Linear Elliptic PDE's

Elliptic partial differential equations frequently arise out of conservation statements of the form

$$\int_{\partial B} \vec{F} \cdot \vec{n}\, d\sigma = \int_B S\, dx \qquad \forall B \text{ contained in bounded open set } U \subset R^n.$$

Here $\vec{F},\ S$ denote respectively, the flux density field and source density field associated with a distribution of some quantity throughout a bounded open set $U \text{ in } R^n,$ and B denotes an arbitrary ball inside U with $\vec{n}$ the unit outward normal to ∂B, the boundary of B. The statement above then asserts that for any ball B contained in U, the flow of material through the boundary of B is exactly balanced by the internal production as presribed by the source term.

Using the divergence theorem, we can convert this expression to

$$\int_B (div \vec{F} - S)\, dx = 0 \qquad \forall B \subset\subset U,$$

which, since B is arbitrary, implies

$$div \vec{F}(x) - S(x) = 0 \qquad \forall x \in U.$$

If there is now some constituitive law which asserts that

$$\vec{F} = \underbrace{-K(x)\nabla u(x)}_{\text{diffusion}} + \underbrace{u(x)\vec{V}(x)}_{\text{convection}} \qquad \forall x \in U,$$

and if, $$S(x) = \underbrace{-c(x)\,u(x)}_{\text{leakage}} + \underbrace{f(x)}_{\text{source term}} \qquad \forall x \in U,$$

then

$$div\left(-K(x)\nabla u(x) + u(x)\vec{V}(x)\right) - (-c(x)\,u(x) + f(x)) = 0 \qquad \forall x \in U,$$

i.e., $$-div\,(K(x)\nabla u(x)) + \vec{V} \cdot \nabla u(x) + c(x)\,u(x) = f(x) \qquad \forall x \in U$$

This last equation is a typical elliptic partial differential equation. The terms which appear in the constituitive laws for flux and source density can frequently be given physical interpretations as diffusive or convective transport terms (in the flux law) and loss due to leakage (in the source equation). Then the equation can be written as

$$L[u(x)] = f(x) \qquad \forall x \in U,$$

where L can be expressed variously as

$$\begin{aligned} L[u(x)] &= -div\,(K(x)\nabla u(x)) + \vec{V} \cdot \nabla u(x) + c(x)\,u(x) \\ &= -\sum_{i,j=1}^{n} \partial_j (K_{ij}(x)\,\partial_i u(x)) + \sum_{i=1}^{n} v_i(x)\partial_i u(x) + c(x)u(x) \\ &= -\vec{\partial}^{\top}[K]\vec{\partial}u(x) + \vec{V}^{\top}\vec{\partial}u(x) + c(x)u(x). \end{aligned}$$

We place the following assumption on the coefficient $K = K(x)$,

i) if K is a scalar, then $\quad K(x) \geq k_0 \quad x \in \bar{U}$

ii) if K is a matrix (tensor), then $\sum_{i,j=1}^{n} K_{ij}(x) z_i z_j \geq k_0 |\vec{z}|^2 \quad \forall \vec{z} \in R^n, \; x \in \bar{U}.$

This is the assumption of uniform ellipticity for the operator L.

In addition to the elliptic equation, which is satisfied throughout U, it is usual to impose certain conditions on the solution values on the boundary of the domain. These boundary conditions are chosen so as to cause the resulting boundary value problem to have a unique solution. At each point on the boundary of U we will assume that one (and only one) of the following conditions holds

$$\left\{ \begin{array}{ll} u(x) = g(x) & x \in \partial U \\ \nabla u \cdot \vec{n}(x) = g(x) & x \in \partial U \\ \nabla u \cdot \vec{n}(x) + p(x)\, u(x) = g(x) & x \in \partial U \end{array} \right\} \left\{ \begin{array}{c} \textit{Dirichlet condition} \\ \textit{Neumann condition} \\ \textit{Robin condition} \end{array} \right\}$$

Here $g(x)$ is a given function defined on ∂U.

Consider the so called Dirichlet boundary value problem,

$$\begin{array}{ll} L[u(x)] = f(x) & x \in U \\ u(x) = 0 & x \in \partial U. \end{array}$$

Then $u = u(x)$ is said to be a **classical solution** of the BVP if $u \in C^2(U) \cap C(\bar{U})$. Clearly it is necessary that the given data function f satisfy $f \in C(U)$ if a classical solution is to have any possibility of existing. There are several inconvenient aspects to classical solutions:

- a classical solution may fail to exist.
  (e.g. if ∂U is not sufficiently smooth or if $f \notin C(U)$)
- approximating a classical solution may be difficult
- even if a classical solution exists, it may be easier to prove the existence of a weak solution and subsequently prove the weak solution is actually classical.

