Fock Spaces and Tropical Curve Counting

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Lecture Notes

Tropical Geometry and Moduli Spaces

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Introduction

The most immediate goal of enumerative geometry of curves is to count the number of curves that satisfy a set of geometric constrains: living in, or mapping to a particular space, passing through a set of subspaces, etc... But one could argue that it is perhaps even more exciting to explore the algebraic structures and connections to other branches of mathematics that arise from such enumerative questions.

In this minicourse, I would like to explore a point of contact of three different approaches to curve counting:

(1) degeneration techniques;
(2) tropical geometry;
(3) Fock space operator techniques.

The philosophy of degeneration techniques is the following: if your goal is to count *complicated* curves (high genus, high degree, whatever it may mean...), try to *break them* into a bunch of *simpler* curves, thus transferring the geometric complexity of the problem into a combinatorial one.

Here is an intuitive description of the **degeneration formula** of [**LR01, Li01, Li02**]: start with an enumerative geometric problem that aims at counting the number of curves mapping to a given (smooth) space X, subject to a series of geometric constrains (passage through points, subvarieties, etc). Continuosly deform X so that it breaks to a union $X_1 \cup_D X_2$ of two smooth spaces attached at a smooth divisor, and make sure that the geometric constrains as well *follow* the deformation to become, in the limit, constrains in the two pieces. Then the intuitive idea is that any curve satisfying the original constrains in X should also follow the deformation, and break to a pair of curves in each of the pieces and satisfying the limit constrains. The content of the degeneration formula is then a precise relation between the original curve count, and counts of curves on X_1 and X_2 satisfying the limit constrains, plus an additional set of compatibility conditions that control the contacts of the curves with the divisor D (these are called **relative conditions**). One typically attempts to *tune* this degeneration in such a way that the discrete invariants controlling the new enumerative problems on X_1 and X_2 are simpler than the original ones, thus giving rise to a recursive approach to the original counting problem.
Quick tangent: in the original degeneration formulas one is able to work only with degenerations where the singuar locus of the degenerated space is smooth. A substantial amount of work has gone into extending the reach of the degeneration formulas to where D may have mild singularities; **logarithmic geometry** has proven a very successful technique for this goal, and the companion mini-course of Dan Abramovich will be exploring this story.

It has long been observed that tropical geometry encodes the combinatorial information of the degeneration formula. In oversimplifying terms, each irreducible component of a degenerated curve corresponds to a vertex of a tropical curve, and each node to an edge. Tropical multiplicities (which sometimes are purely combinatorial, sometimes are forced to contain some geometric *seed* information) are made to coincide with the multiplicities coming from the degeneration formula.

Logarithmic geometry is one of the ways to explain this correspondence, and I will say no more here. Another way to make this correspondence precise is the following: enumerative curve counts that may be degenerated in a *totally ordered way* (whatever this means) can often be seen as arising from products of certain linear operators on the Fock space. Feynman diagram techniques may be used to evaluate quantities arising this way, and one then shows that such Feynman diagrams may be identified with tropical curves for a *tropicalized* counting problem.

Translating a curve counting problem into a question of operators on the Fock space gives access to the structure of generating functions for enumerative invariants, making connections with the theory of integrable hierarchies. Consider the previous sentence as a roadsign for a fork in the road that would lead to a whole different journey. Let me just mention a couple results that still remain fairly close to our story: in [**Joh15**], piecewise polynomiality for double Hurwitz numbers and wall crossing formulas are obtained from commutation properties of operators in the Fock space. The recent article [**HKL17**] generalizes these results to monotone and strictly monotone double Hurwitz numbers.

Fock space techniques arose in physics, and were imported into enumerative geometry by Andrei Okounkov and Rahul Pandharipande ([**OP06**] and a cluster of other works surrounding this one). Part of the reason these techniques are perceived as hard to access is that a lot of the nomenclature comes from physics, and therefore sounds obscure. Another part is the fact that the Fock space can be presented in two very different, but equivalent ways, as a representation of the Heisenberg algebra. The first lecture is dedicated to giving some basic notions about Fock spaces, hopefully in a way that will help get some intuition about these mathematical objects and techniques, and wade through the existing literature on the subject.

Cautions. It became apparent as I compiled these notes that they are at the same time too long and too short. Too long for me to be able to speak about all the material in three hours. And too short to do justice to many of the topics that are touched upon. In my lectures I will choose a beeline through this material, which may very well change if the audience steers me in one particular direction rather than another. However I hope these notes contain a fairly cohesive and still intuitive skeleton illustrating this circle of ideas, and plenty of references for the reader interested in more precise information.

Acknowledgements. Many thanks to Paul Johnson, Hannah Markwig and Dhruv Ranganathan, my wonderful collaborators that have shared this journey with me and made it into a fun (albeit strenuous) mountain hike.

LECTURE 1

Fock Spaces

The class algebra of the symmetric group S_n has two bases both indexed by partitions of n: one basis is easy to understand, the other is easy to work with. The change of basis matrix is given by the character table of the symmetric group. The theory of Fock spaces, philosophically, is attempting to study the class algebras of all symmetric groups at the same time. There are two types of Fock spaces, both naturally endowed with a basis indexed by partitions, with the two bases related by characters of symmetric groups. The two Fock spaces are proven to be isomorphic as representations of the Heisenberg algebra. In this lecture we highlight the key players in this story, with a view towards the applications to curve counting that will be shown in the second and third lectures. More comprehensive introductions can be found in [**Car09, RoZ16**].

1. The Heisenberg Algebra

The Heisenberg algebra $\mathcal{H}$ is a Lie algebra generated by a basis $a_n, n \in \mathbb{Z}$, and c satisfying commutator relations

$$[a_n, a_m] = (n \cdot \delta_{n,-m})c, \quad [c, a_n] = 0 \tag{1}$$

where $\delta_{n,-m}$ is the Kronecker symbol. Note that c, a_0 are central elements in the Heisenberg algebra. For a quick introduction to its origins, and some basic facts about its representation theory, I recommend (and deny I ever did) to give a look at the Wikipedia page [**Wik**].

2. The Bosonic Fock Space

The **bosonic Fock space** $\mathfrak{b}\mathfrak{F}$, as a vector space, is given by a countable copies of a polynomial ring in countably many variables:

$$\mathfrak{b}\mathfrak{F} := \mathbb{C}[p_1, p_2, \ldots; z, \frac{1}{z}] = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}[p_1, p_2, \ldots] \cdot z^m$$

For a monomial $x^I z^m$, the integer m is called the **charge** of the monomial. $\mathfrak{b}\mathfrak{F} = \oplus_{m \in \mathbb{Z}} \mathfrak{b}\mathfrak{F}_m$ is decomposed it its charge m subspaces, and each $\mathfrak{b}\mathfrak{F}_m$ has a basis $\hat{p}^\mu$ indexed by partitions, where

$$\hat{p}^\mu = \left(\prod \mu_i\right) p_{\mu_1} \ldots p_{\mu_r} z^m.$$

The element $z^m \in \mathfrak{b}\mathfrak{F}_m$ is called the charge m **vacuum vector**. It is often denoted v_ϕ rather than $\hat{p}^\phi$.

We define an action of the Heisenberg algebra on $\mathfrak{b}\mathfrak{F}_m$ by the following identifications:

$$a_n = \begin{cases} -np_{|n|}\cdot & n<0 \\ z\dfrac{\partial}{\partial z} & n=0 \\ \dfrac{\partial}{\partial p_n} & n>0 \end{cases}, \qquad c = Id.$$

For any fixed charge k, an alternative way to describe $\mathfrak{b}\mathfrak{F}_k$ is to view it as *generated* by the action of $\mathcal{H}$. Start with a single *vacuum vector*, denoted v_ϕ (which corresponds to z^m). The positive operators annihilate v_ϕ:

$$a_n \cdot v_\phi = 0$$

for $n > 0$; c acts as the identity, a_0 scales by the charge and the negative operators act freely.

The vector $\hat{p}^\mu$ is now obtained as

$$\hat{p}^\mu = a_{-\mu_1} \dots a_{-\mu_r} v_\phi.$$

One may define an inner product on $\mathfrak{b}\mathfrak{F}$ by declaring $\langle v_\phi | v_\phi \rangle = 1$ and a_n to be the adjoint of a_{-n}. We have

$$\langle \hat{p}^\mu | \hat{p}^\nu \rangle = \left(\prod \mu_i\right) |Aut(\mu)| \delta_{\mu,\nu}.$$

At this point I recommend the reader to sit in a quiet room with a pot of coffee and one of chamomille, and verify that indeed we described two isomorphic representations of the Heisenberg algebra.

3. The Fermionic Fock Space

3.1. The vector space. Let V be a countably dimensional vector space with a basis indexed by half-integers[1].

$$V = \bigoplus_{\mathbb{Z}} \mathbb{C} \cdot e_{k+\frac{1}{2}}.$$

The **Fermionic Fock space** $\Lambda^{\frac{\infty}{2}} V$, as a vector space, has a basis given by **semi-infinite wedges** [2], i.e. symbols of the form:

$$w = e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_n} \wedge \dots,$$

where $\{i_j\}$ is a strictly decreasing sequence of half-integers, which eventually has no gaps (the wedge notation suggests that we can write also non-strictly decreasing sequences, and switching the position of two vectors produces a minus sign).

The special vector

$$v_\phi = e_{-\frac{1}{2}} \wedge e_{-\frac{3}{2}} \wedge e_{-\frac{5}{2}} \wedge e_{-\frac{7}{2}} \wedge \dots$$

is called the (charge 0) **vacuum vector**.

A visually appealing and useful way to represent a basis element of the Fermionic Fock space is through a **Maya diagram**: a string of pebbles

[1] Here by half integer we mean any rational number whose fractional part is 1/2, not the more common notion of any integral multiple of 1/2.

[2] It is often the case that the Fermionc Fock Space is referred to as the semi-infinite wedge space. Here we are choosing to call semi-infinite wedges the totally decomposible tensors in the Fermionic Fock Space, as we find it convenient to have a name for the elements of this distinguished basis of the Fock space.

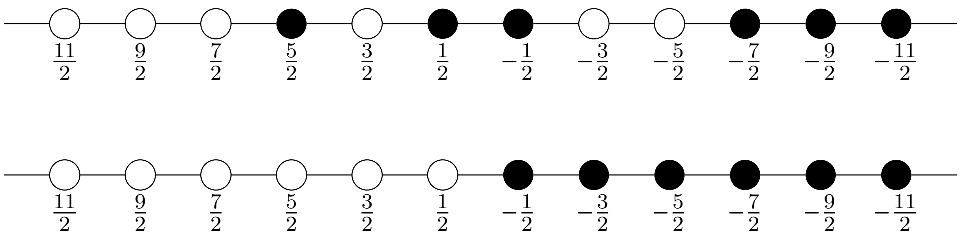


FIGURE 1. In the first line, the Maya diagram corresponding to the wedge $w = e_{\frac{5}{2}} \wedge e_{\frac{1}{2}} \wedge e_{-\frac{1}{2}} \wedge e_{-\frac{7}{2}} \wedge e_{-\frac{9}{2}} \wedge \ldots$. In the second line, the Maya diagram corresponding to the vacuum vector (of charge 0).

positioned at half integer positions on the real line. A pebble is placed in every position j where the corresponding vector e_j appears in the wedge (See Figure 1). It must be that in a Maya diagram eventually all negative positions are filled. The Maya diagram for the vacuum vector has a pebble for every negative half-integer.

