Abstract Hilbert Space Results

We have learned a little about the Hilbert spaces $L_2(U)$ *and* ℓ_2 and we have at least defined $H^1(U)$ and the scale of Hilbert spaces $H^p(U)$. Now we are going to develop additional facts that are true about any Hilbert space. Later these facts will be helpful in formulating and solving partial differential equations in a Hilbert space setting.

1. Subspaces

A subset M of Hilbert space H is a *subspace* of it is closed under the operation of forming linear combinations;i.e.,

$$\forall x, y \in M, \quad C_1 x + C_2 y \in M, \qquad \forall\ C_1, C_2 \in R.$$

The subspace M is said to be ***closed*** if it contains all its limit points; i.e., every sequence of elements of M that is Cauchy for the H-norm, converges to an element of M. In a Euclidean space every subspace is closed but in a Hilbert space this is not the case.

Example 1.1-

1. If U is a bounded open set in R^N then $H = L_2(U)$ is a Hilbert space containing $M = C(U)$ as a subspace. It is easy to find a sequence of functions in M that is Cauchy for the H norm but the sequence converges to a function in H that is discontinuous and hence not in M. For example, if $N = 1,$ and $U = (-1, 1)$ the sequence

$$u_n(x) = \left\{ \begin{array}{ccc} nx & if & |x| \leq 1/n \\ 1 & if & 1/n \leq x \leq 1 \\ -1 & if & -1 \leq x \leq -1/n \end{array} \right\}$$

can be shown to converge in the $L_2 - norm$ to $u(x) = sgn(x)$. Here $sgn(x) \in L_2[-1, 1]$ but $sgn(x) \notin C[-1, 1]$; i.e., the limit does not belong to the subspace of continuous functions. This proves that $M = C[-1, 1]$ is not closed in $H = L_2[-1, 1]$.

2. Every finite dimensional subspace of a Hilbert space H is closed. For example, if M denotes the span of finitely many elements $\{x_1, \ldots x_N\}$ in H, then the set M of all possible linear combinations of these elements is finite dimensional (of dimension N), hence it is closed in H.

3. Let M denote a subspace of Hilbert space H and let $M^\perp$ denote the ***orthogonal complement of M***.

$$M^\perp \stackrel{def}{=} \{x \in H : (x, y)_H = 0, \forall y \in M\}.$$

Then $M^\perp$ is easily seen to be a subspace and it is closed, whether or not M itself is closed. To see this, suppose $\{x_n\}$ is a Cauchy sequence in $M^\perp$ converging to a limit $x \in H$. For arbitrary $y \in M$, $(x_n, y)_H = 0$ for every n, and hence

$$(x, y)_H = (x - x_n, y)_H + (x_n, y)_H = (x - x_n, y)_H + 0 \to 0, \text{ as n tends to infinity.}$$

Then the limit point x is orthogonal to every y in M which is to say, x is in $M^\perp$, and $M^\perp$ is closed.

If M is a subspace of H that is not closed, then M is contained in a closed subspace $\bar{M}$ of H, consisting of M together with all its limit points. $\bar{M}$ is called the ***closure*** or ***completion*** of M and M is said to be ***dense*** in $\bar{M}$. This means that for every x in $\bar{M}$ there is a sequence of elements of M that converge to x in the norm of H. Equivalently, to say M is dense in $\bar{M}$ means that for every x in $\bar{M}$ and every $\epsilon > 0$, there is a y in M such that $\|x - y\|_H < \epsilon$. Then we can show,

Lemma 1.1 Let M denote a subspace of Hilbert space H. Then $(M^\perp)^\perp = \bar{M}$.
Lemma 1.2 A subspace M of Hilbert space H is ***dense in H*** if and only if $M^\perp = \{0\}$.