In deciding how to define a weak solution, consider the special case that $L[u] = -\nabla^2 u$. Then there are the following possibilities for definitions of a weak solution to the Dirichlet BVP:

i) an ultra regular weak solution: $u \in H^2(U) \cap H_0^1(U) \textit{ such that}$

$$\int_U (\nabla^2 u(x) + f(x))\, v(x)\, dx = 0 \quad \forall v \in H^0(U)$$

ii) a weak solution: $u \in H_0^1(U) \textit{ such that}$

$$B[u,v] = \int_U [\nabla u \cdot \nabla v - fv]\, dx = 0 \quad \forall v \in H_0^1(U)$$

iii) an ultra weak solution: $u \in H^0(U) \textit{ such that}$

$$\int_U u(x)(\nabla^2 v(x) + f(x))\, dx = 0 \quad \forall v \in H^2(U) \cap H_0^1(U)$$

In case i) it is difficult to construct a family of approximate solution spaces V_N that tend to $H^2(U) \cap H_0^1(U)$ as N tends to infinity. Similarly, in case iii) it is difficult to construct a sequence of "test function" spaces W_N which tend to $H^2(U) \cap H_0^1(U)$ as N tends to infinity. On the other hand, case ii) is a compromise in which we take $V_N = W_N$ and these spaces are finite dimensional subspaces of $C(U)$. Moreover, the approximate problem contains a (symmetric) positive definite matrix approximating the infinite dimensional operator, L.

Weak Formulation

Consider the following partial differential operator acting on functions defined on a bounded open set $U \subset R^n$

$$L[u(x)] = -\sum_{i,j=1}^{n} \partial_j(K_{ij}(x)\partial_i u(x)) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x), \qquad x \in U$$

where we suppose

i) $K_{ij} \in L^\infty(U) \quad and \quad |K_{ij}(x)| \leq k_1 \; for \;\; x \in U$

also $\displaystyle\sum_{i,j=1}^{n} K_{ij}(x)z_i z_j \geq k_0|\vec{z}|^2 \qquad \forall\vec{z} \in R^n,\; x \in \bar{U}$

ii) $b_i \in L^\infty(U) \quad and \quad |b_i(x)| \leq b_1 \quad for \;\; x \in U$

iii) $c \in L^\infty(U) \quad and \quad c_0 \leq c(x) \leq c_1 \quad for \;\; x \in U.$

Now define, for $u, v \in H_0^1(U)$,

$$B[u,v] = \int_U [\sum_{i,j=1}^{n} K_{ij}(x)\,\partial_i u(x)\,\partial_j v(x) + v(x)\{\sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x)\}]dx$$

Then, for $f \in H^0(U)$, we define u to be a weak solution of the Dirichlet boundary value problem,

$$\begin{aligned} L[u(x)] &= f(x) \qquad && x \in U, \\ u(x) &= 0 && x \in \Gamma \end{aligned}$$

if $u \in H_0^1(U)$ satisfies

$$B[u,v] = (f,v)_0 = \int_U f(x)v(x)dx \qquad \forall v \in H_0^1(U).$$

Definition of $H^{-1}(U)$

If f, v are both elements of $H^0(U)$, then the integral $\int_U fv$ exists since

$$\left|\int_U f(x)v(x)dx\right| \leq \|f\|_0\|v\|_0 < \infty.$$

More generally, suppose $v \in H_0^1(U)$ and that f is of the form

$$f(x) = g_0(x) + \sum_{j=1}^{n} \partial_i g_i(x) \text{ where } g_i \in H^0(U) \text{ for } i = 0,\ldots,n.$$

Then

$$\int_U f(x)v(x)dx = \int_U \left[g_0(x) + \sum_{j=1}^{n} \partial_i g_i(x) \right] v(x)dx =$$

$$= \sum_{j=1}^{n} \int_{\partial U} v g_i dS + \int_U \left[v g_0 - \sum_{j=1}^{n} g_i \partial_i v \right] dx$$

$$= \int_U \left[v g_0 - \sum_{i=1}^{n} g_i \partial_i v \right] dx \qquad \forall v \in H_0^1(U)$$

and

$$\left| \int_U f(x)v(x)dx \right| \leq \| g_0 \|_0 \| v \|_0 + \sum_{i=1}^{n} \| g_i \|_0 \| \partial_i v \|_0 < \infty \qquad \forall v \in H_0^1(U)$$