To each semi-infinite wedge w can be associated an integer called **charge**. Formally this can be defined as the eigenvalue of a particular linear operator. We give here a more intuitive definition: start from the corresponding Maya diagram and move a finite amount of pebbles to obtain a configuration of pebbles with no gaps. Then the integer that separates the pebbles from the empty positions is the charge of w.

The Fermionic Fock space is decomposed by the charge:

$$\Lambda^{\frac{\infty}{2}} V = \bigoplus_{m \in \mathbb{Z}} \Lambda_m^{\frac{\infty}{2}} V,$$

where $\Lambda_m^{\frac{\infty}{2}} V$ denotes the subspace generated by semi-infinite wedges of charge m. The charge m vacuum vector corresponds to the Maya diagram that has pebbles in half-integer position less than m.
Caution: sometimes in the literature the name Fermionic Fock space is used to denote the charge 0 subspace.

Each $\Lambda_m^{\frac{\infty}{2}} V$ has a basis indexed by partitions. How to go from a partition to a Maya diagram and vice-versa is illustrated in Figure 2.

For any fixed charge m, we define an inner product on $\Lambda_m^{\frac{\infty}{2}} V$ by declaring the semi-infinite wedge basis to be orthonormal.

3.2. Operators. There are some natural linear operators acting on the Fermionic Fock space:

wedging: for a half-integer k the operator ψ_k acts on a semi-infinite wedge as follows:

$$\psi_k(w) = e_k \wedge w.$$

In terms of Maya diagrams, ψ_k is attempting to drop a pebble in position k. If the position is already filled, then the result is 0. If the position is empty, we drop the pebble and get a sign depending

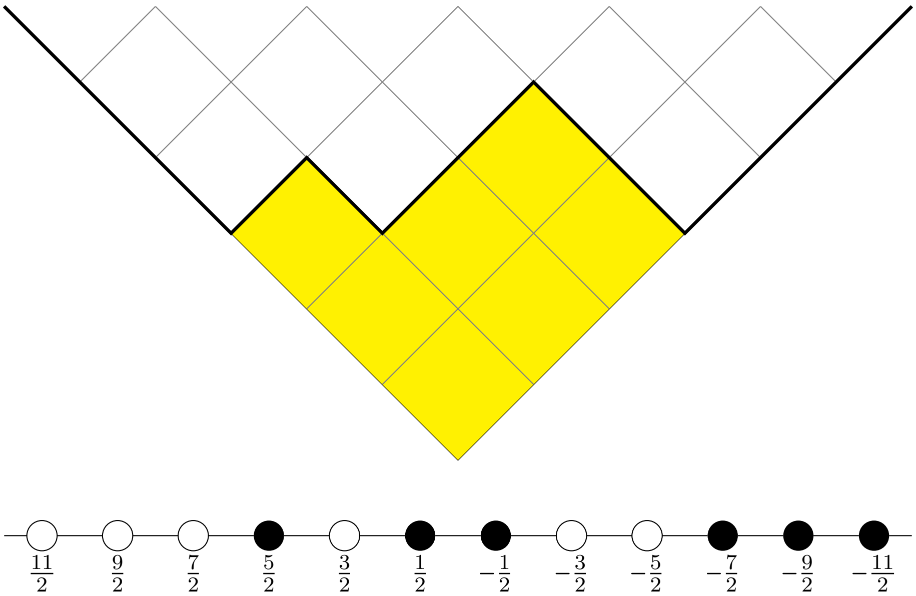


FIGURE 2. The partition corresponding to the wedge $w = e_{\frac{5}{2}} \wedge e_{\frac{1}{2}} \wedge e_{-\frac{1}{2}} \wedge e_{-\frac{7}{2}} \wedge e_{-\frac{9}{2}} \wedge \dots$. Since w has charge 0, the vertex of the tilted quadrant is above 0. We define a piecewise linear, continuous function, as follows: for each integer step above a white dot (empty space) we place a segment of slope -1, and for each integer step above a black dot (stone) we have a segment of slope 1. The partition, colored in yellow, is in this case $\lambda = (3, 2, 2)$.

on how many pebbles you had to jump over if you are coming from the positive direction.

contracting: for a half-integer k the operator ψ_k^* acts on a semi-infinite wedge by attempting to remove a pebble from position k. If there is no pebble, the result is zero. If a pebble can be removed, then the resulting Maya diagram is given a sign according to the parity of the pebbles to the left (aka more positive) then the one removed.

Observations:

- ψ_k and ψ_k^* are adjoint operators with respect to the inner product that declares semi-infinite wedges an orthonormal basis.
- The operator ψ_k adds one pebble to a Maya diagram, therefore it increases the charge of a semi-infinite wedge by one. In other words

$$\psi_k : \Lambda_m^{\frac{\infty}{2}} V \to \Lambda_{m+1}^{\frac{\infty}{2}} V.$$

- The operator ψ_k^* removes one pebble to a Maya diagram, therefore it decreases the charge of a semi-infinite wedge by one:

$$\psi_k^* : \Lambda_{m+1}^{\frac{\infty}{2}} V \to \Lambda_m^{\frac{\infty}{2}} V.$$

- It is easy to verify the following anti-commutation relations:

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}, \quad \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \quad \psi_i\psi_j + \psi_j\psi_i = 0.$$

By composing appropriately wedging and contracting operators we can create operators that act on each fixed charge subspace of the Fermionic Fock space. We first introduce a notation that simplifies bookkeeping. The **normal ordering** of a wedge-contraction product is defined to be:

$$:\psi_i\psi_j^*: \quad = \begin{cases} \psi_i\psi_j^* & j > 0 \\ -\psi_j^*\psi_i & j < 0 \end{cases}$$

The operator $\psi_i\psi_j^*$ is attempting to move a pebble from position j to position i. One part of the content of the normal ordering convention is just to put a minus sign if j is negative. The more interesting feature however happens when $i = j$: in this case for negative positions you try and first put a pebble, and then remove it; for positive positions you first remove it then put it back. Perhaps the best way to motivate this choice is to introduce the **charge operator**

$$C := \sum_{k \in \mathbb{Z}+\frac{1}{2}} :\psi_k\psi_k^*:$$

The normal ordering convention ensures that for any Maya diagram, we are performing only a finite sum of terms. It is a fun exercise to check that semi-infinite wedges are eigenvectors for the charge operator, the corresponding eigenvalue being their charge.

Another fun exercise consists in investigating how the operator $:\psi_i\psi_j^*:$ acts if you index semi-infinite wedges by partitions. You should find that it either adds or removes a strip from the partition, and introduces a sign equal to the height of the strip added/removed.

We conclude by introducing another family of operators, indexed by integers n:

$$\alpha_n := \sum_{k \in \mathbb{Z}+\frac{1}{2}} :\psi_{k-n}\psi_k^*:$$

For non-negative n, the operator α_n is taking each pebble in a Maya diagram and trying to move it n steps in the negative direction (if n is negative, then it is moving the pebble $|n|$ in the positive direction), and correcting with a sign depending on how many pebbles are jumped over in the process.

Important observations:

(1) If n is positive $\alpha_n v_\phi = 0$.
(2) $[\alpha_n, \alpha_m] = n\delta_{-n,m}$.

All α_n for positive n annihilate the vacuum vector. The content of the second line is that the α_n, together with the identity operator which we suggestively call γ, generate a Lie algebra isomorphic to a Heisenberg algebra. Hence, the Fermionic Fock space is a representation of the Heisenberg algebra.

We conclude this whirlwind tour of the Fermionic Fock space with the operator formulation of the Murnaghan-Nakayama rule, an algorithm computing the carachters of the symmetric group ([**Sta99**]). Given a partition

$\mu = (\mu_1, \ldots, \mu_r)$,

$$\alpha_{-\mu_r} \ldots \alpha_{-\mu_2}\alpha_{-\mu_1} v_\phi = \sum_\lambda \chi^\lambda(\mu) v_\lambda.$$

4. The Boson-Fermion Correspondence

The **boson-fermion** correspondence is a linear isomorphism $bf : \Lambda^{\frac{\infty}{2}} V \to \mathfrak{b}\mathfrak{F}$, which turns into an isomorphism of Heisenberg algebra representations when we identify $\alpha_n = a_n$ and $\gamma = c$. Such an isomorphism respects the charge grading, so we may describe its restriction on a fixed charge subspace

$$bf : \Lambda_m^{\frac{\infty}{2}} V \to \mathfrak{b}\mathfrak{F}_m.$$

The boson-fermion correspondence respects the vacuum vector, i.e. $bf(v_\phi) = v_\phi$.

Even though the charge m bosonic and fermionic Fock spaces both have a basis indexed by partitions, unsurprisingly the correspondence matches the two bases in a very interesting way.

From the Murnaghan-Nakayama rule, given a partition $\mu = (\mu_1, \ldots, \mu_r)$:

$$\hat{p}^\mu = \sum_\lambda \chi^\lambda(\mu) w_\lambda.$$

The inverse of this change of basis is compactly expressed as

$$w_\lambda = S_\lambda\left(\frac{p_1}{1}, \frac{p_2}{2}, \ldots, \frac{p_n}{n}, \ldots\right),$$

where the S_λ's are Schur polynomials. There are several definitions of Schur polynomials (the equation we just wrote being one of them), which give a distinguished $\mathbb{Z}$ basis of the ring of symmetric functions. Rather than getting into this story, let me just refer the reader to the standard reference [**Sta99**].

5. Vacuum Expectations, Action Movies and Feynman Diagrams

Let A be a linear operator on the Fock space $\mathfrak{F}$ (we use this notation here because we don't care whether one is looking at the bosonic or fermionic incarnation of the Fock space). Following standard conventions that come from connections to the physics literature, we write $\langle v|A|w\rangle$ for $\langle v, Aw\rangle$ for $v, w \in \mathfrak{F}$ and an operator A that is a product of elements of $\mathcal{H}$. Such expressions are referred to as **matrix elements**. We also write $\langle A\rangle$ for $\langle v_\phi|A|v_\phi\rangle$, such a value is called a **vacuum expectation**.

