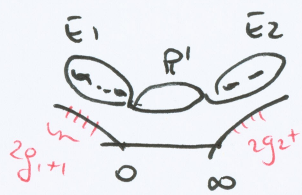
linearize bundles with weights

	0	∞
$\mathcal{O}(-1)$	-1	0
$\mathcal{O}$	1	1

Fixed loci:



$$\longrightarrow (-1)^{g+1}\int_{\bar{H}_{g,1}} \psi^{2g} = \frac{1}{2}$$



$$\longrightarrow (-1)^{g}\,\frac{2}{2}\binom{2g+2}{2g_1+1}\int_{\bar{H}_{g_1}} \psi_1^{2g_1-1}\int_{\bar{H}_{g_2}} \lambda_g\lambda_{g-1} + \dots + \lambda_g\psi^{g-1}$$

In g.f. form: $\frac{1}{2}\sin x \sum_0^{\infty} F_i = \frac{1}{2}(1-\cos x)$

$$\Rightarrow \boxed{\sum_0^{\infty} F_i = \tan\frac{x}{2}} \quad (2)$$

(3) Calculus exercise:

(1) + (2) $\Rightarrow$ $F = \frac{1}{2}\tan\left(\frac{x}{2}\right)$ □

§5. Crepant Resolution Conjecture á la Bryan-Graber

has a $K_{\mathcal{X}}$ line bdle — $I: IB\mathcal{X} \to IB\mathcal{X}$ preserves age

$\mathcal{X}$ a Gorenstein orbifold satisfying Hard Lefschetz and admitting a crepant resolution Y

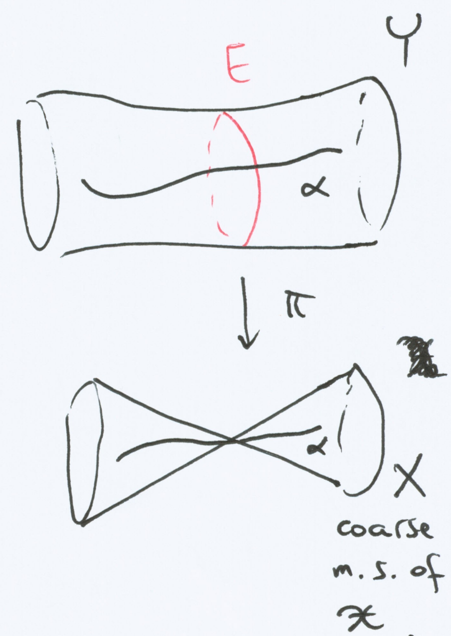
$\pi^* K_X = K_Y$

Then there is a graded linear iso

$$L : H^*(Y) \longrightarrow H^*_{CR}(\mathcal{X})$$

such that:

(1) L behaves naturally w.r.t. non exceptional classes



(2) There is an analytic continuation of F^Y from $q_{exc}=0$ to $q_{exc}=\vec{c}$ ($\vec{c}$ a vector of roots of 1) such that:

(3) $F^{\mathcal{X}} = F^Y$ after substituting $y = L(x)$ and specializing $q_{exc} = \vec{c}$

Example: the pair $[\mathbb{C}^2/\mathbb{Z}_2]$, $\mathcal{O}_{\mathbb{P}^1}(-2)$

(A) The GW theory of $\mathcal{X}$

- $\mathcal{X}$ has no curve classes $\Rightarrow$ only $\beta = 0$
- $I\mathcal{X} = \mathbb{C}^2/\mathbb{Z}_2 \amalg B\mathbb{Z}_2$

$$\begin{array}{cc} \mathbb{C}^2/\mathbb{Z}_2 & B\mathbb{Z}_2 \\ \text{age } 0 & \text{age } 1 \\ \Psi & \Psi \\ 1 & -1 \end{array}$$

$$\langle (-1)^n \rangle_0 = \int_{[\bar{M}_{0,n}([\mathbb{C}^2/\mathbb{Z}_2],0)]^{vir}} \prod ev_i^*(-1) = \int_{[M_{0,n}(B\mathbb{Z}_2,0)]} \prod ev_i^*(-1)\cdot e(Ob) \overset{n=2g+2}{=} -2\int_{H_g} \lambda_g \lambda_{g-1}$$

$$\boxed{F^{\mathcal{X}} = \begin{Bmatrix} \text{3-pt} \\ \text{inv's} \end{Bmatrix} - \iiint \tan\left(\frac{x}{2}\right)}$$

B) GW theory of $Y = \mathcal{O}_{\mathbb{P}^1}(-2)$

- Only 1 curve class (0-section)
- $H^*(Y) = H^0 \oplus H^2$, with $1 \in H^0$ and $pt \in H^2$

DIVISOR EQ'N

$$F^{Y} = \begin{Bmatrix} \text{3-pt} \\ \text{inv'ts} \\ \text{of deg } 0 \end{Bmatrix} + \sum_{d>0} \langle pt^n \rangle_d \frac{y^n}{n!} = \left\{ \quad '' \quad \right\} + \sum_{d>0} \langle \phi \rangle_d e^{dy}$$

$\langle \phi \rangle_d$ - can be computed via localization (Aspinwall-Morrison formula)

$$\langle \phi \rangle_d = \frac{1}{d^3}$$

$$\boxed{F^{Y} = \begin{Bmatrix} \text{3-pt} \\ d=0 \end{Bmatrix} + \sum_{d>0} \frac{1}{d^3} e^{dy}}$$

C) Change of variables

Set $y = ix$

$q = -1$

Calculus exercise:

$$\boxed{\left(\frac{d}{dx}\right)^3 F^{Y}(ix, q=-1) = \left(\frac{d}{dx}\right)^3 F^{\mathcal{X}}(x)}$$