If we define

$$H^{-1}(U) = \left\{ f \in D'(U) : f(x) = g_0(x) + \sum_{j=1}^{n} \partial_i g_i(x) \text{ where } g_i \in H^0(U),\ 0 \leq i \leq n \right\}$$

then $\qquad \left| \int_U f(x)v(x)dx \right| \leq \left(\sum_{i=0}^{n} \| g_i \|_0^2 \right)^{1/2} \| v \|_1$

Evidently, if $v \in H_0^1(U)$ then $(f,v)_0$ is finite for all $f \in H^{-1}(U)$ and we can can consider weak solutions to the Dirichlet problem under the weaker assumption that $f \in H^{-1}(U)$.

Inhomogeneous Boundary Data

To consider weak solutions for the Dirichlet boundary value problem with inhomogeneous boundary data,

$$\begin{aligned} L[u(x)] &= f(x) \qquad & x \in U, \\ u(x) &= g(x) \qquad & x \in \Gamma \end{aligned}$$

where $g \in H^{1/2}(\Gamma)$, we recall that for Γ sufficiently regular, there exists a $G \in H^1(U)$ such that $T_0[G] = g$. Then $w = u - G \in H_0^1(U)$ and w satisfies,

$$\begin{aligned} L[w(x)] &= L[u(x) - G(x)] = f(x) - L[G(x)] = F \qquad & x \in U, \\ w(x) &= u(x) - g(x) = 0 \qquad & x \in \Gamma. \end{aligned}$$

Since $G \in H^1(U)$ *then* $L[G] \in H^{-1}(U)$, *so* $F \in H^{-1}(U)$ and we are back to the weak BVP having homogeneous data. Then the weak solution of the problem with inhomogeneous data is obtained by finding the weak solution for,

$$B[w,v] = (F,v)_0 \qquad \forall v \in H_0^1(U),$$

and then $u = w + G$.

Alternatively, let G be any function such that $G \in H^1(U)$ and $T_0 G = g \in H^{1/2}(\Gamma)$. Then $u \in H^1(U)$ is a weak solution of the Dirichlet boundary value problem with inhomogeneous boundary data, g, if $u - G = w \in H_0^1(U)$ and $B[w,v] = (F,v)_0 \quad \forall v \in H_0^1(U)$.

Existence of Weak Solutions

There are several different ways to prove existence of a weak solution to the Dirichlet problem.

- ***Symmetric Bilinear form*** (Poincare inequality approach)

  Suppose that the bilinear form B has been shown to satisfy

$$
\begin{aligned}
&B[u,v] = B[v,u] \\
&|B[u,v]| \le \alpha \|u\|_1 \|v\|_1 \\
&B[u,u] \ge \beta \|u\|_1^2
\end{aligned}
$$

Then $[[u,v]]_1 := B[u,v]$ defines a new inner product on $H_0^1(U)$ that is equivalent to the standard $H_0^1(U)$ inner product $\big($via the Poincare inequality$\big)$. In this case we can write the weak equation in the form

$$[[u,v]]_1 = (f,v)_0 = F(v) \qquad \forall v \in H_0^1(U).$$

But $F \in H^{-1}(U)$, hence by the Riesz theorem, there exists a unique $z_F \in H_0^1(U)$ such that

$$F(v) = [[z_F,v]]_1 \;\; \forall v \in H_0^1(U).$$

Then $u = z_F$ is the unique weak solution of the BVP.

- ***Non-Symmetric Bilinear form*** (Lax-Milgram approach)

  Now suppose that the bilinear form B is not symmetric but has been shown to satisfy

$$
\begin{aligned}
&|B[u,v]| \le \alpha \|u\|_1 \|v\|_1 \\
&B[u,u] \ge \beta \|u\|_1^2.
\end{aligned}
$$

It follows from the first condition that for any fixed $u \in H_0^1(U)$,

$$H_0^1(U) \ni v \longrightarrow B[u,v] \in R$$

defines a bounded linear functional on $H_0^1(U)$. Then the Riesz theorem implies the existence of a unique element $Au \in H_0^1(U)$ such that

$$B[u,v] = (Au,v)_1 \qquad \forall v \in H_0^1(U).$$

Note that since B is not symmetric, we are using the standard inner product on $H_0^1(U)$. It now follows from the second condition that A is an isomorphism of $H_0^1(U)$ *onto* $H_0^1(U)$.