EXAMPLE 1. *Let us compute a simple expectation in a couple different ways. Consider the quantity:*

$$\langle a_2 a_2 a_3 a_{-2} a_{-2} a_{-3}\rangle.$$

First off, we can just compute the expectation by using the bosonic presentation of the Fock space, which is a polynomial ring in infinitely many variables. We compute the vector:

$$\begin{aligned} a_2 a_2 a_3 a_{-2} a_{-2} a_{-3} v_\phi &= 12 a_2 a_2 a_3 p_3 p_2^2 \\ &= 12 a_2 a_2 p_2^2 \\ &= 24 a_2 p_2 \\ &= 24 v_\phi. \end{aligned}$$

Then the inner product $\langle v_\phi, 24v_\phi \rangle = 24$ *gives the desired expectation.*

As simple as this computation was, this is not the most informative way to perform such expectation. We therefore repeat this simple computation using a thriller Bond style algorithm that we call...

Kill the vacuum.

The storyline is as follows. The positive operators are secret service agents whose mission is to kill the vacuum vector. In order to do so they have to bypass the protection of the bodyguards - the negative operators.

So as the mission starts the first agent, a_3 is able to pass unnoticed by the a_{-2} bodyguards (they commute), but is finally detected by the a_{-3}. After a confrontation full of special effects the action splits into two parallel universes, which in the end need to be added together: in the first universe a_3 has successfully managed to slip by a_{-3}, and is now free to approach the vacuum and accomplish his/her mission (don't assume secret service agents are all men!). In the second universe a_3 and a_{-3} kill each other, and in doing so they produce a coefficient of 3 that will remember the sacrifice of the service men of both factions.

Now, I am hoping that the cheesy dramatization will have helped this stick to your memory better, but really all that I have done is use the commutation relation (1) to substute a_3a_{-3} with $a_{-3}a_3 + 3$.

Anyway, back to our action movie. So in the universe where a_3 failed to accomplish the mission, the first of the a_2 agents leaps into action. It gets immediately detected by the first a_{-2} bodyguard and after punches, judo moves, gunshots and explosions two more scenarios arise, the first in which the agent has passed by, and the second in which the two contenders annhilated each other and produced a 2 in the process. OK, now it is gonna get complicated to keep track of possibilities, so let us call case I and case II these two possibilities.

In case I, the agent is still operative, but it is now immediately confronted by another bodyguard a_{-2}, resulting in:

- **case Ia:** agent a_2 successfully bypasses the second bodyguard, approaches the vacuum and completes the mission;
- **case Ib:** agent a_2 is stopped by the second bodyguard and a coefficient of 2 is produced in the process. We also had a factor of 3 from the action of the first agent, so at this point we have a total coefficient of 6.

Let us continue scenario Ib. At this point the last a_2 agent springs in action and there is only one bodyguard left. In one outcome the agent manages to slip by and complete the mission, and in the other all agents are gone, and the vacuum is still safely there with a coefficient of 12.

Now we go back to scenario II. Rember we are down two pairs of agents and have a coefficient of 6 floating around. The last a_2 plays his role, and in one case the mission succeeds, in the other it fails producing another coefficient of 12.

At the end of the day, the vacuum expectation is the weighted sum of the failed missions.

FIGURE 3. An illustration of the two failed missions in Example 1, each with multiplicity $2 \cdot 2 \cdot 3$.

EXERCISE 1. *Trace back through the silly screenplay and recognize all terms as arising from using the commutation relations* (1)

This way of computing the expectation seems actually way more complicated than the first one, but it has one significant advantage. There is a nice graph theoretic way to keep track of the "failed missions": draw three strands labeled with *weights* $2, 2, 3$ on the left, and $-2, -2, -3$ on the right. A failed mission corresponds to connecting each of the strands on the left with a strand on the right, in a way that the absolute values of the weights of the strands are equal. The *multiplicity* of a failed mission is then the product of the (absolute values) of the weights of the edges of the resulting graph (in this case this is a rather silly graph, as it consists of three disjoint edges, but this will remain true in more complicated situations). The expectation value is then obtained by summing over all possible graphs. See Figure 3 for an illustration.

This algorithm generalizes, let us view this through another example.

EXAMPLE 2. *Consider the vacuum expectation*

$$\langle P \rangle = \langle (a_1^4) \cdot (a_{-1}^3 \cdot a_2 \cdot a_1) \cdot (a_{-2} \cdot a_{-1} \cdot a_1^3) \cdot (a_{-1}^4) \rangle.$$

Observe that we have put parenthesis on the product of operators in such a way that the following conditions are satisfied:

- *the leftmost monomial has all positive operators (secret service agents);*
- *the rightmost monomial has all negative operators (bodyguards);*
- *all remaining monomials are* **normal ordered***, which means that the indices from left to right are not decreasing.*

In this situation, we can associate to each terms in each parenthesis a combinatorial gadget we call **Feynman fragment***. For the two external monomials this consists, as before, of just some segments labeled with the indeces of the operators. To each internal monomial we associate a star graph with edges pointing to the left labeled by the negative indices, and edges pointing to the right by the positive indices. edges that are labeled by the same indices need to be distinguishable, we may do that by decorating them with additional markings. We order all Feynman fragments on the plane in the same way as the product of monomials, as illustrated in Figure 4.*

We call a Feynman graph for P any graph that **completes** *the Feynman fragments above. By this we mean:*

1. *all germs of edges of the fragments are pairwise glued;*
2. *two edge germs that are glued have opposite labelings;*
3. *positive germs can only be glued to negative germs that are to their right.*

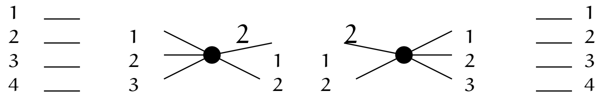


Figure 4. The Feynman fragment corresponding to the product P of Example 2. The small numbers are markings for the edge germs, the bigger ones weights. Edge weights which are one are not indicated in the picture.

Then the expectation $\langle P \rangle$ may be computed as a weighted sum over Feynman diagrams, as follows.

Theorem 1 (Wick's Theorem, see [**BG16**, Proposition 5.2]). *The vacuum expectation $\langle P \rangle$ for a product P as in Example 2 equals the weighted sum of all Feynman graphs for P, where each Feynman graph is weighted by the product of weights of all edges (interior edges and ends).*

LECTURE 2

Maps from Curves to Curves

Maps of Riemann surfaces are topological ramified covers. This hands us a natural set of discrete invariants to play with to create a family of enumerative problems:

- the genera of the source and target Riemann surfaces;
- the degree of the topological cover;
- ramification data: the number and type of ramification points (points on the surface where locally the map is not a homeomorphism);
- branch data: the position of the branch points (the images of ramification points).

An enumerative geometer won't resist fixing values for these invariants and asking how many maps of Riemann Surfaces exist satisfying them: **Hurwitz numbers** are just born! The modern perspective on such enumerative problems is to realize the answers as the degree of an intersection cycle on a moduli space. In this particular case, we have Hurwitz spaces parameterizing covers of Riemann Surfaces fixing genera, degree and ramification data, and a natural branch morphism br which records the branch data. The Hurwitz number is then naturally the degree of the cycle obtained by pulling back a generic point via the branch morphism.

In order to employ cohomological/intersection theoretic techniques, it is essential that we work with proper moduli spaces (otherwise the class of a point has a good chance of being trivial).

Gromov-Witten theory of target curves and Hurwitz theory arise from two different ways of compactifying the Hurwitz space of coverings by smooth curves. For simplicity of exposition let us assume throughout that the target is $\mathbb{P}^1$.

Hurwitz numbers arise as degrees of tautological branch morphisms from a moduli space of admissible covers to the moduli space $\overline{M}_{0,n}$. The boundary of admissible covers welcomes maps where source and target curve have degenerated simultaneously, and such that they are covering spaces away from the inverse images of the special points on the base. Over each of the n marked points on the base one gets to specify the precise ramification profile of the map above that point.

In Gromov-Witten theory the target curve is not allowed to degenerate. The source curve may become nodal, and it may have contracted components. For this reason, from a strictly geometric point of view, spaces of **stable maps** are a fairly redundant and nasty compactifications of the Hurwitz space, consisting of multiple components of different dimension. There is a whole lot of technicalities involved into cooking a decent virtual enumerative theory which we will happily shove under the rug (The friendliest introduction I can think of for the interested reader is [**HKK+03**, Chapters 22-27]). Let us just say in Gromov-Witten theory a cycle in the moduli space that parameterizes maps that over a fixed point of the base *sort of* have a prescribed ramification profile is called a **stationary descendant insertion**. Formally this cycle is:

$$ev_i^*([pt.])\psi_i^k, \tag{2}$$

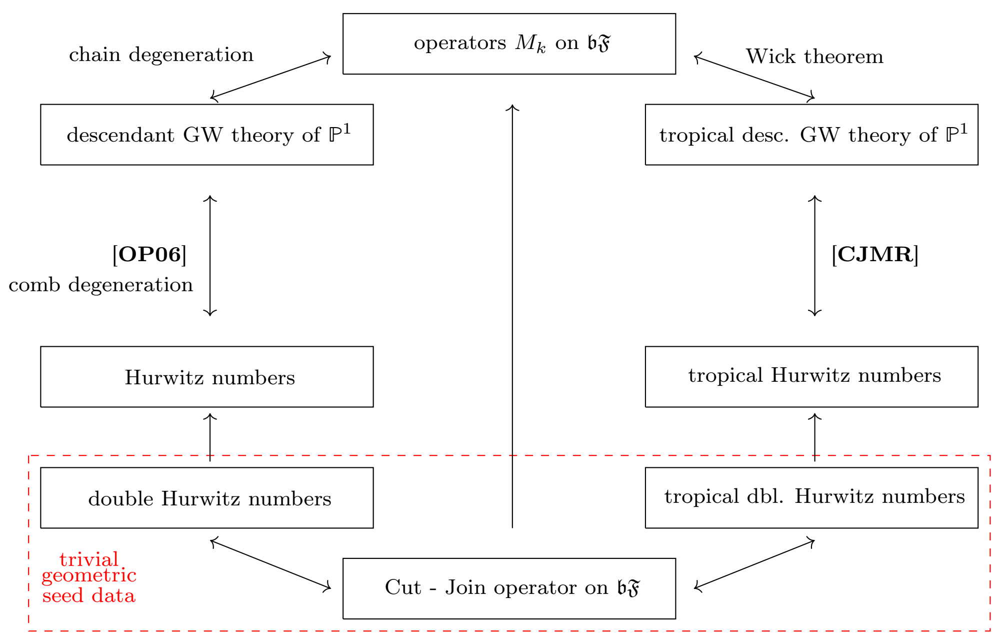
where $ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \to \mathbb{P}^1$ is defined by $ev_i([C, f, p_1, \dots, p_n]) = f(p_i)$ and ψ_i is the first Chern class of a cotangent line bundle on the space of stable maps.