Example 1.2
1.Recall the Hilbert-Sobolev space of order one,

$$H^1(U) = \left\{u(x) \in L_2(U) : \ \partial_{x_i}u(x) \in L_2(U) \ for \ \ i = 1,...,n\right\}$$

where $\partial_{x_i}u(x)$ denotes the distributional derivative of $u(x)$ with respect to x_i. We defined an inner product on $H^1(U)$ as follows,

$$(u,v)_1 = \int_U [u(x)v(x) + \nabla u \bullet \nabla v]\, dx \qquad for \quad u,v \in H^1(U).$$

and showed that $H = H^1(U)$ is complete for the norm induced by this inner product. The linear space $C^\infty(U)$, of infinitely differentiable functions is contained in $H^1(U)$ and is therefore a subspace of $H^1(U)$. We can show, using a technique called regularization, that $M = C^\infty(U)$ is dense in H; i.e., the completion of $C^\infty(U)$ in the norm of $H^1(U)$ is the whole Hilbert space. Then, according to the previous lemma, only the zero function is orthogonal, in the $H^1(U)$ – inner product, to every function in $C^\infty(U)$.

2. The linear space of test functions $M = C_c^\infty(U)$ is also a subspace of $H^1(U)$ The closed subspace $\bar{M}$ obtained by completing the test functions in the norm of $H^1(U)$ is denoted by $H_0^1(U)$ and we refer to this closed subspace as the $H^1(U)$ functions that ”vanish on the boundary of U”. In fact, what we can show is that for every $g \in H_0^1(U)$, and for every point $p \in \partial U$, the integral of g over the ball, $B_\varepsilon(p)$, tends to zero as ε tends to zero. This says that if $g \in H_0^1(U)$, then g has zero mean value near every point on ∂U.

We will show now that the closed subspace $H_0^1(U)$ is not equal to $H^1(U)$. To do this, we need an inequality known as the ***Poincaré inequality***. This inequality asserts that for any bounded subset $U \subset R^n$ there exists a constant, C depending only on U, such that

$$\|\phi\|_0^2 \le C\|\nabla\phi\|_0^2 \qquad \forall\phi \in C_c^\infty(U) \qquad \|\phi\|_0 = \left(\int_U \phi^2 dx\right)^{1/2}$$

or, equivalently

$$\|\phi\|_1^2 \le C\|\nabla\phi\|_0^2 \qquad \forall\phi \in C_c^\infty(U)$$

The proof of this inequality will be given later. We will use it to show there are functions in $H^1(U)$ that are not in $H_0^1(U)$. For example, let $1(x) \in H^1(U)$ denote the constant function having value one at every point. Then

$$\|1-\phi\|_1^2 = \|1-\phi\|_0^2 + \|\nabla(1-\phi)\|_0^2 = \|1-\phi\|_0^2 + \|\nabla\phi\|_0^2 \qquad \forall\phi \in C_c^\infty(U).$$

Clearly then $\quad \|1-\phi\|_1^2 \geq \|1-\phi\|_0^2 \quad and \quad \|1-\phi\|_1^2 \geq \|\nabla\phi\|_0^2.$

The first of these results implies $\quad \|1-\phi\|_1^2 \geq \|1\|_0^2 - \|\phi\|_0^2 \quad$ which leads to

$$\|1-\phi\|_1^2 + \|\phi\|_0^2 \geq \|1\|_0^2 = \int_U 1\,dx = |U| = area\ of\ U$$

Then the Poincaré inequality implies $\|1-\phi\|_1^2 + C\|\nabla\phi\|_0^2 \geq |U|,$

and finally, $\qquad (1+C)\|1-\phi\|_1^2 \geq |U|.$

It is clear from this inequality that there is no sequence of test functions ϕ_n which can converge to $1(x)$ in the norm of $H^1(U)$. Then $1(x)$ does not belong to $H_0^1(U)$ although it does belong to $H^1(U)$.

2. Projections

A Hilbert space H is said to be ***separable*** if H contains a countable dense subset $\{h_n\}$. In this case, for every x in H and every $\epsilon > 0$ there exists an integer N_ϵ and scalars $\{a_n\}$ such that

$$\left\| x - \sum_{n=1}^N a_n h_n \right\|_H < \epsilon \quad \text{for} \quad N > N_\epsilon$$

If H is a separable Hilbert space, then the Gram-Schmidt procedure can be used to construct an orthonormal basis for H out of a countable dense subset. Recall that an ***orthonormal basis for H*** is a set of mutually orthogonal unit vectors, $\{\phi_n\}$ in H with the following property:

