Summary of Semigroup Results

$$\left.\begin{array}{c} \text{A is closed and densely defined} \\ (\lambda I + A)^{-1} \text{ exists for all } \lambda > 0 \\ \|(\lambda I + A)^{-1}\|_{L(H)} \le \lambda^{-1} \quad \textit{for all } \lambda > 0 \end{array}\right\} \quad \text{Hille-Yosida properties}$$

H-Y properties $\Longrightarrow$ $\forall\ u \in H$ and $\forall\ t \ge 0,\ S_n(t)u$ converges strongly in H to $S(t)u$.

$S(t) = \lim_n S_n(t)$ has the following properties:

$$\left.\begin{array}{c} S(t) \text{ is stronly continuous in t, } t \ge 0 \\ S(0) = I \\ S(t+s) = S(t) \circ S(s) \quad s,t \ge 0 \\ \|S(t)\|_{L(H)} \le 1 \end{array}\right\} \quad S(t) \text{ is a } C^0 - s/g \text{ of contractions}$$

Also

$$S(t)\ :\ D_A \to D_A$$

$$AS(t)u = S(t)Au \qquad \forall u \in D_A$$

$$S'(t)u = -AS(t)u = -S(t)Au \qquad \forall u \in D_A$$

$$\lim_{t\to 0} \frac{S(t) - I}{t} u = -Au \qquad \forall u \in D_A$$

$$S(t)u - u = -\int_0^t S(\tau)Au\, d\tau \qquad \forall u \in D_A$$

Then

H-Y properties $\Longrightarrow$ $-A$ generates a $C^0 - s/g$ of contractions

Finally, if $-A$ generates the $C^0 - s/g$ of contractions, $S(t),$ then $-(\lambda I + A)$ generates the semigroup, $J(t) = e^{-\lambda t}S(t)$ and:

$$u = \int_0^\infty e^{-\lambda\tau}S(\tau)(\lambda I + A)u\, d\tau \qquad \forall u \in D_A$$

$$(\lambda I + A)^{-1}v = \int_0^\infty e^{-\lambda\tau}S(\tau)v\, d\tau \qquad \forall v \in H$$

It follows from these last two results that

$-A$ generates a $C^0 - s/g$ of contractions $\Longrightarrow$ H-Y properties

We also have

$$\left.\begin{array}{c} A \text{ is accretive} \\ I + A : D_A \to H \quad is \;\; onto \end{array}\right\} \quad \text{Lumer-Phillips properties}$$

and

$$\text{L-P properties} \Longrightarrow \quad -A \text{ generates a } C^0 - s/g \text{ of contractions}$$

If $-A$ generates a $C^0 - s/g$ of contractions, $S(t)$, then

$\forall u_0 \in D_A$ and $\forall f \in C^1(0,T : H)$

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau \in C[0,T : D_A] \cap C^1(0,T : D_A)$$

solves

$$u'(t) + Au(t) = f(t), \qquad 0 < t < T \qquad and \qquad u(0) = u_0$$

Examples

1.Let $H = L_2(0,\infty) = H^0(R_+)$, $\quad A = -a\,\partial_x$, and $D_A = H^1(0,\infty)$ for $a > 0$.
Note that $u \in D_A$ implies $u(x) \to 0 \;\; as \;\; x \to \infty$. Then

$$(Au,u)_H = -a\int_0^\infty u'(x)u(x)dx = -\tfrac{a}{2}u(x)^2|_0^\infty = \tfrac{a}{2}u(0)^2 \geq 0$$

Then A is accretive for $a > 0$, and for arbitrary $v \in H$

$$(I + A)u = u(x) - au'(x) = v(x)$$

has the solution

$$u(x) = \int_x^\infty e^{\frac{1}{a}(x-y)}v(y)dy$$

This shows that $I + A$ is onto (i.e., $R(I+A) = H$). Then by the L-P theorem, $-A$ generates a $C^0 - s/g$, $S(t)$ and the unique solution of

$$u'(t) + Au(t) = \partial_t u(x,t) - a\partial_x u(x,t) = f(x,t), \qquad u(x,0) = u_0(x) \in D_A;$$

is given by,

$$u(x,t) = S(t)u_0(x) + \int_0^t S(t-\tau)f(x,\tau)d\tau = u_0(x+at) + \int_0^t f(x + a(t-\tau),\tau)d\tau$$

In order to have this solution it is sufficient to suppose $u_0 \in D_A = H^1(0,\infty)$ and f is C^1 in t and L_2 in x.