The reason for the cautionary qualification is while the stationary insertion cycle (2) does indeed contain points that parameterize maps that ramify to order $k+1$ at the i-th marked point, it also contains spurious contributions by maps where the point p_i lies on some contracted component. We denote a disconnected, stationary, descendant GW invariant of $\mathbb{P}^1$ using the correlator notation,

$$\langle \tau_{k_1}(pt) \dots \tau_{k_n}(pt) \rangle_n^{\mathbb{P}^1, d, \bullet}$$

The **Gromov-Witten/Hurwitz** (GW/H) correspondence ([**OP06**]) gives a very precise description of the relation between these two counting theories. The goal of this lecture is to understand the GW/H correspondence as a consequence of the degeneration formulas. We also show a correpondence theorem between classical and tropical stationary descendant GW invariants. This arises because stationary descendants GW invariants may be expressed as matrix elements of certain operators on the Fock space, and tropical stationary descendants GW invariants are the Feynman graphs in the expansion for such operators.

A roadmap to this lecture is contained in the following diagram:



1. Hurwitz Theory

Hurwitz numbers have multiple personalities:

- they are counts of ramified covers of topological surfaces (with specified invariants);
- they count analytic functions of Riemann Surfaces;
- they count monodromy representations, i.e. certain kind of homomorphisms into the symmetric group.
- they are multiplication problems in the class algebra $\mathcal{Z}\mathbb{C}[S_d]$;
- they are expectations of operators on a Fock space;
- they are appropriately weighted counts of graphs.

Keeping in mind that in mathematics multiple personalities is a virtue and not a disorder, let us some zoom in some interesting part of this story.

Definition 1 (Geometry). *Let $(Y, p_1, \dots, p_r, q_1, \dots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus g. Let $\underline{\eta} = (\eta_1, \dots, \eta_s)$ be a vector of partitions of the integer d. We define the Hurwitz number:*

$$H^r_{h\to g,d}(\underline{\eta}) \quad := \quad \textit{weighted number of} \left\{ \begin{array}{c} \textit{degree } d \textit{ covers} \\ X \xrightarrow{f} Y \textit{ such that}: \\ \bullet\ X \textit{ is connected of genus } h; \\ \bullet\ f \textit{ is unramified over} \\ X \smallsetminus \{p_1, \dots, p_r, q_1, \dots, q_s\}; \\ \bullet\ f \textit{ ramifies with profile } \eta_i \textit{ over } q_i; \\ \bullet\ f \textit{ has simple ramification over } p_i; \\ \bullet\ \textit{preimages of each } q_i \textit{ with same} \\ \textit{ramification are distinguished by} \\ \textit{appropriate markings.} \end{array} \right\}$$

Each cover is weighted by the number of its automorphisms. Figure 1 illustrates the features of this definition.

Definition 2 (Representation Theory). *Let $(Y, p_1, \dots, p_r, q_1, \dots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus g, and $\underline{\eta} = (\eta_1, \dots, \eta_s)$*

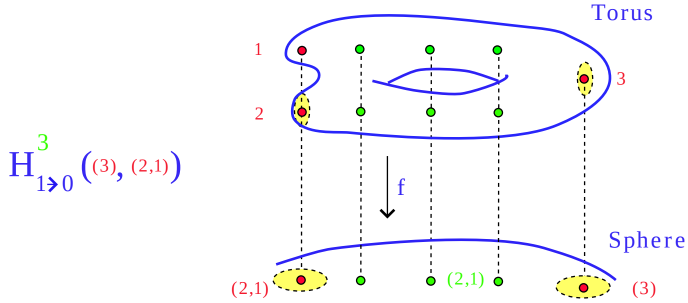


Figure 1. The covers contributing to a given Hurwitz Number.

a vector of partitions of the integer d:

$$H^r_{h\to g,d}(\underline{\eta}) := \frac{|\{\underline{\eta}\text{-monodromy representations } \varphi^{\underline{\eta}}\}|}{|S_d|}\prod |Aut(\eta_i)|, \tag{3}$$

where an $\underline{\eta}$-monodromy representation is a group homomorphism

$$\varphi^{\underline{\eta}} : \pi_1(Y - B, y_0) \to S_d$$

such that:

- *for ρ_{q_i} a little loop winding around q_i once, $\varphi^{\underline{\eta}}(\rho_{q_i})$ has cycle type η_i.*
- *for ρ_{p_i} a little loop winding around p_i once, $\varphi^{\underline{\eta}}(\rho_{p_i})$ is a transposition.*

⋆ *$Im(\varphi^{\underline{\eta}}(\rho_{q_i}))$ acts transitively on the set $\{1, \dots, d\}$.*

1.1. Tropical Hurwitz Numbers. In [**CJM10**], we defined **tropical Hurwitz numbers** as weighted sums of appropriate graphs. Our work is initially restricted to the case where the base is genus zero and we have only two points with arbitrary ramification profile, and all other ramification is requested to be simple. The analysis and the correspondence theorems are then generalized to arbitrary Hurwitz numbers in [**CMR16, BBM11**]; we mantain here the more limited context of double Hurwitz numbers to $\mathbb{P}^1$ to make notation and exposition slicker.

DEFINITION 3. *Fix g and μ, ν two partitions of a positive integer d; a graph Γ is a **monodromy graph** for this data if:*

1. *Γ is a connected, genus g, directed graph.*
2. *Γ has $\ell(\mu) + \ell(\nu)$ 1-valent vertices called **leaves;** the edges leading to them are **ends**. In-ends are labeled by the parts of the partition μ and out-ends are labeled by the parts of the partition ν.*
3. *All other vertices of Γ are 3-valent, and are called **internal vertices**. Edges that are not ends are called **internal edges**.*
4. *Γ does not have directed loops, sinks or sources.*
5. *The internal vertices are ordered compatibly with the partial ordering induced by the directions of the edges.*
6. *Every internal edge e of the graph is equipped with a **weight** $w(e) \in \mathbb{N}$. The weights satisfy the **balancing condition** at each internal vertex: the sum of all weights of incoming edges equals the sum of the weights of all outgoing edges.*

We then define **tropical Hurwitz numbers** *as:*

$$H^{trop}_g(\mu, \nu) = \sum_\Gamma \frac{1}{|Aut(\Gamma)|}\varphi_\Gamma, \tag{4}$$

where the sum is over all monodromy graphs Γ for g and μ, ν, and φ_Γ denotes the product of weights of all internal edges.

Figure 1.1 gives a couple examples of tropical double Hurwitz numbers. We then have a correspondence theorem.

THEOREM 2 ([**CJM10**]).

$$H^{trop}_g(\mu, \nu) = H_g(\mu, \nu)$$

We have at this point four different ways to understand why this theorem holds:

$H_0^{trop}((1,1),(1,1)) = 2$

$H_0^{trop}((2,1),(2,1)) = 4$

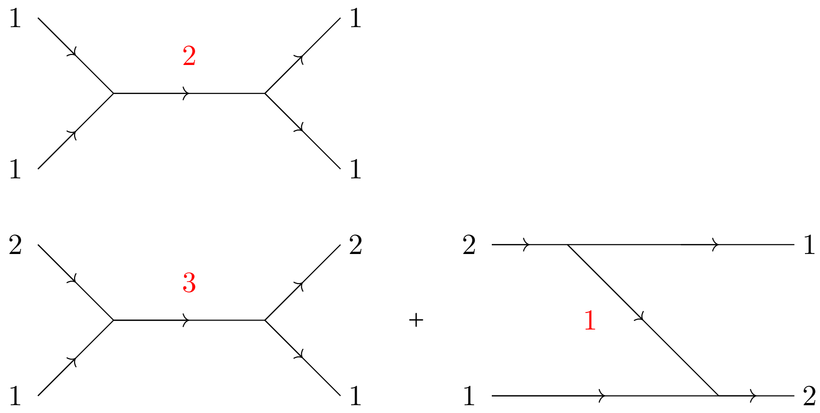


FIGURE 2. Two examples of double tropical Hurwitz numbers. In red, the weights of compact edges give us the contributions of each graph to the computation.

cut and join: the tropical Hurwitz number is a sum over graphs which parallels the classical Hurwitz count through monodromy representations. Each graph corresponds to a sequence of cycle types of all successive products of generators of the fundamental group of the base. The tropical weights just *happen* to be the multiplicities coming from the cut and join equation for multiplication by a transposition in the class algebra of S_d. This was the first proof of the theorem in [**CJM10**].

degeneration: Hurwitz numbers satisfy a degeneration formula (See Figure 3). One may compute the Hurwitz number by degenerating the target $\mathbb{P}^1$ to a chain of rational curves each containing three among branch points and nodes. Then the tropical Hurwitz covers are the dual graphs of the degenerated source curves, and the tropical multiplicities agree with the multiplicities coming from the degeneration formula.

tropicalization: the Hurwitz number is the degree of a branch morphism on appropriate moduli spaces. In [**CMR16**] we show that such branch morphism tropicalizes in the sense of [**ACP15**], and hence the degree is preserved.

Feynman diagrams: double Hurwitz numbers may be written as vacuum expectations for certain operators in the Fock space, and tropical Hurwitz numbers are the Feynman diagrams computing such expectations.

Since we are highlighting connections between enumerative geometric problems and Fock space techniques, let us expand just a bit the last point of view.

The **cut-join** operator is defined by:

$$\mathfrak{t} = \sum_{i+j+k=0} \frac{1}{6} : a_i a_j a_k : = \frac{1}{2} \sum_{k \geq 0} \sum_{\substack{0 \leq i,j \\ i+j=k}} a_{-j} a_{-i} a_k + a_{-k} a_i a_j \tag{5}$$

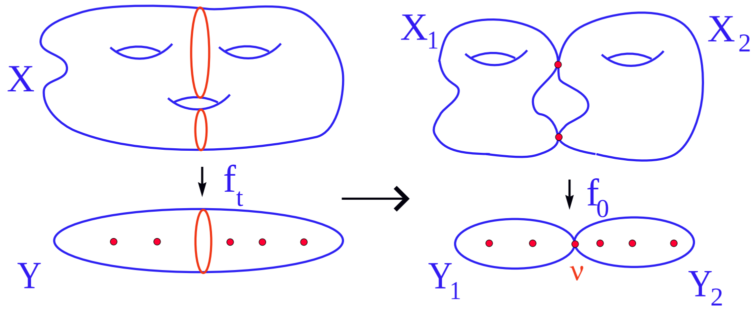


FIGURE 3. Degeneration of a cover to a nodal cover. Note that source and target degenerate simultaneously and the ramification orders on both sides of the node match.

The cut-join operator encodes the effect of multiplication by a transposition in the group algebra of S_n. Intuitively, the meaning of the cut-join operator is that multiplying by a transposition either cuts a cycle of a permutation into two cycles, or joins two cycles into one, which are the content of the first and second terms in the last part (5).

It follows that double Hurwitz numbers can be written as matrix elements of the appropriate power of the cut-join operator:

$$H^{\bullet}_{g\to 0}(\mu,\nu) = \frac{1}{\prod \mu_i}\frac{1}{\prod \nu_i}\langle \hat{p}^{\mu}|\mathfrak{t}^r|\hat{p}^{\nu}\rangle.$$

Monodromy graphs then arise as the Feynman diagrams computing these matrix elements, and the tropical multiplicities agree with the Wick theorem multiplicities.

