Introduction to the Theory of Distributions

Generalized functions or distributions are a generalization of the notion of a function defined on R^n. Distributions are more general than the usual notion of pointwise defined functions and they are even more general than $L_p(U)$ functions. Distributions have many convenient properties with respect to the operations of analysis but in return for these convenient properties we must give up the notion of a function that assumes it values pointwise or even pointwise almost everywhere. Instead these functions assume their values only is a locally averaged sense (to be made precise later). This point of view is consistent, however, with many physical applications and makes possible a coherent description of such things as impulsive forces in mechanics and poles and dipoles in electromagnetic theory. Mathematically these objects are Dirac deltas and its derivatives and these may be accomodated within the theory of distributions.

This development distributions will be based on the notion of duality. We begin with a linear space X that is contained in an inner product space, H, whose inner product we denote by $(\cdot,\cdot)_H$. Now suppose that X is equipped with a notion of convergence that is stronger than the one in H associated with its inner product; i.e., if $\{x_n\}$ is a sequence in X that converges in X then $\{x_n\}$, viewed as a sequence in H, is still convergent. This is the same thing as saying X is continuously embedded in H. Then for each x in X and every y in H, $(x,y)_H$ is a real number and if $x_n \to x \ in\ X$, then $(x_n,y)_H \to (x,y)_H$ for every y in H. If X is properly contained in H (i.e., X is not equal to H) then it may be that

$$x_n \to x,\ in\ X \quad \text{implies} \quad (x_n,y)_H \to (x,y)_H \quad \text{for every y in X'}$$

where X' is a linear space containing H but larger than H. In particular, if we choose

$$X = C_0^\infty(U), \quad H = L^2(U) \quad and \quad (u,v)_H = \int_U u(x)\,v(x)\,dx$$

then, since X is very much smaller than H, the space X', the space of distributions on U, is very much larger than H. We will now make these remarks precise.

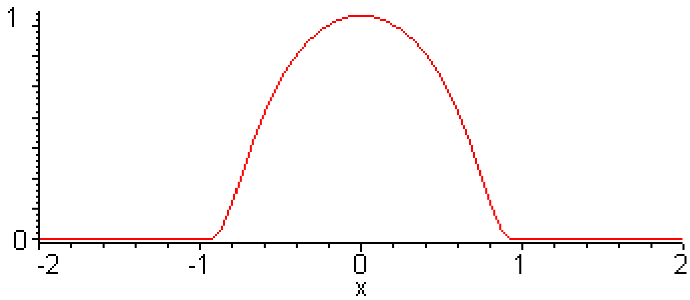
1. Test Functions

In order to compute the locally averaged values of a distribution, the notion of a test function is required. A function $f(x)$ defined on an open set $U \subset R^n$ is said to have ***compact support*** if $f(x) = 0$ for x in the complement of a compact subset of U. In particular, if $U = R^n$, then f has compact support if there is a positive constant, C such that $f(x) = 0\ for\ |x| > C$. We say that $f(x)$ is a ***test function*** if f has compact support and, in addition, f is infinitely differentiable on U. We use the notation $f \in C_c^\infty(U)\ or\ f \in D(U)$ to indicate that f is a test function on U.

The function

$$T(x) = \begin{cases} K\exp\left(\frac{-1}{1-|x|^2}\right) & if\ |x| < 1 \\ 0 & if\ |x| \geq 1 \end{cases}, \qquad x \in R^n$$

where the constant K is chosen such that $\int_{R^n} T(x)dx = 1$, is a test function on R^n. For the case $n = 1$, this test function is pictured below



The Test Function T(x)

Note that $T(x)$ vanishes, together with all its derivatives as $|x| \to 1^-$, so $T(x)$ is infinitely differentiable and has compact support. .For $n = 1$ *and* $\varepsilon > 0$, let

$$S_\varepsilon(x) = \frac{1}{\varepsilon}T(\frac{x}{\varepsilon}) \qquad and \qquad P_\epsilon(x) = T(\frac{x}{\varepsilon}).$$

Then
$$S_\varepsilon(x) \geq 0 \qquad and \qquad P_\epsilon(x) \geq 0 \quad for\ all\ x$$

$$S_\varepsilon(x) = 0 \qquad and \qquad P_\epsilon(x) = 0 \quad for\ |x| > \varepsilon$$

$$\int_R S_\varepsilon(x)\,dx = 1 \qquad \forall\varepsilon > 0, \qquad S_\varepsilon(x) \to +\infty \quad as \quad \varepsilon \to 0,$$

$$\int_R P_\varepsilon(x)\,dx \to 0 \qquad as \quad \varepsilon \to 0, \qquad P_\varepsilon(0) = K/e \qquad \forall\varepsilon > 0,$$

Evidently, $S_\varepsilon(x)$ becomes thinner and higher as ε tends to zero but the area under the graph is constantly equal to one. On the other hand, $P_\epsilon(x)$ has constant height but grows thinner as ε tends to zero. These test functions can be used as the "seeds" from which an infinite variety of other test functions can be constructed by using a technique called regularization to be discussed later.

Convergence in the space of test functions

Clearly $D(U)$ is a linear space of functions but it turns out to be impossible to define a norm on the space. However, it will be sufficient to define the notion of convergence in this space. We say that the sequence $\{\phi_n\} \in D(U)$ ***converges to zero*** in $D(U)$ if

1. there exists a single compact set M in U such that the support of every ϕ_n is contained in M

2. for every multi-index, α, $\max_M |\partial_x^\alpha \phi_n| \to 0$ *as* $n \to \infty$ (ϕ_n and all its derivatives tend uniformly to zero on M)

Then $\{\phi_n\} \in D(U)$ is said to converge to ϕ in $D(U)$ if $\phi_n - \phi$ converges to zero in $D(U)$.

2. Functionals on the space of test functions

A real valued function defined on the space of test functions is called a ***functional*** on $D(U)$. Consider the following examples, where ϕ denotes an arbitrary test function:

$$J_1[\phi] = \phi(a) \qquad for \quad a \in U \quad (a\ fixed\ point\ in\ U)$$

$$J_2[\phi] = \int_V \phi(x)\,dx \quad for \;\; V \subset U \;(a \; fixed \; set)$$

$$J_3[\phi] = \phi'(a) \quad for \;\; a \in U \;\; (fixed)$$

$$J_4[\phi] = \phi'(a)\,\phi(a) \quad for \;\; a \in U \;\; (fixed).$$

Linear Functionals on $D(U)$

Each of the examples above is a real valued function defined on the space of test functions, i.e., a functional. A functional on D is said to be a ***linear functional on D*** if

$$\forall \phi, \psi \in D(U), \qquad J[C_1\phi + C_2\psi] = C_1 J[\phi] + C_2 J[\psi], \qquad \forall C_1, C_2 \in R$$

The functionals $J_1,\; J_2,\; and\; J_3$ are all linear, J_4 is not linear.

Continuous Linear Functionals

A linear functional J on D(U) is said to be ***continuous at zero*** if, for all sequences $\{\phi_n\} \in D(U)$ converging to zero in $D(U)$, we have $J[\phi_n]$ converging to zero as a sequence of real numbers; i.e.,

$$\phi_n \to 0 \;\; in \;\; D(U) \qquad implies \qquad J[\phi_n] \to 0 \;\; in \;\; R.$$

For linear functionals, continuity at any point is equivalent to continuity at every point.