2. Let $H = L_2(0,\infty) = H^0(R_+)$, $\quad A = a\partial_x$, and $D_A = \{u \in H^1(0,\infty) : \; u(0) = 0\}$.
Note that $u \in D_A$ implies $u(x) \to 0 \;\; as \;\; x \to \infty$. Then

$$(Au,u)_H = a\int_0^\infty u'(x)u(x)dx = \tfrac{a}{2}u(x)^2|_0^\infty = 0$$

Then A is accretive (for any real a) and for arbitrary $v \in H$

$$(I+A)u = u(x) + au'(x) = v(x) \qquad for\ v \in H$$

has the solution

$$u(x) = \int_0^x e^{-\frac{1}{a}(x-y)} v(y)dy.$$

Note that it is necessary to have $a > 0$ in order to get $u \in D_A$ for $v \in H$ since a negative value for a would produce an exponentially growing $u(x)$ which is not even in H much less in D_A. This shows that $R(I+A) = H$ for $a > 0$. .Then $-A$ generates a $C^0 - s/g,\ S(t)$ and $u(t) = S(t)u_0$ is the unique solution of

$$u'(t) + Au(t) = \partial_t u(x,t) + a\partial_x u(x,t) = 0, \quad u(0,t) = 0, \qquad u(x,0) = u_0(x) \in D_A;$$

i.e,

$$u(x,t) = u_0(x-at)H(x-at) = S(t)u_0(x).$$

3. Let $H = L_2(R) = H^0(R), \qquad A = \partial_x,$ and $D_A = H^1(R).$ Note that $u \in D_A$ implies $u(x) \to 0\ \ as\ \ x^2 \to \infty$. Then

$$(Au,u)_H = \int_{-\infty}^{\infty} u'(x)u(x)dx = \tfrac{1}{2}u(x)^2|_{-\infty}^{\infty} = 0.$$

so A is accretive. Note that $-A$ is also accretive in this case. For arbitrary $v \in H$

$$(I \pm A)u = u(x) \pm u'(x) = v(x) \qquad for\ v \in H$$

implies via Fourier transformation that

$$(1 \pm i\alpha)U(\alpha) = V(\alpha)$$

Then

$$U(\alpha) = (1 \mp i\alpha)\frac{V(\alpha)}{1+\alpha^2} = W(\alpha) \mp (i\alpha)W(\alpha)$$

and

$$u(x) = w(x) \mp w'(x) \qquad w(x) = T_F^{-1}\left[\frac{V(\alpha)}{1+\alpha^2}\right] = \int_R e^{-|x-y|}v(y)dy$$

Then the ODE has the solution

$$u(x) = \int_0^x e^{-|x-y|}v(y)dy \mp \frac{d}{dx}\left(\int_0^x e^{-|x-y|}v(y)dy\right)$$

which shows that $R(I \pm A) = H$. Then $-A$ generates a $C^0 - s/g,\ S(t)$ and $u(t) = S(t)u_0$ is the unique solution of

$$u'(t) + Au(t) = \partial_t u(x,t) + \partial_x u(x,t) = 0, \quad u(x,0) = u_0(x) \in D_A;$$

i.e,

$$u(x,t) = u_0(x+t) = S(t)u_0(x).$$

But $+A$ also generates a $C^0 - s/g,\ Z(t)$ and then $u(t) = Z(t)u_0$ is the unique solution of

$$u'(t) - Au(t) = \partial_t u(x,t) - \partial_x u(x,t) = 0, \quad u(x,0) = u_0(x) \in D_A;$$

i.e,

$$u(x,t) = u_0(x-t) = Z(t)u_0(x) = S(-t)u_0(x).$$

So in this case, since both A and $-A$ are accretive and $(I \pm A)$ is onto, there are two $C^0 - s/g's$, $S(t)$ and $S(-t)$. Moreover, by the semigroup property, $S(t) \circ S(-t) = S(0) = I$, which is to say $S(t)^{-1} = S(-t)$; i.e., $S(t)$ forms a $\boldsymbol{C}^0$ ***–group*** for $t \in R$.