2. The GW/H correspondence

Let $V = \oplus_{k=0}\mathbb{Q}\tau_k(pt)$ be a countably dimensional rational vector space with a basis given by the symbols $\tau_k(pt)$, and $W = \oplus_{d=0}^{\infty}\oplus_{\mu\vdash d}\mathbb{Q}\mu$ a vector space with a basis given by all partitions of all nonnegative integers (where ϕ is considered the unique partition of 0). The collection of all n-pointed stationary Gromov-Witten invariants of $\mathbb{P}^1$ defines a multilinear function $GW^{\bullet}_n : V^{\otimes n} \to \mathbb{Q}[[q]]$, defined on the elements of the natural basis of $V^{\otimes n}$ by:

$$\tau_{k_1}(pt)\otimes\ldots\otimes\tau_{k_n}(pt) \mapsto \sum_d \langle \tau_{k_1}(pt)\ldots\tau_{k_n}(pt)\rangle_n^{\mathbb{P}^1,d,\bullet} q^d, \tag{6}$$

where q is a formal variable keeping track of degree and we omit the genus subscript since the genus of the source curve is determined by the other discrete invariants.

Similarly, Hurwitz theory for target curve $\mathbb{P}^1$ with n branch points defines a multi-linear function $H^{\bullet}_n : W^{\otimes n} \to \mathbb{Q}[[q]]$:

$$\mu_1\otimes\ldots\otimes\mu_n \mapsto \sum_d H^{\bullet}_d(\mu_1,\ldots,\mu_n)q^d, \tag{7}$$

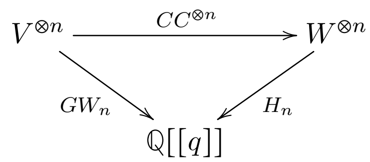
where we omit the genera subscripts since the genus of the base curve is fixed and the genus of the source curve is determined via the Riemann-Hurwitz formula. An important detail to pay attention to is that the partitions μ_i are not necessarily partitions of d, so we adopt the following conventions in order for (7) to make sense:

- $H_0^\bullet(\phi,\dots,\phi) := 1$.
- if some of the μ_i are partitions of an integer strictly greater than d, then we set $H_d^\bullet(\mu_1,\dots,\mu_n) := 0$.
- if μ is a partition of $d-k$, for $k \geq 0$, then we define $\tilde{\mu}$ to be the partition of d consisting of the parts of μ plus k parts of size 1. Moreover let j denote the number of parts of μ of size 1. Then we define:

$$H_d^\bullet(\mu_1,\dots,\mu_n) := \prod_{i=1}^{n} \binom{j_i + k_i}{k_i} H_d^\bullet(\tilde{\mu}_1,\dots,\tilde{\mu}_n) \tag{8}$$

While definition (8) may seem unnatural at first, it has an intuitive geometric interpretation: we are counting Hurwitz covers corresponding to the branching data $\tilde{\mu}_i$'s together with a choice, for each branch point p_i, of a divisor of unramified preimages of p_i of size k_i.

The GW/H correspondence of [**OP06**] may now be stated as the existence of a linear transformation $CC : V \to W$ that gives rise, for every n, to the commutative diagram:



Defining the linear transformation CC in detail requires a fair amount of work and a detour into the connection between character theory of the symmetric group and the theory of shifted symmetric functions. We refer the reader to the elegant treatment in [**OP06**, Section 0.4]) Here we point out some of the features of this transformation. For every $k \geq 0$, CC is defined by

$$\tau_k(pt) \mapsto \frac{1}{k!}\overline{(k+1)}, \tag{9}$$

where $\overline{(k+1)} = (k+1) + \sum \rho_{k+1,\mu}\mu$ is called a *completed cycle*, and it has the following characteristics:

- $\sum \rho_{k+1,\mu}\mu$ is a linear combination of partitions of integers strictly less than $k+1$.
- the *completion coefficients* $\rho_{k+1,\mu}$ with $\mu \neq \phi$ are non-negative rational numbers.
- for every integer $g \geq 0$, the *completion coefficient* $\rho_{2g-1,\phi}$ is equal to the Hodge integral $(-1)^g \int_{\overline{M}_{g,1}} \lambda_g \psi^{2g-2}$.

If the reader is unfamiliar with Hodge integrals, then the content that should be taken away from the last bullet point is that completion coefficients in degree 0 have some natural geometric meaning. This is true in fact of all completion coefficients, which are essentially connected relative 1-point Gromov-Witten invariants of $\mathbb{P}^1$ relative to $0 \in \mathbb{P}^1$.

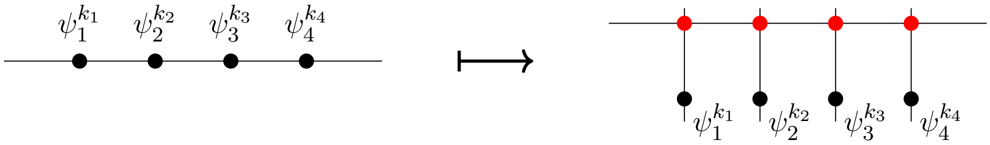


FIGURE 4. The comb degeneration: a smooth $\mathbb{P}^1$ degenerates to a a rational curve where each of the descendant insertions is on a tooth of the comb. The nodes now host relative conditions (denoted in red).

THEOREM 3 ([**OP06**, Proposition 1.6]). *The completion coefficients are:*

$$\frac{\rho_{k+1,\mu}}{k!} = \left(\prod \mu_i\right) \langle \mu | \tau_k(pt) \rangle_{n=1}^{\mathbb{P}^1,d}, \tag{10}$$

where $\mu \vdash d$ and the genus of the source curve is determined by the other discrete invariants.

The GW/H correspondence, and the geometric interpretation of the completion coefficients arise as a consequence of the *degeneration formula* [**LR01, Li01, Li02**]. The *GW/H* correspondence is obtained by degenerating $(\mathbb{P}^1, p_1, \dots, p_n)$ to a "comb" of rational curves with each of the p_i on one of the teeth of the comb, as illustrated in Figure 2. Then the invariants corresponding to the spine of the comb have only relative insertions, and give therefore Hurwitz numbers. For each tooth of the comb, a simple dimension count shows that the disconnected, descendant, one pointed invariant with one relative condition $\tilde{\mu} \vdash d$ must actually consist of a connected, descendant, one pointed invariant with one relative condition $\mu \vdash d - m$, where the parts of $\tilde{\mu}$ consists of the parts of μ plus m parts of size one.

3. Tropical descendant invariants

Tropical stationary descendant invariants are defined in a similar fashion to tropical Hurwitz numbers. The main differences in the two contexts are:

(1) Internal vertices of the graphs are not required to be rational and trivalent; in fact the descendant condition imposes a condition on the vertices.
(2) A vertex multiplicity is introduced, in the form of a connected relative stationary descendant invariant with one descendant insertion and two relative points.

Let us give a precise definition.

DEFINITION 4. *Let g be a non-negative integer, μ, ν two partitions of a positive integer d; let $k_1, \dots, k_n$ be nonnegative integers such that $\sum k_i = 2g + \ell(\mu) + \ell(\nu) - 2$. We define the tropical descendant GW invariant*

$$\langle \mu | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d,trop} = \sum_{\pi:\Gamma \to \mathbb{P}^1_{trop}} \frac{1}{|Aut(\pi)|} \prod_V m_V \prod_e m_e$$

where

(1) *The map $\pi : \Gamma \to \mathbb{P}^1_{trop} = \mathbb{R}$ is a connected tropical cover.*

(2) *The unbounded left (resp. right) pointing ends of* Γ *have expansion factors given by the partition* μ *(resp.* ν*).*
(3) *There is a unique vertex of* Γ *over each one of* n *fixed points* $p_1, \dots, p_n$ *in* $\mathbb{P}^1_{trop}$.
(4) *The unique vertex* v_i *over* p_i *has genus* g_i*, and valence* $k_i + 2 - 2g_i$.
(5) *If the star of* v_i *has right (resp. left) hand side expansion factors given by* μ_i *(resp.* ν_i*), the multiplicity of the vertex* v_i *is defined to be*

$$m_{v_i} = \int_{\overline{M}_{g_i,1}(\mathbb{P}^1,\mu_i,\nu_i)} \psi^{k_i} ev^\star(pt). \tag{11}$$

(6) *The product of the expansion factors* $\omega(e)$ *runs over the set of all bounded edges of* Γ.

One may define *disconnected* tropical descendant invariants simply by removing the requirement that the graph Γ be connected. Note however that the vertex multiplicities m_{v_i} remain connected one-pointed descendant invariants. We now state the correspondence theorem — note that an analogous statement holds in the disconnected case.

Theorem 4. *For any choice of discrete data, tropical and classical descendant invariants agree:*

$$\langle \mu | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d} = \langle \mu | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d,trop}$$

Proof. Consider a degeneration of the base curve $\mathbb{P}^1$ into a chain C of n rational curves, such that the point p_i is on the i-th component. By the degeneration formula, the classical descendant invariant $\langle \mu | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d}$ may be computed in terms of relative stable maps to C, as a sum of contributions from all possible ways of imposing relative conditions at each of the nodes of C. For each choice that gives a non-zero contribution, one may consider the dual graph Γ of a corresponding map to $f : X \to C$. Give each edge e, corresponding to a note of X mapping to a node of C, expansion factor equal to the order of ramification of the map at the node.[1] The key point is that after forgetting two valent genus zero vertices there is a bijection between such dual graphs and the tropical covers in Definition 4: by dimension reasons, over each component of C, all connected components that do not host the marked point must be rational curves mapping to the base trivially (i.e. ramifying only over the nodes). Also, dimension reasons imply that the discrete invariants for the map restricted to the component containing the marked point must satisfy (11). The proof is concluded by noting that the degeneration formula multiplicity for a given type of relative map equals the multiplicity of the corresponding tropical cover. □

4. Stationary descendant invariants and the Fock space

Stationary descendant GW invariants may be expressed as matrix element or vacuum expectations on the Fock space. Each insertion of $\tau_k(pt)$ corresponds to an operator.

[1] More precisely, we should speak about the order of ramification of the normalization of f at either of the shadows of the node of C. Such order is well defined by the *kissing condition* at nodes over relative points.