1) For $f \in H,\ (\phi_n, f)_H = 0\ \forall n \iff f = 0$

When the orthonormal set $\{\phi_n\}$ has property 1, then it is said to be ***dense or complete in H***. Of course, not every orthonormal set in H is complete. Recall that other equivalent ways of characterizing completeness for orthonormal sets can be stated as follows:

2) For all f in H and every $\epsilon > 0$, there exists an integer N_ϵ such that

$$\left\| f - \sum_{n=1}^N (f, \phi_n)_H \phi_n \right\|_H < \epsilon \quad \text{for} \quad N > N_\epsilon$$

3) For every f in H, $\sum_{n=1}^{\infty} f_n^2 = \|f\|_H^2$ where $f_n = (f, \phi_n)_H$

Hilbert Space Projection Theorem
In a Euclidean space,E, where all subspaces M are closed, it is a fact that for each $y \in E$ there is a unique $z \in M$ such that $\|y - z\|_E$ is minimal. This element z, which is just the orthogonal projection of y onto M, is the ”best approximation to y from within M”. In an infinite dimensional Hilbert space, a similar result is true for closed subspaces but for subspaces that are not closed there may fail to be a ”best” approximation in M.

Hilbert Space Projection Theorem Let M be a closed subspace of Hilbert space H and let y in H be given. Then

(i) there exists a unique x_y in M such that $\|y - x_y\|_H \leq \|y - z\|_H$ for all z in M
(x_y is the unique point of M that is closest to y, the best approximation to y in M)

(ii) $(y - x_y, z)_H = 0$ for all z in M; i.e., $y - x_y \perp M$

(iii) every y in H can be uniquely expressed as $y = x_y + z_y$
where

$$Py = x_y \in M, \qquad Qy = z_y \in M^{\perp}$$

and

$$\|y\|_H^2 = \|Py\|_H^2 + \|Qy\|_H^2 \qquad \text{i.e., } H = M \oplus M^{\perp}.$$

The proof of this result will be given later.

3. Linear Functionals and Bilinear Forms
A real valued function defined on H, is said to be a ***functional*** on H. The functional, L, is said to be:

(a) ***Linear*** if, for all x and y in H, $L(C_1x + C_2y) = C_1Lx + C_2Ly,$ for all scalars C_1,C_2.

(b) ***Bounded*** if there exists a constant C such that $|Lx| \leq C\|x\|_H$ for all x in H

(c) ***Continuous*** if $\|x_n - x\|_H \to 0$ implies that $|Lx_n - Lx| \longrightarrow 0$

It is not difficult to show that the only example of a linear functional on a Euclidean space E is $Lx = (x,z)_E$ for some z in E, fixed. For example, if F is a linear functional on E, then for arbitrary x in E,

$$F(x) = F\left(\sum_{i=1}^{n} x_i \bar{e}_i\right) = \sum_{i=1}^{n} x_i F(\bar{e}_i) = \sum_{i=1}^{n} x_i F_i = (x, z_F)_E = x^{\top} z_F$$

where $\{e_i\}$ denotes the standard basis in E and $\bar{z}_F$ denotes the n-tuple whose i-th component is $F_i = F(\bar{e}_i)$. This displays the isomorphism between functionals F and elements, z_F, in E. This isomorphism also exists in an abstract Hilbert space.

Riesz Representation Theorem For every continuous linear functional f on Hilbert space H there exists a unique element, z_f in H such that $f(x) = (x, z_f)_H$ for all x in H.

Proof- Let $N_f = \{x \in H : f(x) = 0\}$.Then N_f is easily seen to be a closed subspace of H. If $N_f = H$ then $z_f = 0$ and we are done. If $N_f \neq H$ then $H = N_f \oplus N_f^{\perp}$ by the Hilbert space projection theorem. Since N_f is not all of H, $N_f^{\perp}$ must contain nonzero vectors, and we denote by z_0 an element of $N_f^{\perp}$ such that $\|z_0\|_H = 1$. Then for any x in H,

$$w = f(x)z_0 - f(z_0)x,$$

belongs to N_f hence $w \perp z_0$. But in that case,

$$(f(x)z_0 - f(z_0)x, z_0)_H = f(x)(z_0, z_0)_H - f(z_0)(x, z_0)_H = 0.$$

This leads to, $f(x) = f(z_0)(x, z_0)_H = (x, f(z_0)z_0)_H$ which is to say $z_f = f(z_0)z_0$.
To see that z_f is unique, suppose that

$$f(x) = (z_f, x)_H = (w_f, x)_H \qquad \forall x \in H$$

Subtracting leads to the result that

$$(z_f - w_f, x)_H = 0 \qquad \forall x \in H.$$

In particular, choosing $x = z_f - w_f$ leads to $\|(z_f - w_f\|_H = 0.$■

A real valued function $a(x,y)$ defined on $H \times H$ is said to be:

(a) ***Bilinear*** if, for all $x_1, x_2, y_1, y_2 \in H$ and all scalars C_1, C_2

$$a(C_1x_1 + C_2x_2, y_1) = C_1a(x_1, y_1) + C_2a(x_2, y_1)$$

$$a(x_1, C_1y_1 + C_2y_2) = C_1a(x_1, y_1) + C_2a(x_1, y_2)$$

(b) ***Bounded*** if there exists a constant b > 0 such that,

$$|a(x,y)| \leq b\|x\|_H\|y\|_H \text{ for all x,y in H}$$

(c) ***Continuous*** if $x_n \to x$, and $y_n \to y$ in H, implies $a(x_n, y_n) \to a(x,y)$ in R

(d) ***Symmetric*** if $a(x,y) = a(y,x)$ for all $x, y \in H$

(e) ***Positive or coercive*** if there exists a constant $a_0 > 0$ such that

$$a(x,x) \geq a_0\|x\|_H^2 \text{ for all x in H}$$

It is not hard to show that for both linear functionals and bilinear forms, boundedness is equivalent to continuity. If $a(x,y)$ is a bilinear form on $H \times H$, and $F(x)$ is a linear functional on H, then $\Phi(x) = \frac{1}{2}a(x,x) - F(x) + Const$ is called a ***quadratic functional on H***. In a Euclidean space a quadratic functional has a unique extreme point located at the point where the gradient of the functional vanishes. This result generalizes to the infinite dimensional situation.

Lemma 3.1 Suppose $a(x,y)$ is a positive, bounded and symmetric bilinear form on Hilbert space H, $F(x)$ is a bounded linear functional on H and C is a constant. Consider the following problems

(a) minimize $\Phi(x) = \frac{1}{2}a(x,x) - F(x) + C$ over H

(b) find x in H satisfying $a(x,y) = F(y)$ for all y in H.

Then

i) x in H solves (a) if and only if x solves (b)

ii) there is at most on x in H solving (b)

iii) there is at least one x in H solving (a)

Proof- For t in R and x,y fixed in H, let $f(t) = \Phi(x+ty)$. Then $f(t)$ is a real valued function of the real variable t and it follows from the symmetry of $a(x,y)$ that

$$f(t) = \tfrac{1}{2}t^2\, a(y,y) + t[a(x,y) - F(y)] + \tfrac{1}{2}a(x,x) - F(x) + C$$

and

$$f'(t) = t\, a(y,y) + [a(x,y) - F(y)]$$

It follows that $\Phi(x)$ has a global minimum at x in H if and only if $f(t)$ has a global minimum at $t = 0$; i.e.,

$$\Phi(x+ty) = \Phi(x) + tf'(0) + t^2/2\; a(x,x) \geq \Phi(x), \quad \forall t \in R \quad and \quad \forall y \in H$$

if and only if

$$f'(0) = a(x,y) - F(y) = 0. \quad \forall y \in H.$$

This establishes the equivalence of (a) and (b).

To show that (b) has at most one solution in H, suppose

$a(x_1,y) = F(y)$ and $a(x_2,y) = F(y)$ for all y in H.