Lemma 2.1 If J is a linear functional on D(U), then J is continuous at zero, if and only if J is continuous at every point in D(U); i.e.,

$$\phi_n \to 0 \;\; in \;\; D(U) \qquad implies \qquad J[\phi_n] \to 0 \;\; in \;\; R.$$

if and only if

$$\forall \phi \in D(U), \;\; \phi_n \to \phi \;\; in \;\; D(U) \;\; implies \;\; |J[\phi_n] - J[\phi]| \to 0 \;\; in \;\; R.$$

examples

1. $J_1[\phi] = \phi(a) \quad for \;\; a \in R \;\; (fixed)$ is a continuous linear functional on $D = D(R^1)$.

Let $\{\phi_n\}$ denote a sequence of test functions converging to zero in $D(R)$. Then there is a closed bounded interval M such that for every n, $\phi_n(x) = 0$ for x not in M. If a is not in M then $J_1[\phi_n] = \phi_n(a) = 0$ for every n so $|J[\phi_n] - J[0]| = 0 \to 0 \;\; in \;\; R.$ in this case. On the other hand, if $a \in M$, then $|J[\phi_n] - J[0]| = |\phi_n(a)| \le \max_M |\phi_n(x)| \to 0 \;\; as \;\; n \to \infty$ so we have convergence in this case as well.

2. $J_2[\phi] = \int_V \phi(x)\,dx \quad for \;\; V \subset U$ is a continuous linear functional on $D(U)$

Let $\{\phi_n\}$ denote a sequence of test functions converging to zero in $D(U)$. Then there is a compact set $M \subset U$ such that for every n, $\phi_n(x) = 0$ for x not in M. Let $W = V \cap M.$ If W is empty, then $J_2[\phi_n] = 0$ for every n so we have convergence in this case. If W is not empty then

$$|J_2[\phi_n]| = |\int_W \phi_n(x)\,dx| \le \max_W \; |\phi_n| \; |W| \le \max_M \; |\phi_n| \; |W| \to 0 \;\; as \;\; n \to \infty$$

and we have convergence in this case as well.

3. Distributions

A continuous linear functional on the space of test functions $D(U)$ will be called a ***distribution on U***. We will denote the set of all distributions on U by $D'(U)$, or, when U is the whole space, by D'. We will soon see the connection between distributions and generalized functions.

If we define $(C_1 J_1 + C_2 J_2)[\phi]$ to equal $C_1 J_1[\phi] + C_2 J_2[\phi]$ for all test functions ϕ, then $D'(U)$ is a linear space.

Lemma 3.1 $D'(U)$ is a linear space over the reals.

For ϕ a test function in $D(U)$, and J a distribution on U, we will use the notations $J[\phi] = \langle J, \phi \rangle$ interchangeably to denote the value of J acting on the test function ϕ, and we refer to this as the ***action of J***. Although J is evaluated at functions in D rather than at points in U, we will still be able to show that distributions can be interpreted as a generalization of the usual pointwise notion of functions.

Locally Integrable Functions

A function $f(x)$ defined on U is said to be ***locally integrable*** if, for every compact subset $M \subset U$, there exists a constant $K = K(M,f)$ such that

$$\int_M |f(x)|\, dx = K < \infty$$

We indicate this by writing $f \in L_1^{loc}(U)$. Note that when U is R^n then $f \in L_1^{loc}(R^n)$ need not be absolutely integrable, it only needs to be integrable on every compact set. This means, for example, that all polynomials are in $L_1^{loc}(U)$ although polynomials are certainly not integrable on R^n.

Lemma 3.2 For $f \in L_1^{loc}$, define $J_f[\phi] = \int_U f(x)\,\phi(x)\,dx \quad for\ \phi \in D(U).$

Then

1. $J_f \in D'(U).$
2. For $f, g \in L_1^{loc}\ such\ that\ \ J_f[\phi] = J_g[\phi], \quad for\ all\ \phi \in D(U), \quad f = g\ a.e.$

Proof- To show 1, let $\{\phi_n\}$ denote a sequence of test functions converging to zero in $D(U)$. Then there is a compact set $M \subset U$ such that for every n, $\phi_n(x) = 0$ for x not in M. For $f \in L_1^{loc}$, and

$$J_f[\phi] = \int_U f(x)\,\phi(x)\,dx \qquad for\ \phi \in D(U),$$

we have

$$|J_f[\phi_n]| = |\int_U f(x)\,\phi_n(x)\,dx| = |\int_M f(x)\,\phi_n(x)\,dx| \leq \max_M|\phi_n(x)|\,|\int_M f(x)\,dx| = K\,\max_M|\phi_n(x)| \to 0\ as\ n \to$$

To show 2, note that for $f, g \in L_1^{loc}\ such\ that\ \ J_f[\phi] = J_g[\phi], \quad for\ all\ \phi \in D(U),$ we have

$$\int_U [f(x) - g(x)]\,\phi(x)\,dx = 0 \qquad for\ all\ \phi \in D(U).$$

Since the test functions are dense in L_1^{loc}, it follows from the last equality that $\int_U |f(x) - g(x)|\,dx = 0;\ i.e.,\quad f = g\ a.e.$ ■

Regular and Singular Distributions

Lemma 3.2(2) asserts that the mapping which associates a function from $L_1^{loc}(U)$ with a distribution J is one to one. Then $L_1^{loc}(U)$ can be treated as a subspace of $D'(U)$, the space of distributions. We call this subspace, the subspace of ***regular distributions*** and we say that each regular distribution is generated by a unique locally integrable function; i.e. for each regular distribution J_f there is a unique $f \in L_1^{loc}$ such that

$$J_f[\phi] = \int_U f(x)\,\phi(x)\,dx \qquad for\ \phi \in D(U).$$

Of course not all distributions are regular distributions. Any distribution that is not a regular distribution is called a ***singular distribution***.

Example 3.1-

1. Let $$H_b(x) = \begin{Bmatrix} 1 & if & x > b \\ 0 & if & x < b \end{Bmatrix}$$

Then clearly, $H_b \in L_1^{loc}(R^1)$ and $J_H[\phi] = \int_b^\infty \phi(x)\,dx$ is the regular distribution generated by this locally integrable function. The function $H_0(x)$ is called the ***Heaviside step function***.

