Pries: 470 Euclidean and non-Euclidean Geometry

Homework 12: Hyperbolic measurement and isometries
Due Friday April 14

Hyperbolic measurement and isometries:

1. Consider the hyperbolic triangle with sides $x = 0$, $x^2 + y^2 = 1$, and $(x + 1/2)^2 + y^2 = 1$.

   (i) Find the coordinates of its vertices.

   (ii) Find the three angles in this triangle.

   (iii) Find the area of the triangle using an integral.

   (iv) Check the hyperbolic Descartes formula.

2. Let $\triangle ABC$ have a right angle at C and sides of lengths a, b, c. The hyperbolic Pythagorean theorem says that $\cosh(c) = \cosh(a)\cosh(b)$. Use Taylor series to show that this approximates the Euclidean Pythagorean theorem when a, b, c are very small.

3. Let $P = (0, 1)$ and $Q = (0, 2)$. Let $V(x, y) = (x, y + 1)$ be vertical translation by 1. Show that $hd(P, Q) \neq hd(V(P), V(Q))$ (so V does not preserve hyperbolic distance).

4. Prove that the dilation $D_k(x, y) = (kx, ky)$ preserves the hyperbolic arclength element.

5. Let $I((x, y)) = (x/(x^2 + y^2), y/(x^2 + y^2))$ be inversion in the unit circle. Writing $z = x + iy$, show that $I(z) = 1/\overline{z}$.

6. Let $P = 2 + 4i$ and $Q = 2 + (4/3)i$. Use the isometry $f(z) = (z + 2)/(-z + 2)$ to find $f(P)$ and $f(Q)$. Check that $hd(P, Q) = hd(f(P), f(Q))$.

7. Let a, b, c, d be real numbers. If $f(z) = (az + b)/(cz + d)$ stabilizes the upper half plane, show that $ad - bc > 0$. Hint: look at the imaginary part of $f(i)$.

8. Suppose $f_1(z) = (a_1z + b_1)/(c_1z + d_1)$ and $f_2(z) = (a_2z + b_2)/(c_2z + d_2)$. Find a formula for the composition $f_2(f_1(z))$ and show it is the same as the multiplication of the two corresponding matrices.