4. Let $H = L_2(U) = H^0(U), \quad A = -\nabla^2,$ and $D_A = H_0^1(U) \cap H^2(U).$ Then

$$(Au,u)_H = -\int_U u\,\nabla^2 u = \int_U |\nabla u|^2 \geq 0 \qquad u \in D_A$$

So A is accretive. For arbitrary $f \in H,$ the elliptic problem

$$(\lambda I + A)u = \lambda u - \nabla^2 u = f \;\; in\; U, \qquad u = 0 \; on\; \partial U$$

has a unique weak solution $u \in D_A$ since

$$b[u,v;\lambda] = \int_U \nabla u \bullet \nabla v + \lambda uv$$

satisfies

$$b[u,u;\lambda] = \|\nabla u\|_0^2 + \lambda \|u\|_0^2 \qquad u \in D_A$$

and, in particular,

$$b[u,u;1] = \|u\|_1^2$$

which implies $b[u,u;1]$ is coercive and $(I + A)$ is then an isomorphism from D_A onto H. Then $-A$ generates a $C^0 - s/g,\ S(t)$ and $u(t) = S(t)u_0$ is the unique solution of

$$\begin{aligned} & u'(t) + Au(t) = \partial_t u(x,t) - \nabla^2 u(x,t) = 0, \\ & u = 0 \;\; on \;\; \partial U \times (0,T) \\ and \quad & u(x,0) = u_0(x) \in D_A; \end{aligned}$$

i.e,

$$u(x,t) = \sum_n (u_0, w_n)_H e^{-\lambda_n t} w_n(x) = S(t)u_0(x).$$

where $\{w_n\}$ are the family of orthonormal eigenfunctions associated with

$$-\nabla^2 w(x) = \lambda w(x) \qquad x \in U, \qquad w \in D_A.$$

5.Consider the problem

$$\begin{aligned} \partial_{tt}u(x,t) - \partial_{xx}u(x,t) &= 0, \qquad 0 < x < 1,\ t > 0 \\ u(x,0) &= f(x), \qquad 0 < x < 1, \\ \partial_t u(x,0) &= g(x) \qquad t > 0, \\ u(0,t) = u(1,t) &= 0, \qquad 0 < x < 1. \end{aligned}$$

Let $\qquad u_1 = \partial_x u \quad$ and $\quad u_2 = \partial_t u$

Then $\qquad \partial_t u_1 = \partial_{xt} u = \partial_x u_2$

$\qquad\qquad \partial_t u_2 = \partial_{tt} u = \partial_x u_1$

i.e.,

$$\partial_t\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\partial_x\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(x,0) = \begin{bmatrix} f'(x) \\ g(x) \end{bmatrix}$$

i.e.,

$$\partial_t\vec{U}(t) + A\vec{U}(t) = 0, \qquad \vec{U}(0) = \vec{U}_0$$

where

$$H = L^2(0,1)^2, \qquad A = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\partial_x$$

$$D_A = \{\vec{U} \in H : u_1 \in H^1(0,1),\ \ u_2 \in H_0^1(0,1)\}$$

Then

$$(A\vec{U},\vec{U})_H = -\int_0^1(\partial_x u_2 \cdot u_1 + u_2 \cdot \partial_x u_1)dx = -\int_0^1 d/dx(u_1u_2)dx$$

$$= -(u_1u_2)|_{x=0}^{x=1} = 0 \quad \text{(since } u_2 \in H_0^1(0,1))$$

This proves A is accretive, in fact, conservative. Now for $\lambda \neq 0,\ \vec{F} \in H,$ consider

$$\lambda\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \lambda u_1 - \partial_x u_2 \\ \lambda u_2 - \partial_x u_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Then

$$\lambda\partial_x u_1 - \partial_{xx}u_2 = \partial_x F_1 \qquad \textit{and} \qquad \partial_x u_1 = \lambda u_2 - F_2,$$

or

$$-\partial_{xx}u_2 + \lambda^2 u_2 = \partial_x F_1 + \lambda F_2$$

Since $\partial_x F_1 + \lambda F_2 \in H^{-1}(0,1),$ this last equation has a unique weak solution $u_2 \in H_0^1(0,1),$ by the previously developed elliptic theory. Then

$$\lambda u_1 = F_1 + \partial_x u_2 \in L^2(0,1), \qquad \partial_x u_1 = \lambda u_2 - F_2 \in L^2(0,1),$$

so $u_1 \in H^1(0,1)$ and $\vec{U} \in D_A.$ This shows that $\lambda + A : D_A \to H$ is surjective for all $\lambda \neq 0.$

Then A generates a $C^0 - s/g,$ actually both $\pm A$ generate $C^0 - s/g's$ so A generates a group of solution operators, $G(t).$

This group $G(t),$(using a previously discovered solution to the wave equation) is seen to be given by

$$G(t)[\vec{U}(x,0)] = \begin{bmatrix} \frac{1}{2}(\tilde{f}'(x+t) + \tilde{f}'(x-t)) + \frac{1}{2}(\tilde{g}(x+t) - \tilde{g}(x-t)) \\ \frac{1}{2}(\tilde{f}'(x+t) - \tilde{f}'(x-t)) + \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) \end{bmatrix}$$

where f̃, g̃ denote the odd 2-periodic extensions of f and g.