2. The distribution $J[\phi] = \phi(0) \in D'(R)$ is a singular distribution. To see that J cannot be generated by any locally integrable function, suppose there exists a $\delta \in L_1^{loc}(R^1)$ such that

$$J[\phi] = \int_R \delta(x)\,\phi(x)\,dx = \phi(0) \qquad for\ all\ \phi \in D(R)$$

For $\varepsilon > 0$ choose $\phi(x) = P_\varepsilon(x)$. Then

$$|J[P_\varepsilon(x)]| = |\int_R \delta(x)\,P_\varepsilon(x)\,dx| \le P_\varepsilon(0)\left|\int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx\right|.$$

The dominated convergence theorem implies that if $\delta \in L_1^{loc}(R^1)$, then

$$\left|\int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx\right|. \to 0\ as\ \varepsilon \to 0.$$

That is,

$$0 < P_\varepsilon(0) = |J[P_\varepsilon(x)]| \le P_\varepsilon(0)\left|\int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx\right| \to 0 \qquad as\ \varepsilon \to 0.$$

This contradiction shows that there can be no locally integrable function that generates this distribution. In spite of this fact, we will nevertheless write

$$\int_R \delta(x)\,\phi(x)\,dx = \phi(0) \qquad for\ all\ \phi \in D(R)$$

and speak of $\delta(x)$ as the generalized function associated with the distribution J. We use the terms generalized function and distribution interchangeably and we often operate on $\delta(x)$ as if it were a function but we must realize that this is just formal notation and that $\delta(x)$ is not a function in the usual sense. We refer to $\delta(x)$ as the ***Dirac delta function*** concentrated at the origin. More generally, the distribution $J[\phi] = \phi(a) \quad for\ all\ \phi \in D(R),$ is written as

$$\int_R \delta(x-a)\,\phi(x)\,dx = \phi(a) \qquad for\ all\ \phi \in D(R)$$

and is called the Dirac delta function concentrated at $x = a.$ We refer to $J[\phi] = \phi(a) \quad for\ all\ \phi \in D(R)$ as the "action" of the distribution and we refer to $\delta(x-a)$ as the generalized function associated with the distribution having this action. Although we will often write equations like

$$\nabla^2 G(x) = \delta(x-a), \qquad x \in U,$$

all we mean by this is that

$$\int_R \nabla^2 G(x)\,\phi(x)\,dx = \phi(a) \qquad for\ all\ \phi \in D(R).$$

Problem 1 Each of the following distributions is defined by its action. Identify the generalized function for each of them.

(a) $J_1[\phi] = a^2\phi(a)$ (b) $J_2[\phi] = -e^{-b}(\phi(b) + \phi'(b))$

(c) $J_3[\phi] = \int_a^b x^2\phi(x)\,dx$ (d) $J_4[\phi] = \int_a^b \phi(x)\,dx$

Problem 2 Describe the action on test functions for the following generalized functions.

(a) $F_1(x) = x\delta(x)$ (b) $F_2(x) = x\ \delta'(x-a)$

(c) $F_3(x) = \delta(3x)$ (d) $F_4(x) = (H_0(x) - H_0(x-1))\sin(\pi x)$

Equality and Values on an Open Set

Although we are not entitled to treat generalized functions as if they have pointwise values, we are still able to talk about distributions that are zero on certain sets as well as defining what is meant by equality of distributions on open sets.

Distributions $J_1\ and\ J_2$ are said to be ***equal in the sense of distributions*** if

$$J_1[\phi] = J_2[\phi] \quad for\ all\ \phi \in D(U);$$

If $F_1\ and\ F_2$ are generalized functions, then $F_1\ and\ F_2$ are said to be equal if

$$\int_U F_1(x)\,\phi(x)\,dx = \int_U F_2(x)\,\phi(x)\,dx \qquad for\ all\ \phi \in D(U).$$

The support of a test function $\phi(x)$ is defined as the smallest closed set K such that $\phi(x)$ is identically zero off K. Then we can say that a distribution J vanishes on an open set, Ω, if $J[\phi] = 0$ for all test functions ϕ having support in Ω. For example, $\delta(x)$ vanishes on the open sets $(-\infty, 0)\ and\ (0, \infty)$. Two distributions, $J_1\ and\ J_2$, can be said to be ***equal on an open set***, Ω if $J_1 - J_2$ vanishes on the open set Ω.

We define the ***support of a distribution*** J to be the complement of the largest open set where J is zero. Then the support of the Dirac delta is just the point $\{0\}$, and any regular distribution generated by a function having compact support, K, is a distribution whose support is equal to K.

Differentiation of Distributions

Let J_f denote the regular distribution generated by the locally integrable function of one variable, $f(x)$ and consider the distribution generated by the derivative $f'(x)$. The action of this distribution is described by

$$\int_R f'(x)\,\phi(x)dx = f(x)\phi(x)\,|_{x=-\infty}^{x=\infty} - \int_R f(x)\,\phi'(x)dx = 0 + \langle J_f, -\phi'\rangle \qquad \forall\phi \in D(R)$$

Here we integrated by parts (formally) and used the fact that a test function has compact support to make the boundary term vanish. It is evident that the derivative generates a distribution $J_{f'}$ whose action on the test function $\phi(x)$ is the same as the action of J_f on $-\phi'(x)$. This motivates the following definition.

distributional derivative For any $J \in D'(R)$, the derivative $dJ/dx \in D'(R)$ is the distribution whose action on test function $\phi(x)$ is given by $dJ/dx[\phi] = J[-\phi']$; $i.e.,$

$$\langle dJ/dx, \phi(x)\rangle = \langle J, -\phi'(x)\rangle \qquad \forall\phi \in D(R)$$

Equivalently, the distribution J_f generated by the generalized function $f(x)$ has as its distributional derivative, the distribution $J_{f'}$ generated by $f'(x)$. For higher order derivatives, we have

$$\langle d^kJ/dx^k, \phi(x)\rangle = (-1)^k\,\langle J, \phi^{(k)}(x)\rangle \qquad \forall\phi \in D(R).$$

More generally, for any $J \in D'(U)$, the derivative $\partial_x^\alpha J \in D'(U)$ is the distribution whose action on test function $\phi(x)$ is given by $\partial_x^\alpha J[\phi] = J[(-1)^{|\alpha|}\partial_x^\alpha\phi]$; $i.e.,$

$$\langle \partial_x^\alpha J, \phi(x)\rangle = (-1)^{|\alpha|}\langle J, \partial_x^\alpha\phi(x)\rangle \qquad \forall\phi \in D(R).$$

Here $\alpha = \alpha_1 \ldots \alpha_n$ denotes a multi-index , hence $\partial_x^\alpha J = (\partial_{x_1}^{\alpha_1})\ldots(\partial_{x_n}^{\alpha_n})J,\ and\ |\alpha| = \alpha_1 + \ldots + \alpha_n.$

Equality of Mixed Partials

Suppose J is a distribution on open set $U \subset R^2$. Then $\partial_{xy}J = \partial_{yx}J$. To see this write

$$\langle \partial_{xy}J, \phi(x,y)\rangle = (-1)^2\langle J, \partial_{xy}\phi(x,y)\rangle \qquad \forall\phi \in D(R).$$
$$\langle \partial_{yx}J, \phi(x,y)\rangle = (-1)^2\langle J, \partial_{yx}\phi(x,y)\rangle \qquad \forall\phi \in D(R).$$

But for a test function, $\partial_{yx}\phi(x,y) = \partial_{xy}\phi(x,y)$, hence $\partial_{xy}J = \partial_{yx}J$. More generally, every distribution is infinitely differentiable and the values of derivatives of higher order are independent of the order of differentiation.

