Hilbert Space Problems

1. Let

$$u_n(x) = \begin{cases} -1 \;\; if \;\; -1 \le x \le -1/n \\ nx \;\; if \;\; -1/n \le x \le 1/n \\ +1 \;\; if \;\; 1/n \le x \le 1 \end{cases}$$

(a) show that $\{u_n(x)\}$ is a sequence in $L_2[-1,1]$ that is Cauchy in the norm of $L_2[-1,1]$.

(b) show that $u_n(x)$ converges (in the norm of $L_2[-1,1]$) to

$$sgn(x) = \begin{cases} -1 \;\; if \;\; -1 \le x < 0 \\ +1 \;\; if \;\; 0 < x \le 1 \end{cases}$$

(c) how does this example show that $C[-1,1]$ is a subspace of $L_2[-1,1]$ that is not closed?

2 Let M denote a subspace of Hilbert space H.

Show that $\bar{M} = (M^{\perp})^{\perp}$.

hint: Show first that $\bar{M} \subset (M^{\perp})^{\perp}$,

Then show that if $x \neq 0,\; x \in (M^{\perp})^{\perp}$

$and.\;\; x \notin \bar{M},$

$then \;\; x \in (M^{\perp})^{\perp} \cap M^{\perp},$

Show that $x \in (M^{\perp})^{\perp} \cap M^{\perp} \;\; imlies \;\; x = 0.$

3. Let F denote a linear functional on Hilbert space H. Show that if F has any one of the following properties, it has all of them.

(a) **Continuous at zero**: $x_n \to 0 \;\; in \;\; H \quad \Rightarrow \quad F(x_n) \to 0 \;\; in \;\; R$

(b) **Continuous on H**: $x_n \to x \;\; in \;\; H \quad \Rightarrow \quad F(x_n) \to F(x) \;\; in \;\; R$

(c) **Bounded** : There exists $C > 0 \;\; such \;\; that \;\; |F(x)| \le C\|x\|_H \quad \forall x \in H$

4. Suppose F is a bounded linear functional an Hilbert space H. Show that $N_F = \{x \in H : F(x) = 0\}$ is a closed subspace in H.

5. Let $a(u,v)$ denote a bilinear functional on Hilbert space H.
Suppose there are positive constants $a_0, a_1 \;\; such \;\; that$

$$|a(u,v)| \le a_1\|u\|_H\|v\|_H \quad \forall u,v \in H$$
$$a(u,u) \ge a_0\|u\|_H{}^2 \quad \forall u, \in H$$

(a) Show $x_n \to 0 \;\; in \;\; H \quad \Rightarrow \quad a(x_n,y) \to 0 \;\; in \;\; R \quad \forall y \in H$

$y_n \to 0 \;\; in \;\; H \quad \Rightarrow \quad a(x,y_n) \to 0 \;\; in \;\; R \quad \forall x \in H$

(b) Show $x_n \to x \quad in\ H \quad \Rightarrow \quad a(x_n, y) \to a(x,y) \quad in\ R \quad \forall y \in H$

$y_n \to y \quad in\ H \quad \Rightarrow \quad a(x, y_n) \to a(x,y) \quad in\ R \quad \forall x \in H$

6. Suppose that

$a(x,y)$ is a positive, bounded and symmetric bilinear form on Hilbert space H,
$F(x)$ is a bounded linear functional on H and C is a constant.

Let

$$\Phi(x) = \tfrac{1}{2}\, a(x,x) - F(x) + C$$

(a) Show that for $\{x_n\} \subset H$,

$$2[a(x_n,x_n) + a(x_m,x_m)] = a(x_n - x_m, x_n - x_m) + a(x_m + x_n, x_m + x_n)$$

(b) Show that if $\alpha = \inf\limits_{x \in H} \Phi(x)$, then

$$\Phi(x_m) + \Phi(x_n) = \frac{1}{4}\, a(x_m - x_n, x_m - x_n) + 2\ \Phi[\frac{x_m + x_n}{2}] \geq \frac{1}{4} a_0 \|x_m - x_n\|_H^2 + 2\alpha.$$

7. We have a lemma that says a quadratic functional on a Hilbert space has a global minimum at a point x_0 if and only if its gradient vanishes at x_0. This is a generalization of a similar result for R^n where the role of the bilinear form is played by a symmetric matrix and the role of the bounded linear functional is played by a vector. The bilinear form $a(u,v)$ in the Hilbert space setting had to be symmetric, positive and bounded.