Definition 5. *For $k \in \mathbb{Z}, k > 0$, define the operator*

$$M_k = \sum_{g,d\geq 0} \sum_{(\mu,\nu)\dashv d} \langle \mu | \tau_k(pt) | \nu \rangle_{g,d}^{\mathbb{P}^1,\bullet} \cdot u^{r-1+g} \cdot a_{-\mu_1} \cdot \ldots \cdot a_{-\mu_r} \cdot a_{\nu_1} \cdot \ldots \cdot a_{\nu_s} \in \mathcal{H}[u],$$

where the second sum goes over all pairs of partitions μ, ν satisfying

$$\ell(\mu) + \ell(\nu) = k + 2 - 2g.$$

Theorem 5. *The disconnected stationary relative descendant Gromov-Witten invariants of $\mathbb{P}^1$ can be expressed as matrix elements in the Fock space as follows:*

$$\langle \mu | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | \nu \rangle_{g,|\mu|}^{\mathbb{P}^1,\bullet} = \tag{12}$$

$$\langle \hat{p}^{\mu} | \mathit{Coeff}_{u^{g+\ell(\mu)-1}}(M_{k_1} \cdot \ldots \cdot M_{k_n}) | \hat{p}^{\nu} \rangle = \tag{13}$$

$$\langle a_{\mu_1} \cdot \ldots \cdot a_{\mu_{\ell(\mu)}} \cdot \mathit{Coeff}_{u^{g+\ell(\mu)-1}}(M_{k_1} \cdot \ldots \cdot M_{k_n}) \cdot a_{-\nu_1} \cdot \ldots \cdot a_{-\nu_{\ell(\nu)}} \rangle. \tag{14}$$

A vacuum expectation as the one appearing in Equation (12) can be computed as the weighted sum over **Feynman graphs**, as we saw in Section 5. In fact the operators M_k are constructed specifically so that their Feynman graphs are the tropical stable maps contributing to tropical Gromov-Witten invariants from Definition 4. Observe in particular the color coding:

blue: each Feynman fragment has valence $k + 2 - 2g$, where g is controlled by the auxiliary variable u. This is condition 4. in Definition 4.

purple: the vertex multiplicity agrees with the multiplicity given in 5. in Definition 4.

LECTURE 3

Maps from Curves to Surfaces

The scope of this lecture is to explore the relationships between the following enumerative and combinatorial geometric theories of surfaces:

(1) decorated floor diagram counting;
(2) logarithmic and relative Gromov–Witten theory of Hirzebruch surfaces;
(3) tropical descendant Gromov–Witten theory of Hirzebruch surfaces;
(4) matrix elements of operators on a bosonic Fock space.

A map of the situation is given in the following diagram:

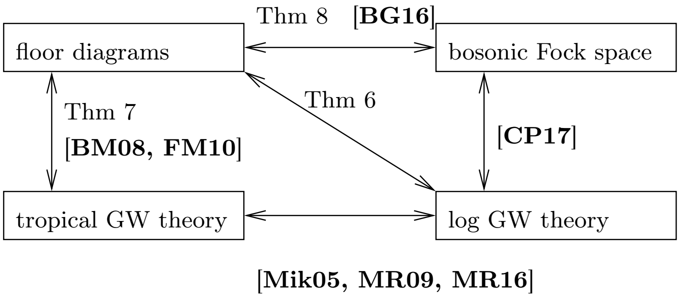


Figure 1. An overview of the content and background.

Correspondence theorems between tropical curve counts and primary Gromov–Witten invariants of surfaces – those with only point conditions and no descendant insertions – were established by Mikhalkin, Nishinou–Siebert, and Gathmann–Markwig in [**Mik05, NS06, GM07a**]; the tropical descendant invariants in genus 0 was first investigated by Markwig–Rau [**MR09**], and correspondence theorems were established independently, using different techniques, by A. Gross [**Gro15**] and by Mandel–Ruddat [**MR16**]. Tropical descendants have also arisen in aspects of the SYZ conjecture [**Gro10, Ove16**].

Cooper and Pandharipande pioneered a Fock space approach to the Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ by using degeneration techniques [**CP17**]. Block and Göttsche generalized their work to a broader class of surfaces (the h-transverse surfaces, see for instance [**AB13**] and [**BM08**]), and to refined curve counts, via quantum commutators on the Fock space side [**BG16**].

1. Counting Curves on Hirzebruch Surfaces

For $k \geq 0$, the **Hirzebruch surface** $\mathbb{F}_k$ is defined to be total space of the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$; it is a smooth projective toric surface, whose fan is depticted in Figure 1 The rays of its fan Σ_k are generated by the four vectors $e_1, \pm e_2, -e_1 + ke_2$. The 2-dimensional cones are spanned by the consecutive rays in the natural counterclockwise ordering. The zero section B, the infinity section E, and the fiber F have intersections

$$B^2 = k, \quad E^2 = -k, \quad BF = EF = 1, \quad \text{and } F^2 = BE = 0.$$

The Picard group of $\mathbb{F}_k$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ generated by the classes of B and F. In particular, we have $E = B - kF$. A curve in $\mathbb{F}_k$ has **bidegree** (a, b) if its class is $aB + bF$. The polygon depicted in Figure 1 defines $\mathbb{F}_k$ as a projective toric surface polarized by an (a, b) curve.

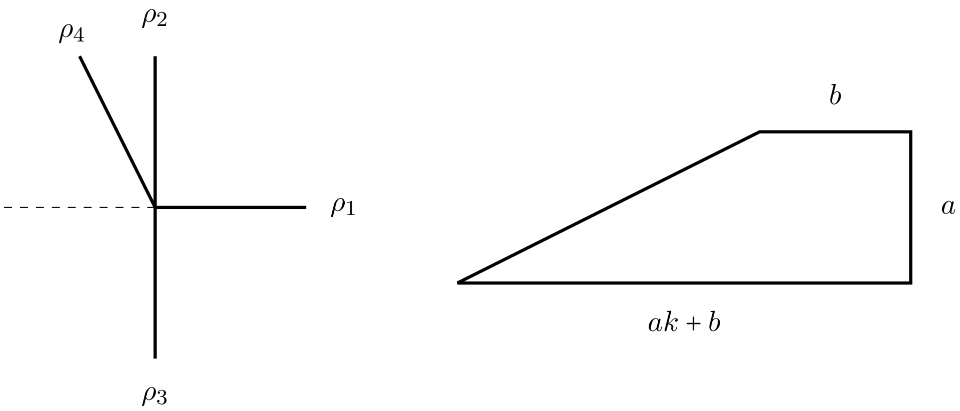


FIGURE 2. On the left, the fan of the Hirzebruch surface $\mathbb{F}_k$. On the right, the polygon defining it as a projective surface with hyperplane section the class of a curve of bidegree (a, b). The horizontal sides correspond to the sections B (down) and E (up), while the other two sides correspond to the torus invariant fibers; the labels indicate the lengths of the sides of the polytope as well as the intersection numbers with a curve of class (a, b).

We state an algebro-geometric and a tropical enumerative problems related to counting curves in Hirzebruch surfaces, and discuss how they are related to each other. Both enumerative problems will depend on the following **discrete data**.

NOTATION 1. *(Discrete data) Fix a Hirzebruch surface $\mathbb{F}_k$. We further fix:*

- *A positive integer n;*
- *Non-negative integers $g, a, k_1, \dots, k_n, n_1, n_2$;*
- *A vector $\underline{\phi} = (\varphi_1, \dots, \varphi_{n_1}) \in (\mathbb{Z} \smallsetminus 0)^{n_1}$;*
- *A vector $\underline{\mu} = (\mu_1, \dots, \mu_{n_2}) \in (\mathbb{Z} \smallsetminus 0)^{n_2}$.*
- *We assume that $\underline{\phi}$ and $\underline{\mu}$ are non-decreasing sequences.*
- *We denote by $(\underline{\phi}^+, \underline{\mu}^+)$ the positive entries of $(\underline{\phi}, \underline{\mu})$, and by $(\underline{\phi}^-, \underline{\mu}^-)$ the negative ones.*

Further, the following two equations must be satisfied:

$$\sum_{i=1}^{n_1} \varphi_i + \sum_{i=1}^{n_2} \mu_i + ka = 0; \tag{15}$$

$$n_2 + 2a + g - 1 = n + \sum_{j=1}^{n} k_j. \tag{16}$$

It is also useful (albeit redundant) to introduce the notion of a Newton fan, which serves as a compact way to encode multiple discrete invariants in the counting problems.

Definition 6. *A **Newton fan** is a sequence $\delta = \{v_1, \dots, v_k\}$ of vectors $v_i \in \mathbb{Z}^2$ satisfying*

$$\sum_{i=1}^{k} v_i = 0.$$

*If $v_i = (v_{i1}, v_{i2})$, then the positive integer $w_i = \gcd(v_{i1}, v_{i2})$ (resp. the vector $\frac{1}{w_i} v_i$) is called the **expansion factor** (resp. the **primitive direction**) of v_i. We use the notation*

$$\delta = \{v_1^{m_1}, \dots, v_k^{m_k}\}$$

to indicate that the vector v_i appears m_i times in δ.

1.1. Algebraic curve count: logarithmic Gromov-Witten invariants. On the algebro-geometric side, the enumerative geometric problem we introduce is *stationary, descendant, logarithmic Gromov–Witten invariants of* $\mathbb{F}_k$, which morally count curves in $\mathbb{F}_k$ with prescribed tangency conditions along the boundary, and satisfying some further geometric constraints, called **descendant insertions** at a number of fixed points in the interior of the surface.

In this context, g is the genus of the curves being counted, n is the number of ordinary marked points on the curves, and the k_i are the degrees of the descendant insertions at each point. The sequences $(\underline{\phi}, \underline{\mu})$ identify a curve class in $H_2(\mathbb{F}_k, \mathbb{Z})$, as well as the required tangency with the toric boundary.

The tuple $(\underline{\phi}, \underline{\mu})$ determines the curve class

$$\beta = aB + \left(\sum_{\varphi_i \in \underline{\phi}^+} \varphi_i + \sum_{\mu_i \in \underline{\mu}^+} \mu_i \right) F. \tag{17}$$

The compatibility condition (15) ensures that β is an effective, integral curve class in $H_2(\mathbb{F}_k, \mathbb{Z})$.

We require curves to have contact orders $|\varphi_i|$ for $\varphi_i < 0$ (resp. $\varphi_i > 0$) with the zero (resp. infinity) section at fixed points, and contact orders $|\mu_i|$ for $\mu_i < 0$ (resp. $\mu_i > 0$) with the zero (resp. infinity) section at arbitrary points.

The Newton fan

$$\delta_{(\underline{\phi}, \underline{\mu})} := \{(1,0)^a, (-1,k)^a, \varphi_1 \cdot (0,1), \dots, \varphi_{n_1} \cdot (0,1), \mu_1 \cdot (0,1), \dots, \mu_{n_2} \cdot (0,1)\}. \tag{18}$$

is an equivalent way to encode at the same time the toric surface $\mathbb{F}_k$, the curve class β, and prescribed tangencies along the toric divisors.

Definition 7. *Fix discrete data as in Notation 1.*

The ***stationary descendant log Gromov–Witten*** *invariant counts curves subject to the geometric constrains explained in the previous paragraph, and is denoted*

$$\langle(\underline{\phi}^-,\underline{\mu}^-)|\tau_{k_1}(pt)\dots\tau_{k_n}(pt)|(\underline{\phi}^+,\underline{\mu}^+)\rangle_g^{\mathsf{log}} \tag{19}$$

This is only a moral definition, but it will suffice for our purposes. For an honest definition, one defines these invariants as intersection cycles on a moduli space of logarithmic stable maps to the Hirzebruch surface (See [**CJMR**] for a precise definition, [**AC14, Che14, GS13**] for a treatment of logarithmic stable maps). Condition (16) comes from equating the expected dimension of the moduli space with the codimension of the intersection cycle, and hence it is a necessary condition for Equation (19) to be non-zero.