Example 3.2-

1. Let J_H denote the distribution generated by the locally integrable function $H_0(x)$. Then

$$\langle dJ_H/dx, \phi(x)\rangle = \langle J_H, -\phi'(x)\rangle = \int_0^\infty -\phi'(x)\,dx \qquad \forall\phi \in D(R)$$

i.e.,

$$\int_0^\infty -\phi'(x)\,dx = -\phi(x)\,|_{x=0}^{x=\infty} = \phi(0).$$

Then

$$\langle dJ_H/dx, \phi(x)\rangle = \phi(0). \qquad \forall\phi \in D(R),$$

and since $\langle \delta(x), \phi(x) \rangle = \phi(0), \quad \forall \phi \in D(R),$ we conclude that $dH_0(x)/dx = \delta(x)$ in the sense of distributions.

2. Consider the discontinuous but locally integrable function

$$f(x) = \left\{ \begin{array}{ll} x^2 + x & if \quad x < 1 \\ e^{-5x} & if \quad x > 1 \end{array} \right\}$$

Then $\qquad \langle dJ_f/dx, \phi(x) \rangle = \langle J_f, -\phi'(x) \rangle = -\int_{-\infty}^{1} (x^2 + x)\phi'(x)\,dx - \int_1^{\infty} e^{-5x}\phi'(x)\,dx \qquad \forall \phi \in D(R)$

and $\qquad -\int_{-\infty}^{1} (x^2 + x)\phi'(x)\,dx = -(x^2 + x)\phi(x)|_{x=-\infty}^{x=1} + \int_{-\infty}^{1} (2x + 1)\phi(x)\,dx$

$$= -2\phi(1) + \int_{-\infty}^{1} (2x + 1)\phi(x)\,dx$$

$$-\int_1^{\infty} e^{-5x}\phi'(x)\,dx = -e^{-5x}\phi(x)|_{x=1}^{x=\infty} + \int_1^{\infty} -5e^{-5x}\phi(x)\,dx = e^{-5}\phi(1) - \int_1^{\infty} 5e^{-5x}\phi(x)\,dx.$$

It follows that

$$\begin{aligned} \langle dJ_f/dx, \phi(x) \rangle &= -2\phi(1) + \int_{-\infty}^{1} (2x + 1)\phi(x)\,dx + e^{-5}\phi(1) - \int_1^{\infty} 5e^{-5x}\phi(x)\,dx \\ &= [e^{-5} - 2]\phi(1) + \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx. \\ &= \ < [f(1+) - f(1-)]\,\delta(x-1), \phi(x) > + \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx, \end{aligned}$$

where

$$f'(x) = \left\{ \begin{array}{ll} 2x + 1 & if \quad x < 1 \\ -5e^{-5x} & if \quad x > 1 \end{array} \right\}.$$

In general, if $f(x)$ is piecewise differentiable with a jump discontinuity at $x = x_0$, then the distributional derivative of $f(x)$ is given by

$$df/dx = \Delta f(x_0)\,\delta(x - x_0) + f'(x)$$

where $\Delta f(x_0) = f(x_0+) - f(x_0-),$ *and* $f'(x)$ denotes the derivative in the classical sense at all points where this derivative exists.

3. Consider the function

$$f(x) = \left\{ \begin{array}{ll} |x| & if \quad |x| < 1 \\ 0 & if \quad |x| > 1 \end{array} \right\}$$

This function, which is clearly not differentiable in the classical sense, was shown earlier not to have a derivative even in the $L_2 - sense.$ This function can be treated as in the previous example to obtain

$$\begin{aligned} df/dx &= \Delta f(-1)\,\delta(x+1) + \Delta f(1)\,\delta(x-1) + f'(x) \\ &= \delta(x+1) - \delta(x-1) + f'(x) \end{aligned}$$

where

$$f'(x) = \left\{ \begin{array}{lll} -1 & if & -1 < x < 0 \\ +1 & if & 0 < x < 1 \\ 0 & if & |x| > 1 \end{array} \right\}$$

Alternatively, using the definition of distributional derivative leads to

$$\langle dJ_f/dx, \phi(x)\rangle = \langle J_f, -\phi'(x)\rangle = \int_{-1}^{0} x\phi'(x)\,dx - \int_{0}^{1} x\phi'(x)\,dx \qquad \forall\phi \in D(R)$$

$$= x\phi(x)|_{x=-1}^{x=0} - \int_{-1}^{0} \phi(x)\,dx - x\phi(x)|_{x=0}^{x=1} + \int_{0}^{1} \phi(x)\,dx$$

i.e., $$\int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx = \phi(-1) - \phi(1) + \int_{-1}^{1} sgn(x)\,\phi(x)\,dx, \qquad \forall\phi \in D(R).$$

This implies $f'(x) = \delta(x+1) - \delta(x-1) + I_1(x)\,sgn(x)$ Of courser this is the derivative in the distributional sense.

4. Recall that the initial value problem

$$\partial_t u(x,t) + a\partial_x u(x,t) = 0, \qquad x \in R, \quad t > 0$$
$$u(x,0) = f(x)$$

has the solution $u(x,t) = f(x - at)$. In the case that the function $f(x)$ is not sufficiently regular for the existence of a classical solution, (i.e., if f is not at least differentiable in the classical sense) we can say that u is a solution in the sense of distributions if, for every test function $\phi(x,t) \in D(R \times R_+)$,

$$\langle \partial_t u(x,t) + a\partial_x u(x,t), \phi(x,t)\rangle = 0.$$

e.g., suppose $f(x) = I_1(x)$ so that $f'(x) = \delta(x+1) - \delta(x-1) + 0$ and then

$$(\partial_t + a\partial_x)f(x - at) =$$

$$= [\delta(x - at + 1) - \delta(x - at - 1) + 0](-a) + a[\delta(x - at + 1) - \delta(x - at - 1) + 0] = 0.$$

This is the same thing as the argument,

$$\langle \partial_t u(x,t) + a\partial_x u(x,t), \phi(x,t)\rangle$$

$$= (-a)\langle \delta(x - at + 1) - \delta(x - at - 1), \phi(x,t)\rangle + a\langle \delta(x - at + 1) - \delta(x - at - 1), \phi(x,t)\rangle$$

$$= (-a)(\phi(at - 1, t) - \phi(at + 1)) + a(\phi(at - 1, t) - \phi(at + 1)) = 0.$$

Note that $\langle \delta(x - at - 1), \phi(x,t)\rangle = \phi(at + 1, t)$ is a special case of $\langle \delta(\psi(x,t)), \phi(x,t)\rangle = \phi(x(t), t) \;\; or \;\; \phi(x, t(x)),$ a result which is only true if $\psi(x,t) = C$ implies one of the variables x, t can be expressed as a C^∞ function of the other, $x = x(t) \;\; or \;\; t = t(x)$. In general, this type of composition is not defined. In this case, however, the verification that $u(x,t) = f(x - at)$ solves the initial value problem is formally the same as in the case where f is smooth. For this discontinuous function f, however, the steps showing that $u(x,t)$ solves the problem can only be justified within the framework of distribution theory.