Then $a(x_1,y) - a(x_2,y) = a(x_1 - x_2,y) = 0$ for al l y in H. In particular, for $y = x_1 - x_2$

$$0 = a(x_1 - x_2, x_1 - x_2) \geq a_0 \| x_1 - x_2 \|_H^2; \text{ i.e., } \; x_1 = x_2.$$

To show that $\Phi(x)$ has at least one minimum in H, let $\alpha = \inf_{x \in H} \Phi(x)$. Now

$$\Phi(x) = \tfrac{1}{2} a(x,x) - F(x) \geq \tfrac{1}{2} \, a_0 \| x \|_H^2 - b \| x \|_H$$

and it is evident that $\Phi(x)$ tends to infinity as $\| x \|_H$ tends to infinity. This means $\alpha > -\infty$ (i.e.," the parabola opens upward rather than downward"). Moreover since α is an infimum, there exists a sequence x_n in H such that $\Phi(x_n) \to \alpha$ as n tends to infinity. Note that

$$2[a(x_n,x_n) + a(x_m,x_m)] = a(x_n - x_m, x_n - x_m) + a(x_m + x_n, x_m + x_n)$$

which leads to the result,

$$\Phi(x_m) + \Phi(x_n) = \frac{1}{4} a(x_m - x_n, x_m - x_n) + 2 \, \Phi[(x_m + x_n)/2] \geq \frac{1}{4} C \| x_m - x_n \|_H^2 + 2\alpha.$$

But $\Phi(x_m) + \Phi(x_n)$ tends to 2α as n tends to infinity and in view of the previous line, the minimizing sequence $\{x_n\}$ must be a Cauchy sequence with limit x in the Hilbert space H. Finally, since $\Phi(x)$ is continuous, $\Phi(x_n) \to \Phi(x) = \alpha$.■

Applications of the lemma-

1. Lemma 3.1 can now be used to prove the ***Hilbert space projection theorem***.

For M a closed subspace in H it follows that M is itself a Hilbert space for the norm and inner product inherited from H.

For y a fixed but arbitrary element in H, we can define

$$a(z,x) = (z,x)_H \qquad \forall x,z \in M$$

$$F(z) = (y,z)_H \qquad \forall z \in M,$$

and $$\Phi(z) = \tfrac{1}{2} \, a(z,z) - F(z) + \tfrac{1}{2} \| y \|_H^2 \qquad \forall z \in M.$$

Note that $\forall x \in M$

$$\tfrac{1}{2} \| x - y \|_H^2 = \tfrac{1}{2} (x - y, x - y)_H$$

$$= \tfrac{1}{2} \{ \| x \|_H^2 - 2(x,y)_H + \| y \|_H^2 \} \; = \Phi(x)$$

Clearly $a(z,x)$ is a positive, bounded and symmetric bilinear form on M, F is a bounded linear functional on M. Then it follows from the lemma that there exists a unique element $x_y \in M$ which minimizes $\Phi(z)$ over M. It follows also form the equivalence of problems (a) and (b) that x_y satisfies

$$a(x_y, z) = F(z), \quad \forall z \in M; \text{ i.e.,} \quad (x_y, z)_H = (y,z)_H \quad \forall z \in M.$$

But this is just the assertion that

$$(x_y - y, z)_H = 0 \qquad \forall z \in M; \text{ i.e., } \quad x_y - y \perp M.$$

Finally, for y in H, fixed, let the unique element x_y in M be denoted by $Py = x_y \in M$. Then

$$y - Py \perp M, \text{ and } \; z = y - Py \in M^{\perp}.$$

To see that this decomposition of elements of H is unique, suppose

$$y = x_y + z, \quad x_y \in M, \qquad z \in M^{\perp},$$

and
$$y = X_y + Z, \quad X_y \in M, \qquad Z \in M^{\perp},$$

Then $\quad x_y + z = X_y + Z$, and $x_y - X_y = Z - z$.
But $\quad x_y - X_y \in M, \quad Z - z \in M^{\perp}, \quad M \cap M^{\perp} = \{0\}$,
and it follows that $\quad x_y - X_y = Z - z = 0.$ ■