i.e.,the two conditions together imply $\beta \|u\|_1 \le \|Au\|_1 \le \alpha \|u\|_1$

Then $B[u,v] = F(v)$ becomes

$$B[u,v] = (Au,v)_1 = F(v) = (z_F,v)_1 \qquad \forall v \in H_0^1(U),$$

and $Au = z_F$ has a unique solution for every $z_F \in H_0^1(U)$ and this solution is the unique weak solution of the BVP.

A slight variation on the Lax-Milgram approach is offered by the following argument. Define a mapping $T : H_0^1(U)$ *into* $H_0^1(U)$ by

$$Tu = u - \rho(Au - z_F)$$

where ρ is a real number, A denotes the previously defined bounded linear mapping associated with the bilinear form $B(u,v)$ and z_F is associated with F via the Riesz theorem. Then

$$\begin{aligned}\|Tu_1 - Tu_2\|_1^2 &= \|(u_1-u_2) - \rho(Au_1 - Au_2)\|_1^2 \\ &= \|u_1-u_2\|_1^2 - 2\rho B[u_1-u_2, u_1-u_2] + \rho^2\|A(u_1-u_2)\|_1^2 \\ &\leq (1 - 2\rho\beta + \rho^2\alpha^2)\|u_1-u_2\|_1^2\end{aligned}$$

and for $$0 < \rho < \frac{2\beta}{\alpha^2}$$

it is clear that T is a strict contraction on $H_0^1(U)$. Then T has a unique fixed point which must satisfy $Au = z_F$, or equivalently, $B[u,v] = F(v)\ \forall v \in H_0^1(U)$.
None of these abstract approaches gives any insight into how a solution might be constructed. Therefore, we consider an additional approach to existence under the assumptions of the second example above.

- ***Galerkine Proof of Existence***

We assume that the bilinear form is bounded and coercive but not necessarily symmetric. Then we begin by defining a sequence of approximate solutions.

a) **Approximate Solution**

We let $\{\phi_k\}$ denote an ON basis for $H_0^1(U)$, and for each positive integer N let

$$u_N(x) = \sum_{k=1}^{N} C_{k,N}\phi_k(x)$$

where $\quad B[u_N, \phi_j] = F(\phi_j) \qquad 1 \leq j \leq N.$

Then

$$[B_{jk}]\vec{C}_N = \vec{F}_N,$$

where

$$\begin{aligned}[B_{jk}] &= B[\phi_j, \phi_k] \\ \vec{C}_N &= [C_{1,N}, \ldots, C_{N,N}]^\top \\ \vec{F}_N &= [F(\phi_1), \ldots, F(\phi_N)]^\top.\end{aligned}$$

Note that $$\vec{C}_N^\top[B_{jk}]\vec{C}_N = B[u_N, u_N] \geq \beta\|u_N\|_1^2 = \beta\left|\vec{C}_N\right|^2$$

and hence $[B_{jk}]$ is positive definite so that a unique approximate solution exists for every N.

We now obtain estimates on the $H_0^1(U)$ norm of the sequence of approximate solutions

b) **A-priori Estimate**

It follows from $\quad B[u_N, \phi_j] = F(\phi_j) \qquad 1 \leq j \leq N,$
that

$$\beta\|u_N\|_1^2 \leq |F(u_N)| \leq C_F\|u_N\|_1$$

hence

$$\|u_N\|_1 \le \frac{1}{\beta} C_F \qquad \forall N$$

Then the uniform (in N) bound for the norms implies that the sequence $\{u_N\}$ contains a weakly convergent subsequence $\{u_\nu\}$;

$$i.e., \qquad (u_\nu, v)_1 \to (u,v)_1 \qquad \forall v \in H_0^1(U).$$

We must now show that the weak limit of this subsequence is, in fact, a weak solution.

