

Pries: M466-Groups, Rings, and Fields

Homework 3: Ideals and quotient rings.

Due Friday 9/7

Read: Gallian 14, Reid 2.2.

Throughout this assignment, R is an integral domain (a commutative ring with identity, with no zero-divisors) and I is an ideal of R .

Problems:

1. If $a \in R$, let $(a) = \{ar \mid r \in R\}$. Show that (a) is an ideal of R .
2. If $\phi : R \rightarrow S$ is a ring homomorphism, let $\ker(\phi) = \{x \in R \mid \phi(x) = 0\}$. Show that $\ker(\phi)$ is an ideal.
3. Let $R = \mathbb{R}[x]$, let $a \in \mathbb{R}$, and let $I = (x - a)$. Find a ring S and a ring homomorphism $\phi : R \rightarrow S$ so that $I = \ker(\phi)$.
4. Let $R = \mathbb{R}[x]$ and let $I = (x^2 + 1)$.
 - (i) Show that R/I is a field by explicitly finding a multiplicative inverse for a non-zero equivalence class $z + I$ (where $z = a + bx$ for some $a, b \in \mathbb{R}$).
 - (ii) Show that R/I is a field using a theorem about quotient rings. (Make sure to check that the hypotheses of the theorem are true).
 - (iii) Show that R/I is isomorphic to the field of complex numbers \mathbb{C} by finding a surjective ring homomorphism $\phi : R \rightarrow \mathbb{C}$ with kernel I .
5. Let p be prime and $\mathbb{Z}/p = \{0, 1, \dots, p-1\}$. Let $R = \mathbb{Z}/p[x]$ be the ring of polynomials with coefficients in \mathbb{Z}/p . If $f(x) \in R$, let $I = (f(x))$ be the ideal generated by $f(x)$.
 - (i) If $f(x)$ has degree d , what is the size of the quotient ring R/I ?
 - (ii) If $p = 2$ and $f(x) = x^2 + x + 1$, show that R/I is a field.
 - (iii) If $p = 3$ and $f(x) = x^2 + x + 1$, show that R/I is not a field.
6. Let $R = \mathbb{Z}[\sqrt{2}]$. Let $I = (3)$.
 - a. Describe the elements of I .
 - b. Find representatives of the equivalence classes of $\mathbb{Z}[\sqrt{2}]/(3)$. How many are there?
 - c. Find inverses for all non-zero equivalence classes of $\mathbb{Z}[\sqrt{2}]/(3)$.