The Hilbert-Sobolev Space of Order One

We are going to define a new function space which will be of use in the weak formulation of

PDE's. For $U \subset R^n$, we define

$$H^1(U) = \left\{u(x) \in L_2(U) : \ \partial_{x_i} u(x) \in L_2(U) \ for \ \ i = 1,\ldots,n\right\}$$

Here $\partial_{x_i} u(x)$ denotes the distributional derivative of $u(x)$ with respect to x_i. We define an inner product on $H^1(U)$ as follows,

$$(u,v)_1 = \int_U [u(x)v(x) + \nabla u \cdot \nabla v]\, dx \qquad for \ \ u,v \in H^1(U).$$

That this is an inner product is simple to check. That $H^1(U)$ is complete for the norm induced by the inner product must be proved.

Proposition 3.3 $H^1(U)$ is a Hilbert space for the norm $\|u\|_1 = (u,u)_1^{1/2}$

proof- Suppose $\{u_n\} \in H^1(U)$ is a Cauchy sequence for the norm $\|u\|_1$. This means $\|u_m - u_n\|_1 \to 0 \ \ as \ \ m,n \to \infty$, which is the same thing as saying

$$\|u_m - u_n\|_{L_2} \to 0 \ \ and \ \ \|\partial_{x_j}(u_m - u_n)\|_{L_2} \to 0 \quad for \ each \ j, \qquad as \ \ m,n \to \infty.$$

But this means that each of the (n+1) sequences $\{u_m\},\{\partial_{x_1} u_m\},\ldots,\{\partial_{x_n} u_m\}$ is convergent in the complete space $L_2(U)$ to limits $\tilde{u}_0,\ \tilde{u}_1,\ldots,\ \tilde{u}_n$, respectively. That is,

$$\|u_m - \tilde{u}_0\|_{L_2} \to 0 \ \ and \ \ \|\partial_{x_j} u_m - \tilde{u}_j\|_{L_2} \to 0 \quad for \ j = 1,\ldots,n \qquad as \ \ m \to \infty.$$

and it only remains to show that for each j, $\tilde{u}_j = \partial_{x_j}\tilde{u}_0$. This will prove that $\|u_m - \tilde{u}_0\|_1 \to 0$ and, for each j, the distributional derivative $\partial_{x_j}\tilde{u}_0 \in L_2(U)$ so $\tilde{u}_0 \in H^1(U)$. To show that $\tilde{u}_j = \partial_{x_j}\tilde{u}_0$, note that the continuity of the inner product implies that for each j,

$$\begin{array}{lcl} \int_U \partial_{x_j} u_m \phi(x)\, dx = -\int_U u_m\, \partial_{x_j}\phi(x)\, dx & & \forall \phi \in D(U) \\ \quad\downarrow \qquad\qquad\qquad\qquad \downarrow \quad as\ m \to \infty & & \\ \int_U \tilde{u}_j(x)\phi(x)\, dx = -\int_U \tilde{u}_0(x)\, \partial_{x_j}\phi(x)\, dx & & \forall \phi \in D(U) \end{array}$$

Then $$-\int_U \tilde{u}_0(x)\, \partial_{x_j}\phi(x)\, dx = \int_U \partial_{x_j}\tilde{u}_0(x)\phi(x)\, dx = \int_U \tilde{u}_j(x)\phi(x)\, dx \qquad \forall \phi \in D(U).\blacksquare$$

Convergence in the Space of Distributions

For reasons that are beyond the scope of this course, it is not possible to define a norm on the space of distributions. However, it is possible to define the notion of convergence, which is sufficient for our purposes anyway. We say that a sequence $\{T_n\}$ of distributions is c***onvergent in the sense of distributions*** if $T_n[\phi]$ is a convergent sequence of reals for every test function ϕ.

The following theorem may appear to be self evident but it isn't. It is known as the Banach-Steinhaus theorem.

Theorem 3.4 Suppose $\{T_n\}$ is a convergent sequence of distributions and define

$$T[\phi] = Lim_{n\to\infty} T_n[\phi] \quad \text{for all test functions } \phi$$

Then T is a continuous linear functional on the space of test functions; i.e., T is a distribution.

Delta Sequences

One type of distributional sequence we are particularly interested in are the so called ***delta sequences***.

Theorem 3.5 Suppose $\{f_n\}$ is a sequence of locally integrable functions such that for suitable constants M and C, we have

1. $f_n(x) = 0 \quad for \quad |x| > \frac{M}{n}$
2. $\int_R f_n(x)\,dx = 1 \quad \forall n$
3. $\int_R |f_n(x)|\,dx \le C \quad \forall n$

Then the sequence $\{J_n\}$ of distributions generated by the functions $f_n(x)$ converge to $\delta(x)$ in the sense of distributions.

Proof-Suppose $\{f_n\}$ is a sequence of locally integrable functions with properties 1,2, and 3 and let $\{J_n\}$ denote the sequence of distributions generated by the functions $f_n(x)$. For an arbitrary test function $\phi(x)$, it follows from 2 that

$$J_n[\phi(x)] = \int_R f_n(x)\,\phi(x)\,dx = \phi(0) + \int_R f_n(x)\,\phi(x)\,dx - \phi(0)\int_R f_n(x)\,dx$$

and

$$J_n[\phi(x)] - \phi(0) = \int_R f_n(x)\,[\phi(x) - \phi(0)]\,dx.$$

Now properties 1 and 3 show that

$$|J_n[\phi(x)] - \phi(0)| \le \max\{|\phi(x) - \phi(0)| : |x| < \tfrac{M}{n}\}\int_R |f_n(x)|\,dx$$

$$\le C\max\{|\phi(x) - \phi(0)| : |x| < \tfrac{M}{n}\} \to 0 \;\; as \;\; n \to \infty.$$

We just showed

$$\forall\phi \in D \qquad J_n[\phi(x)] \to \phi(0) \quad as \;\; n \to \infty; \;\; i.e. \quad J_n \to \delta(x) \;\; in \;\; D' \blacksquare$$

Note that this is a result for distributions on R^1.

Example 3.3

1. The following are all delta sequences.

$$f_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}, \qquad g_n(x) = \left\{\begin{array}{lll} -n & if & |x| < \frac{1}{2n} \\ 2n & if & \frac{1}{2n} < x < \frac{1}{n} \\ 0 & if & |x| > \frac{1}{n} \end{array}\right\} \qquad h_n(x) = S_{1/n}(x)$$

Note that $f_n(0)$ *and* $h_n(0)$ both tend to plus infinity as n tends to infinity. This is consistent with the commonly held impression that the delta function is zero everywhere except at zero where it has the value plus infinity. However, $g_n(x)$ is also a delta sequence and $g_n(0)$ tends to minus infinity as n tends to infinity. This illustrates the difficulty in assigning a pointwise value to a singular distribution.

2. Consider the following sequences of locally integrable functions.

$$F_n(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(nx), \qquad f_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}, \qquad g_n(x) = \frac{-2n^3}{\pi}\frac{x}{(1+n^2x^2)^2} .$$

Then $F'(x) = f_n(x)$ *and* $f'_n(x) = g_n(x)$. Moreover

$$\lim_{n\to\infty} F_n(x) = \left\{ \begin{array}{lll} 1 & if & x>0 \\ \frac{1}{2} & if & x=0 \\ 0 & if & x<0 \end{array} \right\} = H_0(x) \quad so \qquad \int_R F_n\phi \to \int_0^\infty \phi$$

hence

$$\int_R f_n\phi = \int_R F'_n\phi = \int_R -F_n\phi' \to \int_0^\infty -\phi' = \phi(0) \qquad i.e., \quad f_n \to \delta(x)$$

and

$$\int_R g_n\phi = \int_R f'_n\phi = \int_R -f_n\phi' = \int_R -F_n{}'\phi' = \int_R F_n\phi'' \to \int_0^\infty -\phi'' = -\phi'(0) \qquad i.e., \quad g_n \to \delta'(x)$$

This is an illustration of the following result.

