Pries: M466-Groups, Rings, and Fields

Homework 5: More on Irreducible polynomials.
Due Wednesday 9/19

Read: Wilkons 4.1-4.3, Reid 2.3-2.4

Problems:

1. Which of these polynomials are irreducible over $\mathbb{Q}$? Explain your reasoning.
   (i) $x^5 + 9x^4 + 12x^2 + 6$;
   (ii) $x^4 + x + 1$.
   (iii) $(5/2)x^5 + (9/2)x^4 + 15x^3 + (3/7)x^2 + 6x + (3/14)$.

2. If $n = wm$, divide $x^n - 1$ by $x^m - 1$ and find the quotient and remainder.

3. Let $f(x) = x^4 + 1$.
   (i) Factor $f(x)$ over $\mathbb{C}$.
   (ii) Find three quadratic fields $K = \mathbb{Q}(\sqrt{d})$ so that $f(x)$ factors over K and write down the factorizations.
   (iii) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
   (iv) Extra credit: Show that $x^4 + 1$ is reducible over $\mathbb{Z}/p$ for every prime p.

4. Let $\phi : R \to S$ be a surjective ring homomorphism. If J is an ideal of S, let $\phi^{-1}(J) = \{r \in R \mid \phi(r) \in J\}$. To shorten this problem, let me tell you that $\phi^{-1}(J)$ is an ideal of R and that $\ker(\phi) \subset I$.
   (i) If I is an ideal of R and $\ker(\phi) \subset I$, show that $\phi(I)$ is an ideal of S.
   (ii) Briefly explain why this gives a bijection between ideals of S and ideals of R that contain $\ker(\phi)$.
   (iii) If S is a field, show the only ideals of S are $J = 0$ and $J = S$.
   (iv) Let $S = R/I$ and let ϕ be reduction mod I. Using earlier parts of this problem, show that R/I is a field if and only if I is a maximal ideal.

5. Let $R = \mathbb{Z}[i]$ and $I = (1 + 3i)$.
   (i) Draw the lattice in R with intersection points at every $z \in I$.
   (ii) Find a factorization $1 + 3i = \alpha\beta$ with $\alpha, \beta \in R$ and α, β not units.
   (iii) Find two ideals J so that $I \subsetneq J \subsetneq R$.
   (iv) Find representatives of the 10 equivalence classes of $\mathbb{Z}[i]/(1 + 3i)$ that are all in the same square of the lattice.
   (v) Find two ideals $\bar{J}$ of $\mathbb{Z}[i]/(1 + 3i)$ so that $0 \subsetneq \bar{J} \subsetneq \mathbb{Z}[i]/(1 + 3i)$. Match them up with the ideals from part (iii).
   (vi) Explain why $\mathbb{Z}[i]/(1 + 3i)$ is not a field.