Where Parallel Lines Meet

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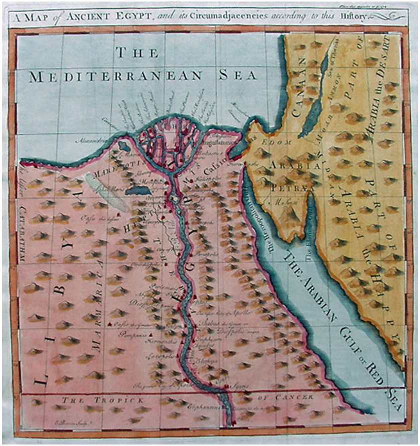
Oct 22, 2014

Geometry

$$\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$$

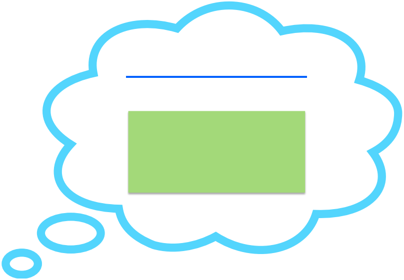
"Geos = earth" + "Metron = measure"

Was developed by King Sesostris as a way to counter tax fraud!



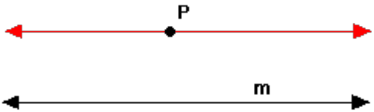
Abstraction

In order to study in broader generality the properties of spaces, mathematicians created concepts that abstract and generalize our physical experience.
For example, a line or a plane do not really exist in our world!



Euclid's Posulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Given a line m and a point P not belonging to m, there is a unique line through P that never intersects m.



The parallel postulate

The parallel postulate was first shaken by the following puzzle:

A hunter gets off her pickup and walks south for one mile. Then she makes a 90° *left turn and walks east for another mile. At this point she sees a bear and shoots it. She loads the bear on her back, makes another* 90° *left turn and walks north another mile and gets back to her pickup.*

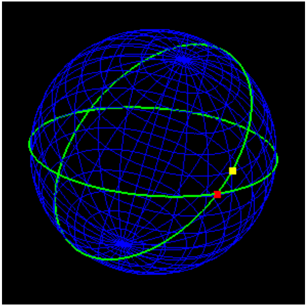
What color was the bear?

Spherical geometry

In spherical geometry, lines are great circles.

The first four postulates of Euclid can be shown to hold, but the fifth fails.

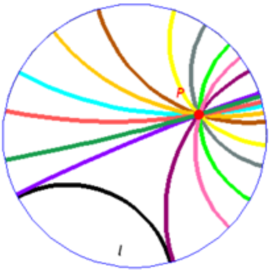
any two great circles intersect in two points.



Hyperbolic geometry

In the Poincaré model of the hyperbolic plane, lines are

- straight lines through the center of the disk.
- circles which intersect the boundary of the disk at 90° angles.

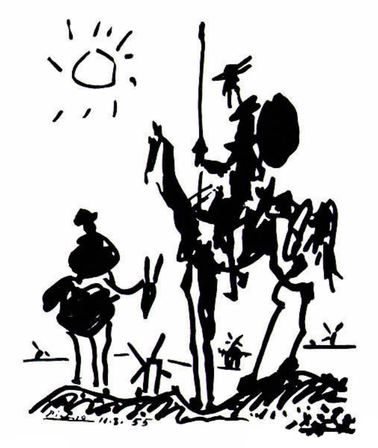


Again, the first four postulates of Euclid can be shown to hold, but the fifth fails.

There are infinitely many lines through a point P outside l which do not intersect l.

A don Quixotian idea

Could parallel lines exist, and intersect?

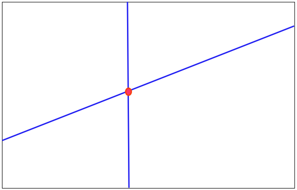


A simple question with an annoying answer

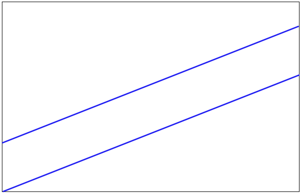
Q: *How many points of intersection do two distinct lines in the plane have?*

A: *well it depends...*

typically one



sometimes none



We artificially make the plane just a little larger by adding one line (*m*-line) and one point (Mr. V). Then we declare:

- any line with slope α contains the point $m = \alpha$ in the *m*-line.
- any vertical line contains Mr. V.