2. Recall that for U open and bounded in R^n, the Hilbert Sobolev space of order one, $H^1(U) = H$ is a Hilbert space which contains $C^{\infty}(U)$ as a dense subspace and also contains the closed subspace $H_0^1(U)$, obtained by completing the subspace of test functions in the H^1-norm. We showed in an earlier example, that $H_0^1(U)$ is not equal to $H^1(U)$. Then by the projection theorem, every y in H can be uniquely expressed as a sum, $y = x_y + z$, with $x_y \in H_0^1(U)$, and $z \in (H_0^1(U))^{\perp}$. To characterize the subspace $(H_0^1(U))^{\perp}$, choose arbitrary $\phi \in C_0^{\infty}(U)$ and $\psi \in C^{\infty}(U)$ and write

$$(\phi,\psi)_H = \int_U [\phi\psi + \nabla\phi \bullet \nabla\psi]\,dx = \int_U \phi[\psi - \nabla^2\psi]dx + \int_{\partial U} \phi\, \partial_N \psi\, dS$$

$$= (\phi, \psi - \nabla^2\psi)_0 + 0. \text{ (Here } (u,v)_0 \text{ denotes the } H^0(U) = L_2(U) \text{ inner product).}$$

Now suppose $\psi \in C^{\infty}(U) \cap (H_0^1(U))^{\perp}$. Then $(\phi,\psi)_H = 0$, for all $\phi \in C_0^{\infty}(U)$, and since $C_0^{\infty}(U)$ is dense in $H_0^1(U)$, $(u,\psi)_H = 0$, for all $u \in H_0^1(U)$. That is, $(u, \psi - \nabla^2\psi)_0 = 0 \quad \forall\, u \in H_0^1(U)$. But this implies that $\psi \in C^{\infty}(U) \cap (H_0^1(U))^{\perp}$ satisfies $\psi - \nabla^2\psi = 0$, in $H^0(U)$. Then, since $C^{\infty}(U)$ is dense in $H = H^1(U)$ it follows that

$$(H_0^1(U))^{\perp} = \{z \in H^1(U) : z - \nabla^2 z \in H^0(U), \text{ and } z - \nabla^2 z = 0\}.$$

The Lax-Milgram Lemma

Lemma 3.1 requires that the bilinear form $a(x,y)$ be symmetric. For application to existence theorems for partial differential equations, this is an unacceptable restriction. Fortunately, the most important part of the result remains true even when the form is not symmetric.

Lax-Milgram Lemma- Suppose $a(u,v)$ is a bounded and positive bilinear form on Hilbert space H; i.e., for positive constants a_0, a_1

i) $|a(u,v)| \leq a_1 \|u\|_H \|v\|_H \qquad \forall u,v \in H$
ii) $a(u,u) \geq a_0 \|u\|_H^2 \qquad \forall u \in H.$

Suppose also that $F(v)$ is a bounded linear functional on H. Then there exists a unique $u_F \in H$ such that

$$a(u_F, v) = F(v) \quad \forall v \in H.$$

Proof- For each fixed $u \in H$, the mapping $v \rightsquigarrow a(u,v)$ is a bounded linear functional on H. It follows that there exists a unique $z_u \in H$ such that

$$a(u,v) = (z_u, v)_H \quad \forall v \in H.$$

Let $Au = z_u$; i.e., $a(u,v) = (Au,v)_H \;\; \forall u \in H.$ Clearly A is a linear mapping of H into H, and since

$$\|Au\|_H^2 = |(Au,Au)_H| = |a(u,Au)| \leq a_1 \|u\|_H \|Au\|_H$$

it is evident that A is also bounded. Note further, that

$$a_0 \|u\|_H^2 \leq a(u,u) = (Au,u)_H \leq \|Au\|_H \|u\|_H$$

i.e., $\quad a_0 \|u\|_H \leq \|Au\|_H \;\; \forall u \in H.$

This estimate implies that A is one-to one and that R_A, the range of A, is closed in H. Finally, we will show that $R_A = H$. Since the range is closed, we can use the projection theorem to write, $H = R_A \oplus R_A^{\perp}$. If $u \in R_A^{\perp}$, then

$$0 = (Au,u)_H = a(u,u) \geq a_0 \|u\|_H^2; \quad \text{i.e., } R_A^{\perp} = \{0\}.$$

Since $F(v)$ is a bounded linear functional on H, it follows from the Riesz theorem that there is a unique $z_F \in H$ such that $F(v) = (z_F, v)_H$ for all $v \in H$. Then the equation $a(u,v) = F(v)$ can be expressed as

$$(Au,v)_H = (z_F,v)_H \;\; \forall v \in H; \text{ i.e., } Au = z_F.$$

But A has been seen to be one-to-one and onto and it follows that there exists a unique $u_F \in H$ such that $Au_F = z_F$.■