Theorem 3.6 Suppose $\{T_n\}$ is a sequence of distributions converging to a limit T in the distributional sense. Then

1. the derivatives $\{T'_n\}$ form a sequence of distributions converging to the limit T' in the distributional sense.
2. the antiderivatives $\left\{\int^x T_n\right\}$ form a sequence of distributions converging to the limit $\int^x T$ in the distributional sense

A connection between pointwise and distributional convergence is contained in the following corollary of the dominated convergence theorem.

Theorem 3.7 Suppose $\{f_n\}$ is a sequence of locally integrable functions such that

$$|f_n| \le g \in L_1^{loc} \qquad and \qquad f_n \to f \text{ pointwise almost everywhere.}$$

Then the sequence of regular distributions, J_{f_n}, converges in the distributional sense to J_f.

In example 3.3(2) it is straightforward to show that $|F_n(x)| \le 1$ and that $F_n(x)$ converges pointwise to $H_0(x)$. Then the distributional convergence of F_n to H_0 follows and theorem 3.6 can be applied as illustrated in example 3.3(2).

Applications to PDE's

1). We are going to show first that for $0 < x,y < \pi$, $\delta(x-y) \in D'[0,\pi]$, can be expanded in terms of the eigenfunctions $\{\sin(nx)\}$. Formally,

$$\delta(x-y) = \sum_{n=1}^\infty d_n(y)\sin(nx),$$

leads to

$$\langle \delta(x-y), \sin(mx) \rangle = \left\langle \sum_{n=1}^{\infty} d_n(y) \sin(nx), \sin(mx) \right\rangle$$
$$= \sum_{n=1}^{\infty} d_n(y) \int_0^{\pi} \sin(nx) \sin(mx)\, dx = \frac{1}{2} d_m(y).$$

But

$$\langle \delta(x-y), \sin(mx) \rangle = \sin(my)$$

hence

$$d_m(y) = 2\sin(my) \quad \textit{and} \quad \delta(x-y) = 2\sum_{n=1}^{\infty} \sin(ny)\sin(nx).$$

This result is only formal since expansion in terms of eigenfunctions was developed for functions in $L_2[0,\pi]$ and has not been shown to extend to generalized functions. In fact, it does extend but this is what we must now show.

One way to show that this result extends to distributions, is to note that for each fixed $y \in [0,\pi]$, the series

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(ny)\sin(nx)$$

converges uniformly for $0 \le x \le \pi$, to a limit we will denote by $G(x,y)$. It follows then from theorem 3.6 that

$$-\partial_{xx}\left(2\sum_{n=1}^{N} \frac{1}{n^2} \sin(ny)\sin(nx)\right) = 2\sum_{n=1}^{N} \sin(ny)\sin(nx) \to -\partial_{xx}G(x,y) \quad \textit{as } N \to \infty$$

where the convergence is in the sense of distributions on $[0,\pi]$. Since the differentiated series is the series we found for $\delta(x-y)$, the validity of the series will be established if we can show that $-\partial_{xx}G(x,y) = \delta(x-y)$. To see this, note that any test function $\phi(x) \in D[0,\pi]$ can be expanded in a series of the form

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(nx) = \sum_{n=1}^{\infty}\left(2\int_0^{\pi} \phi(y)\sin(ny)\,dy\right)\sin(nx)$$
$$= \int_0^{\pi} \phi(y)\left(\sum_{n=1}^{\infty} 2\sin(nx)\sin(ny)\right) dy = \langle \phi(y), -\partial_{xx}G(x,y) \rangle.$$

Since

$$\langle \phi(y), -\partial_{xx}G(x,y) \rangle = \phi(x) \qquad \text{for any test function } \phi(x) \in D[0,\pi],$$

it follows that $-\partial_{xx}G(x,y) = \delta(x-y)$.

2. For each fixed $y \in [0,\pi]$, let $H(x,y,t)$ denote the (distributional) solution of

$$(\partial_t - \partial_{xx})H(x,y,t) = 0, \quad 0 < x < \pi,\ t > 0$$

$$\textit{and} \qquad H(x,y,0) = \delta(x-y),$$
$$H(0,y,t) = H(\pi,y,t) = 0.$$

Then, as we have seen previously, the solution to this problem is given by

$$H(x,y,t) = \sum_{n=1}^{\infty} d_n(y)\, e^{-n^2 t} \sin(nx) = \sum_{n=1}^{\infty} 2\sin(ny)\, e^{-n^2 t}\sin(nx).$$

Here using the expansion for $\delta(x-y)$ is rigorous not just formal.
Now if $u = u(x,t)$ solves

$$(\partial_t - \partial_{xx})u(x,t) = 0, \qquad 0 < x < \pi,\ t > 0$$

$$
\begin{aligned}
\textit{and} \qquad & u(x,0) = f(x), \qquad 0 < x < \pi, \\
& u(0,t) = u(\pi,t) = 0, \qquad t > 0,
\end{aligned}
$$

then $\qquad u(x,t) = \int_0^\pi H(x,y,t)\, f(y)\, dy.$

To see this, simply note that $u(0,t) = u(\pi,t) = 0,$ follows from $H(0,y,t) = H(\pi,y,t) = 0,$ and in addition,

$$
(\partial_t - \partial_{xx}) u(x,t) = \int_0^\pi (\partial_t - \partial_{xx}) H(x,y,t)\, f(y)\, dy = 0,
$$

and

$$
u(x,0) = \int_0^\pi H(x,y,0)\, f(y)\, dy = \int_0^\pi \delta(x-y)\, f(y)\, dy = f(x).
$$

Evidently, $H(x,y,t)$ is what we have previously called the Green's function for the heat equation. Now we will see an equivalent alternative description of the Green's function.