c) **Passing to the** (**weak**) **Limit**

Let $V_N = span\{\phi_1, \ldots, \phi_N\}$. Then for each ν,

$$\begin{aligned} &B[u_\nu, v] = F(v) \qquad \forall v \in V_\nu \\ &\quad \downarrow \ \nu \to \infty \\ &B[u,v] = F(v) \qquad \forall v \in \bigcup_{\nu>0} V_\nu, \end{aligned}$$

where the convergence $B[u_\nu, v] \to B[u,v]$ follows from the weak convergence of $\{u_\nu\}$. Since the $\phi's$ form a basis for $V = H_0^1(U)$, $\bigcup_{N>0} V_N$ is dense in V, and we have that

$$B[u,v] = F(v) \qquad \forall v \in H_0^1(U).$$

We have just shown that if $\{u_\nu\}$ is a subsequence of approximate solutions having weak limit, u, in $H_0^1(U)$ then u must be a weak solution of the BVP. This can be done for any weakly convergent subsequence of the sequence of approximate solutions, and since the weak solution can be shown to be unique, it follows that all sub-sequences have the same weak limit, u. But then it follows that the sequence of approximate solutions, $\{u_N\}$, must itself converge weakly to u. In fact, the sequence converges strongly to u, as we shall now show.

d) **Passing to the** (**strong**) **Limit**

To see the strong convergence, let $\{z_N\}$ denote a sequence with $z_N \in V_N,\ and\ \|z_N - u\|_1 \to 0\ as\ N \to \infty$. For each N

$$\begin{aligned} B[u_N, v] &= F(v) \qquad \forall v \in V_N \\ B[u,v] &= F(v) \qquad \forall v \in H_0^1(U), \end{aligned}$$

hence

$$B[u - u_N, v] = 0 \qquad \forall v \in V_N.$$

In particular, for $v = z_N - u_N \in V_N$,

$$B[u-u_N, z_N - u_N] = B[u - u_N, z_N - u] + B[u - u_N, u - u_N] = 0,$$

i.e., $\quad B[u - u_N, u - z_N] = B[u - u_N, u - u_N],$

$$\alpha\|u - u_N\|_1 \|u - z_N\|_1 \ge \beta\|u - u_N\|_1^2.$$

Then $\qquad \|u - u_N\|_1 \le \dfrac{\alpha}{\beta}\|u - z_N\|_1 \to 0\ as\ N \to \infty,$

which shows that $\{u_N\}$ converges strongly to u.

To see that the weak solution is unique, suppose there are two weak solutions, u_1, u_2. Then their difference satisfies

$$B[u_1 - u_2, v] = 0 \qquad \text{for all } v \text{ in } H_0^1(U).$$

In particular, choosing $v = u_1 - u_2$, and using the coercivity of B leads to $u_1 = u_2$.

Coercivity and V-H Coercivity

The conditions i), ii) and iii) on K_{ij}, b_j and c are sufficient to show that operator L generates a bounded bilinear form B; i.e., the form B satisfies,

$$|B[u,v]| \leq \alpha \|u\|_1 \|v\|_1.$$

These conditions do not, in general imply that the bilinear form B is coercive; i.e., they **do not** imply the existence of $\beta > 0$ such that

$$B[u,u] \geq \beta \|u\|_1^2.$$

However, they do imply that B is $V-H$ *coercive* for $V = H_0^1(U)$ *and* $H = H^0(U)$; *i.e.*, they imply the existence of $\beta, \lambda > 0$ such that

$$B[u,u] \geq \beta \|u\|_1^2 - \lambda \|u\|_0^2.$$

To see this, write

$$B[u,v] = \int_U [\sum_{i,j=1}^n K_{ij}(x)\, \partial_i u(x)\, \partial_j v(x) + v(x)\{\sum_{i=1}^n b_i(x) \partial_i u(x) + c(x) u(x)\}]dx$$

and note that conditions i), ii) and iii) imply

(a) $\vec{z}^\top K(x) \vec{z} \geq k_0 |\vec{z}|^2 \qquad \forall \vec{z} \in R^n, x \in U$

(b) $\|K_{ij}\|_\infty \leq k_1,\ \|b_j\|_\infty \leq b_1, \quad c_0 \leq c(x) \leq c_1.$

Then $$|B[u,v]| \leq |\int_U [\sum_{i,j=1}^n K_{ij}(x)\, \partial_i u(x)\, \partial_j v(x) dx| + |\int_U v(x) \sum_{i=1}^n b_i(x) \partial_i u(x) dx| + |\int_U c(x) u(x) dx|$$

$$\leq k_1 \int_U |\nabla u \bullet \nabla v|\, dx + b_1 \int_U |v \nabla u|\, dx + \|c\|_\infty \int_U |uv|\, dx$$

$$\leq C[\|\nabla u\|_0 \|\nabla v\|_0 + \|\nabla u\|_0 \|v\|_0 + \|u\|_0 \|v\|_0] \leq \alpha\, \|u\|_1 \|v\|_1.$$