1.2. Tropical curve count: tropical Gromov-Witten invariants. The corresponding tropical counting problem has two ingredients: the tropical objects that are enumerated, and their multiplicities.

Definition 8. *A* ***tropical stable map to*** $\mathbb{F}_k^{\mathsf{trop}}$ *is a tuple* (Γ, f) *where* Γ *is a marked abstract tropical curve and* $f:\Gamma\to\mathbb{R}^2$ *is a piecewise integer-affine map satisfying:*

- *On each edge* e *of* Γ*,* f *is of the form*

$$t\mapsto a+t\cdot v \text{ with } v\in\mathbb{Z}^2,$$

  where we parametrize e *as an interval of size the length* $l(e)$ *of* e*.*
- *The* ***balancing condition*** *holds at every vertex, i.e.*

$$\sum_{e\in\partial V} v(V,e)=0.$$

- *the infinite ends give a Newton fan dual to the polygons of Figure 1. Furthermore, we require the horizontal and diagonal ends to be non-marked and of expansion factor* 1*. The vertical ends can have any expansion factor, and are marked.*

Some technical notions that are crucially delicate and important are the notions of **superabundance** and **rigidity**. Superabundance is an attribute of a combinatorial type, meaning that the cone of tropical stable maps exceeds the expected dimension $2+\#\{\text{bounded edges}\}-2b^1(\Gamma)$. The simplest examples of superabundant combinatorial types consists of maps where cycles are collapsed, however there are interesting examples of superabundant maps where all cycles are visible (see Example 3.10 in [**GM07b**]). Impose geometric constrains in such a way that you are expecting a zero-dimensional cycle of stable maps satisfying such constrains. Then a map (Γ, f) is said to be **rigid** if (Γ, f) is not contained in any positive dimensional family of tropical curves having the same combinatorial type.

Definition 9. *Fix discrete invariants as in Notation 1. Let*

$$\Delta=\delta_{(\underline{\phi},\underline{\mu})}\cup\{0^n\}$$

identify a degree for tropical stable maps.

Fix n points $p_1, \dots, p_n \in \mathbb{R}^2$ in general position, and two sets E_0 and E_∞ of pairwise distinct real numbers together with bijections $E_0 \to \{\varphi_i | \varphi_i < 0\}$ (resp. $E_\infty \to \{\varphi_i | \varphi_i > 0\}$).

*The **tropical descendant Gromov–Witten invariant***

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1) \dots \tau_{k_n}(p_n) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\mathsf{trop}}$$

is the weighted number of marked tropical stable maps (Γ, f) of degree Δ and genus g satisfying:

- **Incidence:** *For $j = 1, \dots, n$, the marked end j is contracted to the point $p_j \in \mathbb{R}^2$.*
- **Descendant insertions:** *The end j is adjacent to a vertex V in Γ of valence $\mathrm{val}(V) = k_j + 3 - g(V)$.*
- **Conditions at infinity:** *E_0 and E_∞ are the y-coordinates of ends marked by the set ϕ.*

*Each such tropical stable map is counted with a **multiplicity** $\frac{1}{\mathrm{Aut}(f)} m_{(\Gamma,f)}$, which is defined via logarithmic GW theory. It is beyond our purposes to be precise here, but we refer the interested reader to* [**CJMR**].

Correspondence results take the form

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1) \dots \tau_{k_n}(p_n) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\log} = \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1) \dots \tau_{k_n}(p_n) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\mathsf{trop}};$$

we are presenting them here in the style and degree of generality of []. Before we move on here is a quick review or related literature:

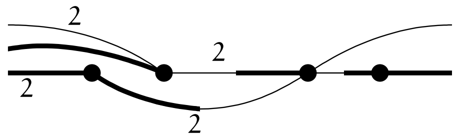
- **Primary invariants:** In [**NS06**], the authors establish a correspondence theorem for the case where there are no descendant insertions ($k_i = 0$).
- **Genus 0:** in [**MR09**], tropical, rational, stationary, descendant invariants are studied.
- **No super-abundance:** the authors of [**MR16**] prove a correspondence theorem between logarithmic and tropical invariants for all geometries that do not give rise to any superabundant tropical maps.

It is important to remark that in these cases the tropical multiplicities are reduced to being purely combinatorial.

2. Floor Diagrams

The simplest conceptual way to relate logarithmic and tropical invariants is to have both of them correspond to **floor diagrams**. Floor diagrams were introduced for counts of curves in $\mathbb{P}^2$ by Brugallé-Mikhalkin [**BM07**], and further investigated by Fomin-Mikhalkin [**FM10**], leading to new results about node polynomials. The results were generalized to other toric surfaces, including Hirzebruch surfaces, in [**AB13**]. For the sake of referencing, we start with the definition of Floor diagrams, but we encourage the reader to skip over it the first time through, and come back later to appreciate how this definition is naturally informed from two different organizations of the curve counting problem, as discussed in Sections 2.1,2.2.

DEFINITION 10. *A **floor diagram** for $\mathbb{F}_k$ of degree $(\underline{\phi}, \underline{\mu})$ consists of the following data:*



k_i	0	1	3	0
s_i	0	1	2	0

FIGURE 3. An example of a floor diagram for $\mathbb{F}_1$ of degree $((-2,1),(-2,-1,1))$ and genus 0. The genus at all vertices is 0. Strictly speaking, the figure should be rotated 90 degrees, but it has become common to draw Floor diagrams horizontally.

(1) *A loop-free connected graph D on a linearly ordered vertex set.*
(2) *Three non-negative integers assigned to each vertex V: g_V (called the **genus** of V), s_V (called the **size** of V) and k_V (called the ψ-**power** of V).*
(3) *Each flag may be decorated with a thickening. For each compact edge precisely one of its two half-edges is thickened.*
(4) *At each vertex V, $k_V + 2 - 2s_V - g_V$ adjacent half-edges are thickened.*
(5) *Each edge e comes with an **expansion factor** $w(e) \in \mathbb{N}_{>0}$.*
(6) *At each vertex V, the signed sum of expansion factors of the adjacent edges (where we use negative signs for edges pointing to the left and positive signs for edges pointing to the right) equals $-k \cdot s_V$.*
(7) *The sequence of expansion factors of non-thick ends (where we use negative signs for the ends pointing to the left and positive signs for the ends pointing to the right) is $(\underline{\phi})$, and the sequence of expansion factors of thickened ends (with the analogous sign convention) is $(\underline{\mu})$.*
(8) *The ends of the graph are marked by the parts of $(\underline{\phi}, \underline{\mu})$.*

*The **genus** of a floor diagram is defined to be the first Betti number of the graph plus the sum of the genera at all vertices.*

Floor diagrams must be counted with appropriate multiplicities. We introduce a vertex multiplicity.

DEFINITION 11. *Given a floor diagram for $\mathbb{F}_k$, let V be a vertex of genus g_V, size s_V and with ψ-power k_V. Let $(\underline{\phi}_V, \underline{\mu}_V)$ denote the expansion factors of the flags adjacent to V; the first sequence encodes the normal half edges, the second the thickened ones. We define the multiplicity $mult(V)$ of V to be the one-point stationary relative descendant invariant*

$$mult(V) = \langle (\underline{\phi}_V^-, \underline{\mu}_V^-) | \tau_{k_V}(pt) | (\underline{\phi}_V^+, \underline{\mu}_V^+) \rangle_{g_V}^{\mathsf{rel}}.$$

Finally, we have a weighted enumeration of Floor diagrams corresponding to our set of discrete invariants.

DEFINITION 12. *Fix discrete data as in Notation 1. We define:*

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\mathsf{floor}}$$

to be the weighted count of floor diagrams D for $\mathbb{F}_k$ of degree $(\underline{\phi}, \underline{\mu})$ and genus g, with n vertices with ψ-powers $k_1, \dots, k_n$, where a equals the sum of all sizes of vertices, $a = \sum_{V=1}^{n} s_V$.

Each floor diagram is counted with multiplicity

$$mult(D) = \prod_{e \in C.E.} w(e) \cdot \prod_{V} \mathsf{mult}(V).$$

2.1. Floor diagrams from degeneration. One of the most successful techniques for computing, or observing the structure of Gromov-Witten invariants is the degeneration formula of Li-Ruan, Li [**?, ?**]. Philosophically, suppose one has fixed some discrete invariants and is interested in counting curves on a smooth surface S. Let S be the generic fiber of a flat family whose central fiber consists of a collection of smooth surfaces glued along smooth divisors. The idea of the degeneration formula is that the curves being counted in the general fiber get *broken up* into curves in the various components of the central fiber. By setting things up appropriately, one can arrange the discrete invariants for the various pieces in the central fiber to be simpler than the original discrete invariants, thus giving rise to a recursive approach to the curve counting problem. We now sketch how to implement this strategy in the case of logarithmic stationary, descendant, invariants of Hirzebruch surfaces.

We wish to construct a family of surfaces such that the generic fiber is $\mathbb{F}_k$, whereas the central fiber consists of a union of n surfaces $S_1 \cup_{D_1} S_2 \cup_{D_2} \dots \cup_{D_{n-1}} S_n$; all surfaces S_i are isomorphic to $\mathbb{F}_k$, and S_i and S_{i+1} meet transversely along the divisor $D_i = E_{S_i} = B_{S_{i+1}}$. Further we wish to have n sections s_i that are general over the generic fiber, but such that $s_i(0) \in S_i$ (this is sometimes called an **accordion degeneration** of the Hirzebruch surface). Start from a trivial family $\mathbb{F}_k \times \mathbb{A}^1 \to \mathbb{A}^1$ together with a section s_1, meeting but not tangent to E at $t = 0$; blow-up $E \times \{0\}$ and consider the proper transform of the section. The exceptional divisor (which is isomorphic to the projectivization of the normal bundle to $E \times \{0\}$) is also a Hirzebruch surface $\mathbb{F}_k$ and the section s_1 specializes to a point of the exceptional divisor. We obtain the family we are after by iterating this construction $n-1$ more times.

The degeneration formula now expresses our original curve counting problem in terms of maps from nodal curves to the accordion, counted with appropriate multiplicities. Floor diagrams arise from this as follows:

- The graph D is the dual graph to the source curve of the map (where we stabilyze away 2-valent rational vertices);
- The i-th vertex is dual to the component of the source curve that hosts the i-th marked point. It is decorated with the genus of such component and the power of the descendant insertion at the i-th point; the size of the floor s_i corresponds to the coefficient of B_{S_i} in the degree of the map (restricted to the component).
- Expansion factors of edges correspond to contact orders with the sections $D_i = E_{S_i} = B_{S_{i+1}}$.
- Thick/thin ends correspond to moving/fixed relative conditions.
- The multiplicity of the floor diagram is precisely the multiplicity coming from the degeneration formula.