3. For each fixed $y \in [0,\pi]$, *and* $t > s > 0$, let $H(x,y,t-s)$ denote the solution of

$$
\begin{aligned}
& (\partial_t - \partial_{xx}) H(x,y,t-s) = \delta(x-y)\,\delta(t-s), \quad 0 < x < \pi,\ t > s > 0 \\
\textit{and} \qquad & H(x,y,0) = 0, \\
& H(0,y,t) = H(\pi,y,t) = 0.
\end{aligned}
$$

In this case, the solution to this problem can be written

$$
H(x,y,t-s) = \sum_{n=1}^\infty H_n(y,t-s)\, \sin(nx)
$$

where

$$
\partial_t H_n(y,t-s) + n^2 H_n(y,t-s) = d_n(y)\,\delta(t-s), \quad H_n(y,0) = 0.
$$

Then

$$
H_n(y,t-s) = d_n(y) \int_0^t e^{-n^2(t-\tau)}\, \delta(\tau - s)\, d\tau = 2\sin(ny)\, e^{-n^2(t-s)}
$$

and

$$
H(x,y,t-s) = 2 \sum_{n=1}^\infty e^{-n^2(t-s)} \sin(ny)\, \sin(nx).
$$

Now if $u = u(x,t)$ solves

$$
\begin{aligned}
& (\partial_t - \partial_{xx}) u(x,t) = F(x,t), \quad 0 < x < \pi,\ t > 0 \\
\textit{and} \qquad & u(x,0) = 0, \qquad u(0,t) = u(\pi,t) = 0,
\end{aligned}
$$

then

$$
u(x,t) = \int_0^t \int_0^\pi H(x,y,t-s)\, F(y,s)\, dy\, ds.
$$

To see this, simply note that $u(0,t) = u(\pi,t) = 0,$ follows from $H(0,y,t) = H(\pi,y,t) = 0,$ and furthermore

$$
\begin{aligned}
(\partial_t - \partial_{xx}) u(x,t) &= \int_0^t \int_0^\pi (\partial_t - \partial_{xx}) H(x,y,t-s)\, F(y,s)\, dy\ ds \\
&= \int_0^t \int_0^\pi \delta(x-y)\delta(t-s)\, F(y,s)\, dyds = F(x,t).
\end{aligned}
$$

and

$$
u(x,0) = \int_0^t \int_0^\pi H(x,y,0)\, F(y,s)\, dyds = 0.
$$

Evidently, $H(x,y,t-s)$ is also what we have previously called the Green's function for the heat equation, and we can characterize the Green's function equivalently as $H(x,y,t)$, the (distributional) solution of

$$(\partial_t - \partial_{xx})H(x,y,t) = 0, \quad 0 < x < \pi, \; t > 0 \qquad and \quad H(x,y,0) = \delta(x-y),$$

or as $H(x,y,t-s)$, the solution in the distributional sense of

$$(\partial_t - \partial_{xx})H(x,y,t) = \delta(x-y)\,\delta(t-s), \quad 0 < x < \pi, \; t > s > 0 \qquad and \quad H(x,y,0) = 0.$$

4. The Distributional Fourier Transform

In order to extend the definition of the Fourier transform to generalized functions, we first recall that an infinitely differentiable function $u(x)$ is said to be ***rapidly decreasing*** if

$$\forall \text{ integers } m,n, \qquad |x^m u^{(n)}(x)| \to 0, \qquad as \; |x| \to \infty$$

Then the rapidly decreasing functions are smooth functions that tend to zero, together with their derivatives of all orders, as $|x|$ tends to infinity, more rapidly than any negative power of $|x|$. We denote this class of functions by $S(R)$ and note that the test functions, $C_c^\infty(R)$ are contained in $S(R)$, and $S(R)$ is contained in $L_2(R)$ and in $L_1(R)$. Then every $f \in S(R)$ has a Fourier transform which can be shown to also belong to $S(R)$. Thus $S(R)$ is another space, like $L_2(R)$, which is symmetric with respect to the Fourier transform in the sense that both $f \text{ and } F$ belong to the same space of functions.

Proposition 4.1 For each $f \in S(R)$, $F = T_F[f] \in S(R)$ and $T_F^{-1}[F] = f(x)$ where the equality here is in the pointwise sense.

.

We now define the sense of ***convergence in*** $S(R)$. A sequence $\{\phi_n\}$ of rapidly decreasing functions is said to converge to zero in $S(R)$ if, for all integers p and q,

$$\max_{x\in R}\left| x^p \phi_n^{(q)}(x) \right| \to 0 \qquad as \qquad n \to \infty$$

Then $\phi_n \to \phi \; in \; S(R)$ if $(\phi_n - \phi) \to 0 \; in \; S(R)$. Since the space of rapidly decreasing functions has no norm, we cannot say that the Fourier transform is an isometry on $S(R)$. However, we have the following result.

Proposition 4.2 For $f_n \in S(R)$, with $F_n = T_F[f_n] \in S(R)$, $\phi_n \to \phi \; in \; S(R)$ implies $F_n \to F = T_F[\phi] \; in \; S(R)$.

Tempered Distributions

A linear functional T defined on $S(R)$ is said to be a ***tempered distribution*** if, for every sequence $\{\phi_n\}$ of rapidly decreasing functions that is converging to zero in $S(R)$, it follows that $T[\phi_n]$ converges to zero as a sequence of real numbers. In this case we write $T \in S'(R)$.

Note that convergence in the sense of test functions is stronger than convergence in the sense of rapidly decreasing functions. Then every $J \in D'(R)$ is also a tempered distribution but the converse is false.

Proposition 4.3 If $f \in L_1^{loc}$ is such that for some positive integer, m

$$\int_R \frac{f(x)}{(1+x^2)^m} dx = C < \infty,$$

then f generates a tempered distribution; i.e., $J_f[\phi] = \langle f, \phi \rangle$ is a continuous linear functional on $S(R)$ as well as on $D(R)$.

Proof- Suppose $f \in L_1^{loc}$ satisfies the condition above and let $\{\phi_n\}$ denote a sequence of rapidly decreasing functions such that $\phi_n \to 0 \ in \ S(R)$. Then

$$J_f[\phi_n] = \int_R f(x)\phi_n(x)dx = \int_R \frac{f(x)}{(1+x^2)^m}(1+x^2)^m \phi_n(x)dx$$

$$\leq \max_{x\in R} |(1+x^2)^m \phi_n(x)| \int_R \frac{f(x)}{(1+x^2)^m} dx = C \max_{x\in R} |(1+x^2)^m \phi_n(x)| \to 0 \ as \ n \to \infty$$

We have proved that $\phi_n \to 0 \ in \ S(R)$ implies $J_f[\phi_n] \to 0$ in R.■

Any locally integrable function which satisfies the condition in proposition 4.3 is said to be a ***tempered function*** or a function of ***slow growth***. Then, according to the proposition, any tempered function generates a tempered distribution. Thus any constant function, any polynomial in x and $\sin x \ and \ \cos x$ all generate tempered distributions.

Definition The Fourier transform of the tempered distribution J is the tempered distribution K whose action on S is defined by $K[\psi] = J[\Psi]$ where $\Psi = T_F[\psi]; \ i.e.,$

$$\langle T_F[J], \psi \rangle = \langle J, T_F[\psi] \rangle \qquad for \ all \quad \psi \in S$$

Example 4.1

1. Formally

$$T_F[\delta(x)] = \frac{1}{2\pi} \int_R \delta(x) e^{-ix\alpha} dx = \frac{1}{2\pi}.$$

This result is only formal since this integral serves to define the Fourier transform only for absolutely integrable functions, and the delta is certainly not in this class. In order to compute the transform of the delta, we have to use the distributional definition. Write,

$$\langle T_F[\delta], \psi \rangle = \langle \delta, T_F[\psi] \rangle = \Psi(0) \qquad for \ all \quad \psi \in S.$$

But

$$\Psi(0) = \frac{1}{2\pi} \int_R \psi(x) e^{-ix\alpha}|_{\alpha=0} dx = \frac{1}{2\pi} \int_R \psi(x)\, dx = \left\langle \frac{1}{2\pi}, \psi \right\rangle.$$

