Pries: 466 Groups, rings, and fields:
Midterm 1, October 6, 2006.

Name:________________________

This midterm has 6 short problems worth 4 points each, 3 medium problems worth 8 points each, and 1 hard problem worth 2 points. Show all your work. Explain all your answers to receive full credit.

Short problems:

1. Let $\phi : \mathbb{R}[x] \to \mathbb{C}$ be the ring homomorphism so that $\phi(x) = i$. What is $\mathrm{Ker}(\phi)$?

2. Find a unit of $\mathbb{Z}[\sqrt{3}]$ which is not a unit of $\mathbb{Z}$.

3. Find the minimal polynomial of $\alpha = \sqrt{1 + \sqrt{5}}$ over $\mathbb{Q}$.

4. Give an example of a ring R and an ideal I so that I is not principal.

5. Find a polynomial $f(x) \in \mathbb{Q}[x]$ so that $(f(x))$ is a maximal ideal of $\mathbb{Q}[x]$ but not a maximal ideal of $\mathbb{R}[x]$.

6. Explain why $\mathbb{Z}[i]/(5)$ is or is not an integral domain.

Medium problems: Try parts (b) and (c) even if you have trouble with part (a)

(I) Let $\lambda_p = e^{2\pi i/p}$ be a primitive pth root of unity.

(a) Prove that $\lambda_p + \lambda_p^2 + \cdots + \lambda_p^{p-1} = 0$.

(b) Prove that $\lambda_p \lambda_p^2 \cdots \lambda_p^{p-1} = 1$.

(II) Let $\alpha \in \mathbb{C}$ be algebraic and let $m(x) = m_{\alpha,\mathbb{Q}}(x)$ be its minimal polynomial over $\mathbb{Q}$. There is a ring homomorphism $\mathbb{Q}[x] \to \mathbb{Q}(\alpha)$ so that $\phi(x) = \alpha$.

(a) Show that $(m(x)) \subset \mathrm{Ker}(\phi)$.

(b) Show that $\mathrm{Ker}(\phi) \subset (m(x))$.

(III) Let $\alpha \in \mathbb{C}$ be an algebraic number.

(a) In 1-2 sentences, explain why $\mathbb{Q}(\alpha^2) \subset \mathbb{Q}(\alpha)$.

(b) If $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\alpha^2)$, what is a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}(\alpha^2)$?

(b) If the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is odd, prove that $\mathbb{Q}(\alpha^2) = \mathbb{Q}(\alpha)$.

Hard problem: Let K be the field $\mathbb{Q}(\zeta_p) \cap \mathbb{R}$. What is the degree $[K, \mathbb{Q}]$?