Similarly,

$$|B[u,u]| = \left| \int_U [\sum_{i,j=1}^n K_{ij}(x)\, \partial_i u(x)\, \partial_j u(x) + u(x)\{\sum_{i=1}^n b_i(x) \partial_i u(x) + c(x) u(x)\}]dx \right|$$

$$\geq k_0 \int_U |\nabla u \bullet \nabla u|\, dx + c_0 \int_U |u|^2\, dx - b_1 \int_U |u|\, |\nabla u|\, dx.$$

Now $$\int_U |u|\, |\nabla u|\, dx \leq \int_U \left(\frac{\varepsilon}{b_1} |\nabla u|^2 + \frac{b_0}{4\varepsilon} |u|^2\right) dx = \frac{\varepsilon}{b_1} \|\nabla u\|_0^2 + \frac{b_0}{4\varepsilon}\, \|u\|_0^2$$

and

$$|B[u,u]| \geq (k_0 - \varepsilon) \|\nabla u\|_0^2 + \left(c_0 - \frac{b_0^2}{4\varepsilon}\right) \|u\|_0^2 \geq \beta \|u\|_1^2 - \lambda \|u\|_0^2.$$

Under these conditions we then are able to prove the Fredholm alternative theorem for the ellliptic boundary value problem. When the space V is chosen to be $H^1(U)$ instead of $H_0^1(U)$ then one obtains the Neumann boundary value problem. By choosing V to be a space lying between $H^1(U)$ and $H_0^1(U),$ one obtains various mixed boundary value problems. (See the notes #8)

The Fredholm Alternative Theorem

What happens in the case that B is $V-H$ coercive but not coercive (i.e., V elliptic)? In this case we can gain insight from looking back at the finite dimensional situation.

Suppose A is an $n \times n$ real matrix. Then

$$R^n = R_{A^\top} \oplus N_A = N_{A^\top} \oplus R_A$$

$$\dim N_A = n - rank(A) = \dim N_{A^\top}$$

$$\dim R_A = n - \dim N_{A^\top} = rank(A)$$

and we have the following alternatives for the problem $Ax = b$:

a) $rank(A) = n$ there is a unique solution for every $b \in R^n$

b) $rank(A) = m < n$ there is no solution unless $b \perp N_{A^\top}$
in which case there is an $n-m$ parameter family of (nonunique) solutions

This is the so called Fredholm alternative stated for a system of linear algebraic equations. We will find a similar result for elliptic boundary value problems.

Suppose the bilinear form B satisfies

$\exists$ positive β, λ such that $|B[u,u]| + \lambda \|u\|_0^2 \geq \beta \|u\|_1^2 \quad \forall u \in H_0^1(U)$

Then B is not coercive but the modified form $B_\mu[u,v] = B[u,v] + \mu(u,v)_0$ is coercive for $\mu \geq \lambda.$ This implies that

$\forall f \in H^{-1}(U)$ there exists a unique $u \in H_0^1(U)$ such that

$$B_\mu[u,v] = \langle f,v \rangle \qquad \forall v \in H_0^1(U)$$

This is equivalent to the statement

$L_\mu = L + \mu I : H_0^1(U) \to H^{-1}(U)$ is an isomorphism

Denote the unique weak solution of the boundary value problem with B_μ by $u = L_\mu^{-1} f$ and recall that we want to solve $Lu = f,$ not $L_\mu u = f.$ If we write

$$L_\mu u = Lu + \mu u = f + \mu u := g,$$

then

$$u = L_\mu^{-1} g = L_\mu^{-1}(f + \mu u), \qquad \text{or} \qquad u - \mu L_\mu^{-1} u = L_\mu^{-1} f.$$

Write $K = \mu L_\mu^{-1},$ and $F = L_\mu^{-1} f,$

so our equation can be written $(I-K)u = F,$
where

$K : H^0(U) \to H_0^1(U)$ is bounded.