Convergence in H

In R^N convergence of x_n to x means

$$\|x_n - x\|_{R^N} = \left[\sum_{i=1}^{N}[(x_n - x) \bullet e_i]^2\right]^{1/2} \to 0 \text{ as } n \to \infty.$$

Here e_i denotes the i-th vector in the standard basis.This is equivalent to,

$$(x_n - x) \bullet e_i \to 0 \text{ as } n \to \infty, \text{ for } i = 1, ..., N,$$

and to $\quad (x_n - x) \bullet z \to 0 \text{ as } n \to \infty$, for every $z \in R^N$

In an infinite dimensional Hilbert space H, convergence of x_n to $x\ in\ H$ means

$$\|x_n - x\|_H \to 0 \text{ as } n \to \infty.$$

This is called ***strong convergence*** in H and it implies that

$$(x_n - x, v)_H \to 0 \text{ as } n \to \infty \quad \forall v \in H.$$

This last mode of convergence is referred to as ***weak convergence*** and, in a general Hilbert space, weak convergence does not imply strong convergence. Thus while there is no distinction between weak and strong convergence in a finite dimensional space, the two notions of convergence are not the same in a space of infinite dimensions.

In R^N the so called Bolzano-Weierstrass theorem asserts that every bounded sequence $\{x_n\}$ contains a convergent subsequence. The theorem is proved by noting that $\{x_n \bullet e_1\}$ is a bounded sequence of real numbers and hence contains a subsequence $\{x_{n,1} \bullet e_1\}$ that is convergent. Similarly, $\{x_{n,1} \bullet e_2\}$ is also a bounded sequence of real numbers and thus contains a subsequence $\{x_{n,2} \bullet e_2\}$ that is convergent. Proceeding in this way, we can generate a sequence of subsequences, $\{x_{n,k}\} \subset \{x_n\}$ such that $\{x_{n,k} \bullet e_j\}$ is convergent for $j \leq k$. Then the diagonal sequence $\{x_{n,n}\}$ is such that $\{\leq x_{n,n} \bullet ej\}$ is convergent for $1 \leq j \leq N$, which is to say, $\{x_{n,n}\}$ is convergent. In a general Hilbert space we have a weaker result,

In H every bounded sequence $\{x_n\}$ contains a subsequence that is *weakly* convergent.

To see this, suppose that $\|x_n\| \leq M$ for all n and let $\{\phi_j\}$ denote a complete orthonormal family in H. Proceeding as we did in R^N, let $\{x_{n,k}\} \subset \{x_n\}$ denote a subsequence such that $\{(x_{n,k}, \phi_j)_H\}$ is convergent (in R) for $j \leq k$. Then for each j, $(x_{n,j}, \phi_j)_H$ converges to a real limit a_j as n tends to infinity. It follows that the diagonal subsequence $\{x_{n,n}\}$ is such that $(x_{n,n}, \phi_j)_H$ converges to a_j for $j \geq 1$. Now define

$$F(v) = Lim_n\ (x_{n,n}, v)_H \quad for\ \ v \in H.$$

Then $\quad |F(v)| \leq |Lim_n(x_{n,n}, v)_H| \leq M\|v\|_H$

from which it follows that F is a continuous linear functional on H. By the Riesz theorem, there exists an element, z_F in H such that

$$F(v) = (z_F, v)_H \quad \text{for all v in H.}$$

But $$F(v) = F\left(\sum_i (v,\phi_i)_H \phi_i\right) = \lim_n (x_{n,n}, \sum_i (v,\phi_i)_H \phi_i)_H$$
$$= \sum_i \lim_n (x_{n,n}, \phi_i)_H (v,\phi_i)_H = \sum_i a_i (v,\phi_i)_H \;;$$

That is, $$F(v) = (z_F, v)_H = \sum_i a_i (v,\phi_i)_H \quad \text{for all v in H.}$$

Then by the Parseval-Plancherel identity, it follows that

$$z_F = \sum_i a_i \phi_i$$

and

$$(x_{n,n}, v)_H \rightarrow (z_F, v)_H \quad \text{for all v in H.}$$