Now the answer to the previous question is always 1.

This enlarged version of the plane is called the Projective Plane, the *m*-line and Mr. V together are called the line at infinity.

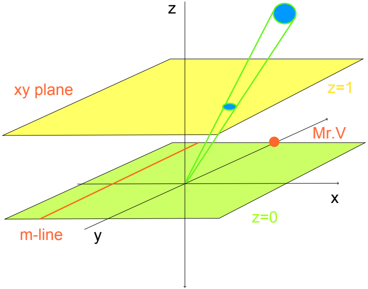
Raffaello Sanzio
Sposalizio della Vergine
(1504)

Imagine you are a painter. You observe the three-dimensional world through your eyes and then reproduce it on a two-dimensional canvas.

Mathematically, you are drawing a straight line between the point you are observing and the eye, and marking the point of intersection with the canvas.

The plane is the canvas and the projective plane is the space of rays of light going to the painters eye. The line at infinity is the set of rays of light that are parallel to the canvas.

The modern formulation



The points of the projective plane correspond to lines through the origin in three-dimensional space.

$$\mathbb{P}^2 = \{< v >_{\mathbb{R}} | v \in \mathbb{R}^3 \setminus (0,0,0)\}$$

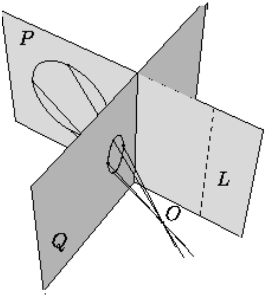
- The usual xy plane is identified with the plane $Z = 1$.
- The m-line with $X = 1, Z = 0$.
- Mr. V is the point $(0:1:0)$.

The "canvas"

If the points of the projective plane are lines through the origin in $\mathbb{R}^3$, there is nothing special about the plane $Z = 1$... it is just one possibility of where to position the canvas.

In fact any plane not through the origin can be chosen to be the screen on which we project our $3d$ world.

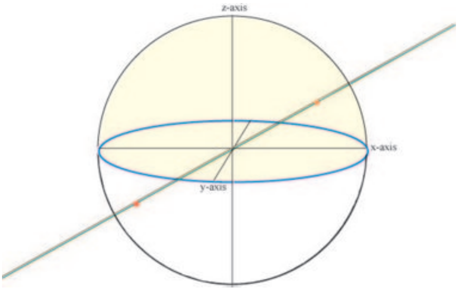
A given shape in $\mathbb{R}^3$ will yield different projections on different canvases. These different shapes are related by a projective transformation, but this is another story...



A symmetric model for $\mathbb{P}^2$

A more symmetric way to capture lines through the origin in $3d$ space is to use as a canvas a sphere centered at the origin. Now all lines are treated equally. However, each line intersects the sphere in two points, which are the same point in $\mathbb{P}^2$.

So we can think of the projective plane as being obtained from a sphere by gluing together all pairs of antipodal points.



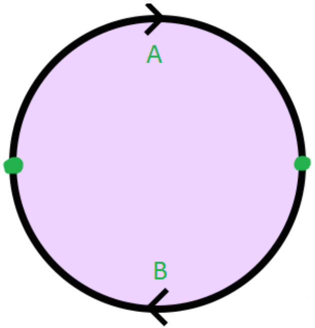
The bigon model for $\mathbb{P}^2$

A hemispherical canvas already captures all lines through the origin in $\mathbb{R}^3$. And pretty efficiently:

1. non-horizontal lines intersect the northern hemisphere in exactly one point.
2. horizontal lines intersect the equator in two opposite points.

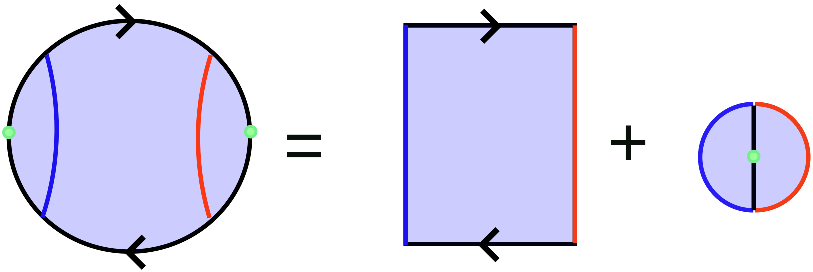
Therefore the projective plane is obtained from a disk by identifying pairs of opposite points on its boundary.