To see this write

$$B_\mu[u,u] = (g,u)_0$$

which leads to

$$\beta\|u\|_1^2 \leq |B_\mu[u,u]| \leq \|g\|_0\|u\|_0 \leq \|g\|_0\|u\|_1;$$

i.e., $$\|u\|_1 = \|L_\mu^{-1}g\|_1 \leq \frac{\|g\|_0}{\beta};$$

But then the definition of K leads to the estimate

$$\|Kg\|_1 = \|\mu L_\mu^{-1}g\|_1 \leq \frac{\mu}{\beta}\|g\|_0$$

and since $K : H^0(U) \to H_0^1(U)$ is bounded, and the embedding $i : H_0^1(U) \to H^0(U)$ is compact, it follows that

$$K : H^0(U) \to H^0(U) \qquad \text{is compact.}$$

Now we list for convenient reference, several equivalent problems:

u is a weak solution for $\quad L[u(x)] = f(x) \quad x \in U$

$$u = 0 \qquad x \in \Gamma$$

if and only if

$(a) \quad u \in H_0^1(U),$ satisfies $\quad B[u,v] = (f,v)_0 \quad \forall v \in H_0^1(U)$

if and only if

$(b) \quad u \in H_0^1(U),$ satisfies $\quad B_\mu[u,v] = (f,v)_0 + \mu(u,v)_0 \quad \forall v \in H_0^1(U)$

if and only if

$(c) \quad u = L_\mu^{-1}f + \mu L_\mu^{-1}u,$

if and only if

$(d) \quad (I-K)u = L_\mu^{-1}f,$

The problems (a) through (d) are all equivalent formulations and $u \in H_0^1(U)$ solves one if and only if it solves all the others. However, the operator $(I-K)$ is an example of what is called a ***Fredholm operator*** on $H^0(U)$. This means

$$H^0(U) = Rng(I-K) \oplus N(I-K^*) = N(I-K) \oplus Rng(I-K^*)$$

$$\dim\, N(I-K) = \dim\, N(I-K^*) < \infty$$

$Rng(I-K)$ is closed and $Rng(I-K) = N(I-K^*)^\perp$

$Rng(I-K^*)$ is closed and $Rng(I-K^*) = N(I-K)^\perp$

The analogy with the statements at the beginning of the section are obvious. We can now state without proof the Fredholm alternative theorem for a compact operator on a Hilbert space.

Fredholm Alternative Theorem- Consider the equation $(I-K)u = F \in H,$ for K a compact operator on Hilbert space, $H.$ Then exactly one of the following alternatives must hold:

(i) $N(I-K) = \{0\}$: in this case $\forall F \in H$ there exists a unique $u \in H$ satisfying

$$(I-K)u = F$$

(ii) $N(I-K) = span\{z_1,\dots,z_p\}$ and $N(I-K^*) = span\{w_1,\dots,w_p\}$ for a positive integer $p.$ In this case $(I-K)u = F$ has no solution unless

$$(F,w_j)_H = 0,\ 1 \leq j \leq p.$$

Then $\quad (I-K)[u_0 + C_1z_1 + \cdots + C_pz_p] = F$

for all constants, C_j and any $u_0 \in H$ such that $(I-K)u_0 = F.$

Here, the operator K^* on H is defined as follows. For $v \in H,$ fixed we can define a linear functional on H by

$$\lambda[u] =: (Ku,v)_H \qquad \forall u \in H.$$

Then the Riesz theorem implies the existence of an element, denoted by K^*v to indicate its dependence on $v,$ such that

$$\lambda[u] =: (u,K^*v)_H \qquad \forall u \in H.$$

i.e., $\quad (Ku,v)_H = (u,K^*v)_H \qquad \forall u,v \in H.$

The operator $\quad K^* : H \to H \quad$ is easily seen to be bounded and linear and it can be shown to be compact of K is compact. The operator is called the ***adjoint operator*** for $K.$

The Adjoint Boundary Value Problem

Suppose that u is a weak solution for

$$L[u(x)] = f(x) \qquad x \in U$$

$$u = 0 \qquad x \in \Gamma$$

That is,

$u \in H_0^1(U),$ satisfies $\quad B[u,v] = (f,v)_0 \qquad \forall v \in H_0^1(U)$