Let A denote an n by n symmetric real matrix, and let $\vec{b}$ denote a fixed vector in R^n. Define a quadratic functional from R^n to $R.$ by $Q(\vec{x}) = \tfrac{1}{2}\vec{x} \bullet A\vec{x} - \vec{b} \bullet \vec{x} = Q(x_1, \ldots x_n)$

(a) Compute expressions for $\dfrac{\partial \vec{x}}{\partial x_j}$, $\dfrac{\partial\left(\vec{b} \bullet \vec{x}\right)}{\partial x_j}$ and $\dfrac{\partial (A\vec{x})}{\partial x_j}$ and use them to derive an expression for $\dfrac{\partial Q}{\partial x_j},\ j = 1, \ldots, n$

(b) Show that $grad\, Q(\vec{x}) = \left[\dfrac{\partial Q}{\partial x_1}, \ldots, \dfrac{\partial Q}{\partial x_n}\right] = A\vec{x} - \vec{b}$

(c) Show that if A is positive definite, then Q has an absolute minimum at the point where $A\vec{x} = \vec{b}$.

8. Suppose $a = a(x,y)$ is a positive, bounded symmetric bilinear form on Hilbert space H and F denotes a bounded linear functional on $H.$ Let

$$\Phi(x) = \tfrac{1}{2}\, a(x,x) - F(x) + C.$$

(a) Show that $\forall x, y \in H \quad 2[a(x,x) + a(y,y)] = a(x-y, x-y) + a(x+y, x+y)$

(b) Use (a) to show that $\Phi(x) + \Phi(y) = \tfrac{1}{4} a(x-y, x-y) + 2\Phi\left(\dfrac{x+y}{2}\right)$

Problems on $H^1(U)$ *and* $H_0^1(U)$

1. $H^1(a,b) = \{f(x) \in L_2(a,b) : f'(x) \in L_2(a,b)\}$ is a special (1-dimensional) special case of $H^1(U)$ for $U \subset R^n$. Modify the proof that $H^1(U)$ is complete for the norm

$$\|u\|_1 = \sqrt{\|u\|_0^2 + \|\nabla u\|_0^2}$$

in order to show that $H^1(a,b)$ is complete for the norm

$$\|f\|_1^2 = (f,f)_0 + (f',f')_0 = \int_a^b [f(x)^2 + f'(x)^2]dx$$

2. Let $H^2(a,b) = \{f(x) \in L_2(a,b) : f'(x)$ and $f''(x) \in L_2(a,b)\}$ .Modify the proof that $H^1(a,b)$ is complete for the norm

$$\|f\|_1^2 = (f,f)_0 + (f',f')_0 = \int_a^b [f(x)^2 + f'(x)^2]dx$$

in order to show that $H^2(a,b)$ is complete for the norm

$$\|f\|_2^2 = (f,f)_0 + (f',f')_0 + +(f'',f'')_0 = \int_a^b [f(x)^2 + f'(x)^2 + f''(x)^2]dx$$

3. If $f \in H^1(a,b)$ then for $a \leq x < y \leq b$,

$$f(y) = f(x) + \int_x^y f'(s)ds$$

(a) show that $\quad f \in C[a,b] \quad$ *and* $\quad |f(y) - f(x)| \leq \|f\|_1 \sqrt{|y-x|}$

(b) $\quad \|f\|_\infty =: \max_{x \in [a,b]} |f(x)| \leq \|f\|_1$

4. Suppose $f \in H^1(a,b)$ and $f(a) = 0,$ or $f(b) = 0,$ or both. Then show that

$$\|f\|_0^2 = \int_a^b f(x)^2 dx \leq C \int_a^b f'(x)^2 dx$$

where $C > 0$ does not depend on *f*.