This is kind of like a grocery bag with a very weird zipper!



$\mathbb{P}^2$ and the Moebius band

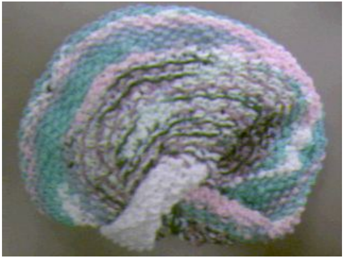
From the bigon model of the projective plane, with some scissors and glue one can see that the projective plane can also be obtained by gluing a disk and a Moebius strip along their boundaries.



$\mathbb{P}^2$ in $3d$

Since the projective plane contains a Moebius strip, it is a non-orientable surface, and a serious theorem says that it cannot exist inside $3d$ space.

In order to visualize $\mathbb{P}^2$ in $3d$, we have to allow it to go through itself. The results are cool and confusing at the same time:



Homogeneous coordinates

Each line through $(0,0,0)$ in $\mathbb{R}^3$ is identified by the coordinates (X, Y, Z) of any point on the line.

This means that each point of $\mathbb{P}^2$ gets a whole class of coordinates, all subject to the following relation:

$$p = [l] = (X, Y, Z) \sim (\lambda X, \lambda Y, \lambda Z)$$

These are called homogeneous coordinates and denoted $(X : Y : Z)$.

In simple terms, all this means is that

$$p = (1:1:1) = (2:2:2) = (\pi : \pi : \pi)$$

are different names for the same point of $\mathbb{P}^2$.

Intersecting two parallel lines in $\mathbb{P}^2$

Given two lines in the usual xy plane, we can:

1. put them in $\mathbb{P}^2$. Their equations are linear, homogeneous.
2. intersect them by solving a homogenous linear system.

Example:

$$y - 3x + 2 = 0 \qquad \text{and} \qquad 2y - 6x + 9 = 0$$

become

$$Y - 3X + 2Z = 0 \qquad \text{and} \qquad 2Y - 6X + 9Z = 0$$

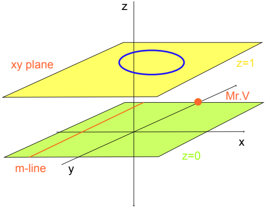
The system has solution:

$$(X : Y : Z) = (1 : 3 : 0),$$

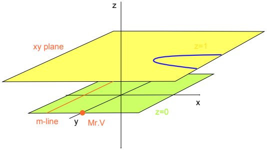
which is indeed one point on the line at infinity.

Conics

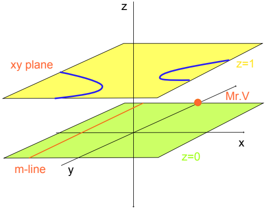
There are three types of conics in the plane:



ELLIPSE



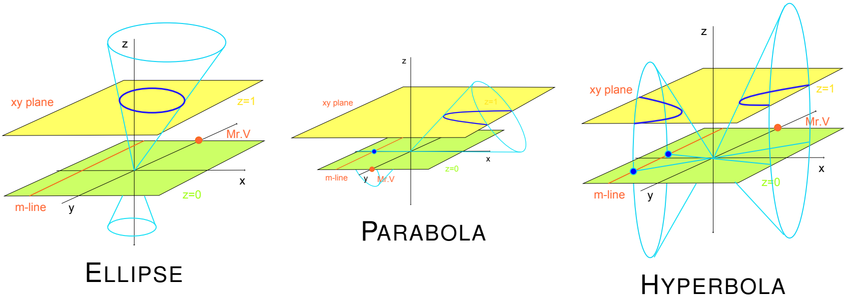
PARABOLA



HYPERBOLA

...but in the Projective Plane...

...they are all sections of a cone!



...according to how the cone is positioned with respect to the canvas, it intersects differently the line at infinity!

Simple!

1. Put it inside the projective plane.
2. Intersect it with the line at infinity!

Example:

$$34x^2 + 17xy + 3x + 457y - \pi = 0$$

1. $$34X^2 + 17XY + 3XZ + 457YZ - \pi Z^2 = 0$$
2. set $Z = 0$:
$$34X^2 + 17XY = 0$$

This equation has solutions $(0:1:0)$ and $(1:-2:0)$. The conic intersects the line at infinity in two distinct points, and therefore it is a hyperbola!

The end

Thankyou!