Note that for $\phi,\psi \in C_c^\infty(\bar{U})$ we have

$$(L[\phi],\psi)_0 = B[\phi,\psi] - \int_\Gamma \psi\ [K]\nabla\phi \cdot n\ dS = B[\phi,\psi] - 0$$

and if we extend this by continuity to $u \in H^1(U),$ this becomes

$$(L[u],\psi)_0 = B[u,\psi] = (f,\psi)_0 \qquad \forall \psi \in C_c^\infty(U)$$

Evidently, if u is a weak solution of the BVP, then

$L[u] = f \quad$ in the sense of distributions on $U.$

If we integrate by parts again in order to move all of the differentiation onto $\psi,$ we get,

$$(L[u],\psi)_0 = B[u,\psi]$$

$$= \int_U [\sum_{i,j=1}^{n} K_{ij}(x)\,\partial_i u(x)\,\partial_j \psi(x) + \psi(x)\{\sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x)\}]dx$$

$$= \int_U [\nabla\psi K \nabla u + \psi b \cdot \nabla u + c\psi u]\; dx$$

$$\int_U u[-\nabla \cdot (K\nabla\psi) - \nabla \cdot (\psi b) + c\psi]\; dx = (u, L^*[\psi])_0$$

where

$$L^*[\psi] = -\nabla(K\nabla\psi) - \nabla(\psi b) + c\psi$$

$$= -\sum_{i,j=1}^{n} \partial_i (K_{ij}(x)\,\partial_j \psi(x)) - \sum_{i=1}^{n} \partial_i (b_i(x)\psi(x)) + c(x)\psi(x)$$

Then

$$(L[u], \psi)_0 = B[u, \psi] = (u, L^*[\psi])_0$$

and by analogy with

u is a weak solution for $\quad L[u(x)] = f(x) \quad x \in U$

$$u = 0 \qquad x \in \Gamma$$

if and only if $\quad u \in H_0^1(U),$ satisfies $\quad B[u,v] = (f,v)_0 \qquad \forall v \in H_0^1(U)$

we have

v is a weak solution for $\quad L^*[v(x)] = g(x) \quad x \in U$

$$v = 0 \qquad x \in \Gamma$$

if and only if $\quad v \in H_0^1(U),$ satisfies $\quad B^*[v,u] = (g,u)_0 \qquad \forall u \in H_0^1(U)$

where $\qquad B^*[v,u] = B[u,v].$

We refer to the BVP as the ***adjoint problem*** to the original BVP.

Proceeding just as we did above, we have that v is a weak solution of the adjoint problem if and only if,

$(a) \qquad v \in H_0^1(U),$ satisfies $\quad B^*[v,u] = (g,u)_0 \qquad \forall u \in H_0^1(U)$

if and only if

$(b) \qquad v \in H_0^1(U),$ satisfies $\quad B_\mu^*[v,u] = (g,u)_0 + \mu(u,v)_0 \qquad \forall u \in H_0^1(U)$

if and only if

$$(c) \qquad u = (L_\mu^*)^{-1} f + \mu (L_\mu^*)^{-1} u,$$

if and only if

$$(d) \qquad (I - K^*)u = (L_\mu^*)^{-1} f,$$

Since $\quad (L[u], v)_0 = B[u,v] = (u, L^*[v])_0 \qquad for\;\; u,v \in H_0^1(U)$

then $\quad \left(\mu (L_\mu)^{-1} f, g\right)_0 = \left(f, \mu (L_\mu^*)^{-1} g\right)_0 \qquad for\;\; f,g \in H^0(U)$

i.e., $$(Kf,g)_0 = (f,K^*g)_0 \qquad for\ f,g \in H^0(U)$$

This proves that K^* is the adjoint mapping for K.

Then we can restate the Fredholm alternative theorem as it applies to the elliptic boundary value problem as follows:

Fredholm Alternative Theorem- Consider the elliptic boundary value problem

$$L[u(x)] = f(x) \qquad x \in U$$

$$u = 0 \qquad x \in \Gamma$$

Exactly one of the following alternatives must apply to the associated weak problem:

(i) $N = \left\{u \in H_0^1(U) \,:\, B[u,v] = 0,\ \forall v \in H_0^1(U)\right\} = \{0\}$:
$N^* = \left\{v \in H_0^1(U) \,:\, B^*[u,v] = 0,\ \forall u \in H_0^1(U)\right\} = \{0\}$
In this case $\forall f \in H^{-1}(U)$ there exists a unique weak solution, $u \in H_0^1(U)$.

(ii) $N = span\{z_1,\ldots,z_p\}$ and $N^* = span\{w_1,\ldots,w_p\}$ for a positive integer p.
In this case there is no weak solution for the BVP unless

$$(f,w_j)0 = 0,\ 1 \le j \le p.$$

Then $$B[u_0 + C_1 z_1 + \cdots + C_p z_{p,v}] = (f,v)_0$$

for all constants, C_j and any weak solution $u_0 \in H_0^1(U)$.