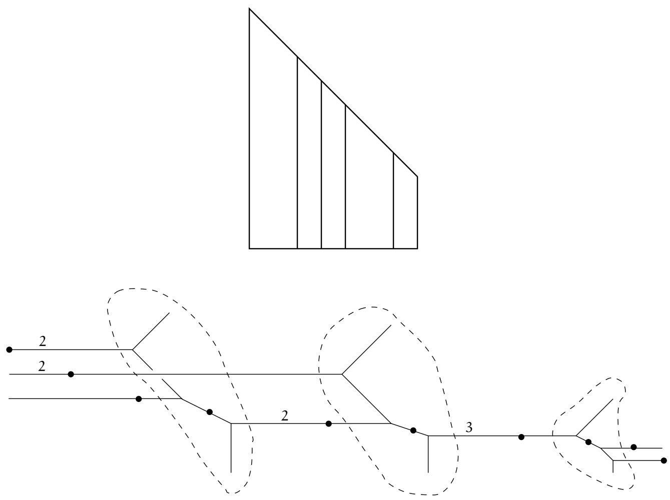


FIGURE 4. On top, the subdivision of a floor decomposed tropical curve refines the sliced Newton polygon. Each strip corresponds to a floor, and the integral width of the strip is called the size of the floor. On the bottom, a floor decomposed tropical curve with the dashed bubbles identifying the floors.

The following theorem is now just a matter of careful bookkeeping.

THEOREM 6. *Fix a Hirzebruch surface $\mathbb{F}_k$ and discrete data as in Notation 1. The descendant log Gromov–Witten invariant coincides with the weighted count of floor diagrams from Definition 12:*

(20)
$$\langle(\underline{\phi}^-,\underline{\mu}^-)|\tau_{k_1}(pt),\ldots,\tau_{k_n}(pt)|(\underline{\phi}^+,\underline{\mu}^+)\rangle_g^{\bullet,\mathsf{floor}} = \langle(\underline{\phi}^-,\underline{\mu}^-)|\tau_{k_1}(pt),\ldots,\tau_{k_n}(pt)|(\underline{\phi}^+,\underline{\mu}^+)\rangle_g^{\mathsf{rel},\bullet}.$$

2.2. Floor diagrams from tropical curves. Tropical curve counts are invariant under moving the points where incidence is required. It is then an interesting (p)art in tropical enumerative geometry to find appropriate configurations of points that allows one to actually perform the tropical curve count via a sensible algorithm. In this case, by picking *horizontally stretched point conditions*, the images of tropical stable maps contributing to a Gromov–Witten invariant become floor decomposed: this means that the dual subdivision of the Newton polygon is *sliced* (i.e. a refinement of a subdivision of the trapezoid by parallel vertical lines — see Figure 4). Floor diagrams are then obtained by shrinking each **floor** (i.e. a part of the plane tropical curve which is dual to a (Minkowski summand of a) slice in the Newton polygon) to a vertex. Each floor contains precisely one marked point. Further marked points lie on horizontal edges which connect floors, the so-called *elevators*.[1]

[1]The bizarre nomenclature makes intuitive sense if everything is rotated by 90°.

A little more precisely, we associate a floor diagram to a tropical curve arising from our stationary descendant tropical count via the following procedure. Let (Γ, f) be a non-superabundant floor decomposed tropical stable map contributing to $\langle(\underline{\phi}^-, \underline{\mu}^-)|\tau_{k_1}(pt)\dots\tau_{k_n}(pt)|(\underline{\phi}^+, \underline{\mu}^+)\rangle_g^{\mathsf{trop}}$.

We associate a floor diagram D contributing to $\langle(\underline{\phi}^-, \underline{\mu}^-)|\tau_{k_1}(pt)\dots\tau_{k_n}(pt)|(\underline{\phi}^+, \underline{\mu}^+)\rangle_g^{\mathsf{floor}}$ to (Γ, f) by contracting each floor to a vertex; also marked points adjacent to only horizontal edges are considered vertices. The vertices are equipped with:

- the ψ-power k_i of the adjacent marked point i,
- the size s_i (i.e. the width) of the dual slice of the Newton polygon for vertices corresponding to a floor; $s_i = 0$ for marked points on elevators,
- the genus g_i of the vertex adjacent to the marked end i in the tropical curve.

We thicken flags if they come from half-edges of $f(\Gamma)$ which are adjacent to a marked point.

At this point, one has a weighted bijection between tropical curves and floor diagrams contributing to our counting problems. After the appropriate amount of careful bookkeeping[2] one obtains the following theorem.

Theorem 7. *Fixing all discrete invariants as in Notation 1, the weighted count of floor diagrams equals the tropical descendant log Gromov–Witten invariant, i.e. we have*

(21)
$$\langle(\underline{\phi}^-, \underline{\mu}^-)|\tau_{k_1}(pt)\dots\tau_{k_n}(pt)|(\underline{\phi}^+, \underline{\mu}^+)\rangle_g^{\mathsf{floor}} = \langle(\underline{\phi}^-, \underline{\mu}^-)|\tau_{k_1}(pt)\dots\tau_{k_n}(pt)|(\underline{\phi}^+, \underline{\mu}^+)\rangle_g^{\mathsf{trop}}.$$

3. Matrix Elements and Feynman Diagrams

In this section we construct operators on a bosonic Fock space in such a way that:

(1) Stationary descendant logarithmic invariants may be expressed as vacuum expectations using these operators;
(2) The Feynman diagrams associated to these vacuum expectations are Floor diagrams.

As often (always?) in mathematics, what we wish for informs our constructions and definitions. Let us observe how the features of the Floor diagrams determine how to proceed.

We have two types of flags in Floor diagram, thick and thin, that get decorated by natural numbers. For this reason we introduce a Heisenberg algebra $\mathcal{H}$ presented with generators a_n, b_n for $n \in \mathbb{Z}$, satisfying the commutator relations

(22)
$$[a_n, a_m] = 0, \quad [b_n, b_m] = 0, \quad [a_n, b_m] = n \cdot \delta_{n,-m},$$

where $\delta_{n,-m}$ is the Kronecker symbol. We let $a_0 = b_0 = 0$.

The Fock space F is the vector space generated by letting the generators a_n, b_n for $n < 0$ act freely (as linear operators) on the so-called vacuum vector

[2]A delicate technical ingredient in this part of the proof is that horizontally stretched constrains rule out contributions from superabundant tropical curves.

v_ϕ. We define $a_n \cdot v_\phi = b_n \cdot v_\phi = 0$ for $n > 0$. A basis for F is indexed by a pair of partitions $\underline{\phi} = (\varphi_1, \ldots, \varphi_{n_1}), \underline{\mu} = (\mu_1, \ldots, \mu_{n_2})$:

$$v_{\underline{\phi},\underline{\mu}} = a_{-\varphi_1} \cdot \ldots \cdot a_{-\varphi_{n_1}} \cdot b_{-\mu_1} \cdot \ldots \cdot b_{-\mu_{n_2}} \cdot v_\phi. \tag{23}$$

Thick flags must attach to thin flags and vice-cersa. Further the multiplicity of a Floor diagram involves a product of all compact edge weights Therefore we define an inner product on F defined in the two-partition basis by:

$$\langle v_{\underline{\phi},\underline{\mu}} | v_{\underline{\phi}',\underline{\mu}'} \rangle = \prod \varphi_i \cdot \prod \mu_i \cdot |\mathrm{Aut}(\underline{\phi})| \cdot |\mathrm{Aut}(\underline{\mu})| \cdot \delta_{\underline{\phi},\underline{\mu}'} \cdot \delta_{\underline{\mu},\underline{\phi}'}. \tag{24}$$

The remaining conditions in the definition of Floor diagram inform the construction of the operators. Let us give a formal definition of the operator first and then analyze how various parts correspond to each definition of Floor diagram. We have an operator M_l for each power of descendant insertion at a point.

Definition 13. *Let $m \in \mathbb{N}_{>0}$, l, s and $g \in \mathbb{N}$ be given. Let $\mathbf{z} \in (\mathbb{Z} \setminus \{0\})^m$ satisfy $\sum_{i=1}^m z_i = -k \cdot s$. Denote $\underline{\mu} = (z_1, \ldots, z_{l+2-2s-g})$ and $\underline{\phi} = (z_{l+2-2s-g+1} \ldots, z_m)$, and let superscripts $\pm$ denote the subsets of positive (resp. negative) entries.*

Define

$$\hat{a}_n = \begin{cases} u a_n & \text{if } n < 0 \\ a_n & \text{if } n > 0 \end{cases} \quad \text{and} \quad \hat{b}_n = \begin{cases} u b_n & \text{if } n < 0 \\ b_n & \text{if } n > 0 \end{cases}.$$

We define the following series of operators in $\mathcal{H}[t,u]$, indexed by $l \in \mathbb{N}$:

$$M_l = \sum_{g \in \mathbb{N}} u^{g-1} \sum_{s \in \mathbb{N}} t^s \sum_{m \in \mathbb{N}_{>0}} \sum_{\mathbf{z} \in \mathbb{Z}^\mathbf{m}} \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_l(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\mathsf{rel}} \cdot$$
$$: \hat{b}_{z_1} \cdot \ldots \cdot \hat{b}_{z_{l+2-2s-g}} \cdot \hat{a}_{z_{l+2-2s-g+1}} \cdot \ldots \cdot \hat{a}_{z_m} :$$

About the color coding:

- **red:** is condition 6 in the definition of Floor diagram;
- **blue:** is condition 4 in the definition of Floor diagram;
- **purple:** encodes the vertex multiplicities for a Floor diagram.

The u variable is a bookkeeping variable that tracks the genus of the Floor diagram. The t variable keeps track of the size of each floor (and in the end, of the B-degree of the curves we are counting).

We constructed these operators precisely so that the Feynman diagram for vacuum expectations of products of these operators are Floor diagrams. We therefore obtain the following theorem that wraps up our story.

Theorem 8. *With discrete data fixed as in Notation 1, the disconnected relative descendant Gromov–Witten invariant $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\mathsf{rel},\bullet}$ equals the vacuum expectation*

$$\begin{aligned} &\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\mathsf{rel},\bullet} \\ &= \frac{1}{\prod |\mu_i|} \frac{1}{\prod |\phi_i|} \left\langle v_{\underline{\mu}^-,\underline{\phi}^-} \middle| \mathit{Coeff}_{t^a u^{g+\ell(\underline{\phi}^-)+\ell(\underline{\mu}^-)-1}} \left(\prod_{i=1}^n M_{k_i} \right) \middle| v_{\underline{\mu}^+,\underline{\phi}^+} \right\rangle, \end{aligned} \tag{25}$$

where the operators M_{k_i} are as defined in Definition 13, and for a series of operators $M \in \mathcal{H}[t,u]$ $\mathit{Coeff}_{t^a u^h}(M) \in \mathcal{H}$ denotes the $t^a u^h$-coefficient.

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