Then

$$\langle T_F[\delta], \psi \rangle = \left\langle \frac{1}{2\pi}, \psi \right\rangle \qquad for \ all \quad \psi \in S,$$

i.e.,

$$T_F[\delta(x)] = \frac{1}{2\pi}.$$

2. The constant function $f(x) = 1 \ \forall x,$ is a tempered function and generates a tempered distribution which we will denote by $I.$ Then for arbitrary $\psi \in S,$

$$\langle T_F[I], \psi \rangle = \langle I, T_F[\psi] \rangle = \int_R \Psi(\alpha)\, d\alpha = \int_R \Psi(\alpha)\, e^{ix\alpha} d\alpha|_{x=0} = \psi(0);$$

i.e.,

$$\langle T_F[I], \psi \rangle = \psi(0) \qquad \textit{for all} \quad \psi \in S.$$

This is equivalent to the assertion that $T_F[I] = \delta.$

3. Using these two results we now can see that

$$T_F[\delta^{(n)}(x)] = (i\alpha)^n \frac{1}{2\pi}, \qquad T_F[x^n I] = i^n \delta^{(n)}(\alpha)$$

$$T_F[\delta(x-p)] = \frac{1}{2\pi}.e^{-ip\alpha}, \qquad T_F[e^{ixp} I] = \delta(x-p),$$

$$T_F[\cos(px)] = T_F[(e^{ixp}I + e^{-ixp}I)/2] = (\delta(x-p) + \delta(x+p))/2$$

$$T_F[\sin(px)] = T_F[(e^{ixp}I - e^{-ixp}I)/2i] = (\delta(x-p) - \delta(x+p))/2i$$

Applications to PDE's

1. For fixed $y \in R, \ \textit{and} \ s \geq 0,$ consider the initial value problem

$$(\partial_t - \partial_{xx})H(x,y,t-s) = \delta(x-y)\delta(t-s) \qquad x \in R, \ t > s \geq 0,$$
$$H(x,y,t-s) = 0 \quad \textit{for} \ s > t.$$

If $\hat{H}(\alpha,y,t-s)$ denotes the Fourier transform in x of $H(x,y,t-s),$ then

$$\tfrac{d}{dt}\hat{H}(\alpha,y,t-s) + \alpha^2 \hat{H}(\alpha,y,t-s) = \frac{1}{2\pi}.e^{-iy\alpha}\,\delta(t-s),$$
$$\hat{H}(\alpha,y,t-s) = 0 \quad \textit{for} \quad s > t.$$

It follows that

$$\hat{H}(\alpha,y,t-s) = \int_0^t e^{-\alpha^2(t-\tau)} \frac{1}{2\pi}.e^{-iy\alpha}\,\delta(\tau-s)d\tau = \frac{1}{2\pi}.e^{-iy\alpha}e^{-\alpha^2(t-s)}$$

Since $\quad T_F^{-1}\left[e^{-\alpha^2(t-s)}\right] = \sqrt{\frac{\pi}{t-s}}\; e^{-\frac{x^2}{4(t-s)}}$ it follows that

$$H(x,y,t-s) = \frac{1}{\sqrt{4\pi(t-s)}}\; e^{-\frac{(x-y)^2}{4(t-s)}}, \qquad t > s \geq 0.$$

This is the previously obtained fundamental solution for the heat equation on $R \times R_+.$ Note that similar arguments show that the solution of the initial value problem

$$(\partial_t - \partial_{xx})H(x,y,t) = 0 \qquad x \in R, \ t > 0,$$
$$H(x,y,0) = \delta(x-y),$$

is given by $\quad H(x,y,t) = \frac{1}{\sqrt{4\pi t}}\; e^{-\frac{(x-y)^2}{4t}}, \qquad t > 0.$

2. For fixed $y \in R,\ and\ s \geq 0,$ consider the initial value problem

$$(\partial_{tt} - \partial_{xx})W(x,y,t-s) = \delta(x-y)\delta(t-s) \qquad x \in R,\ t > s \geq 0,$$
$$W(x,y,t-s) = 0 \quad for\ s > t.$$

If $\hat{W}(\alpha,y,t-s)$ denotes the Fourier transform in x of $\hat{W}(x,y,t-s),$ then

$$\frac{d^2}{dt^2}\hat{W}(\alpha,y,t-s) + \alpha^2\hat{W}(\alpha,y,t-s) = \frac{1}{2\pi}.e^{-iy\alpha}\,\delta(t-s),$$
$$\hat{W}(\alpha,y,t-s) = 0 \quad for \quad s > t.$$

There are various ways to solve this equation, one of them is to apply the Laplace transform in t to get

$$(\beta^2 + \alpha^2)\hat{w}(\alpha,y,\beta-s) = \frac{1}{2\pi}.e^{-iy\alpha}e^{-\beta s}$$

where

$$\hat{w}(\alpha,y,\beta-s) = \pounds\left[\hat{W}(\alpha,y,t-s)\right] = Laplace\ transform\ in\ t\ of\ \ \hat{W}(\alpha,y,t-s)$$

and

$$\pounds[\delta(t-s)] = \int_0^\infty e^{-\beta t}\delta(t-s)dt = e^{-\beta s}$$

Then

$$\hat{w}(\alpha,y,\beta-s) = \frac{1}{2\pi}.e^{-iy\alpha}\frac{\beta}{\beta^2+\alpha^2}\ \frac{e^{-\beta s}}{\beta}.$$

Now

$$T_F^{-1}\left[\frac{1}{2\pi}.e^{-iy\alpha}\frac{\beta}{\beta^2+\alpha^2}\right] = \tfrac{1}{2}e^{-\beta|x-y|} \quad and \quad \pounds^{-1}\left[\frac{e^{-\beta A}}{\beta}\right] = H_0(t-A)$$

hence $\quad W(x,y,t-s) = \tfrac{1}{2}H_0(t-s-|x-y|).$

This is the so called ***causal fundamental solution for the wave equation***. If $u = u(x,t)$ solves

$$(\partial_{tt} - \partial_{xx})u(x,t) = F(x,t) \qquad x \in R,\ t > 0,$$
$$u(x,0) = \partial_t u(x,0) = 0 \qquad for\ x \in R$$

then $\qquad u(x,t) = \int_0^t\int_R W(x,y,t-s)\,F(y,s)\,dy\,ds.$

3. Using the Laplace transform, the solution of the initial boundary value problem,

$$\partial_t v(x,t) - \partial_{xx}v(x,t) = 0, \qquad x > 0,\ t > 0,$$
$$v(x,0) = 0, \qquad x > 0,$$
$$\partial_x v(0,t) = g(t), \qquad t > 0,$$

is found to be

$$v(x,t) = \int_0^t \frac{-1}{\sqrt{\pi(t-s)}}e^{-\frac{x^2}{4(t-s)}}g(s)\,ds = -\int_0^t K(x,t-s)\,g(s)\,ds.$$

while the solution of

$$\partial_t w(x,t) - \partial_{xx} w(x,t) = 0, \qquad x > 0,\ t > 0,$$
$$w(x,0) = 0, \qquad x > 0,$$
$$w(0,t) = f(t), \qquad t > 0,$$

is given by

$$w(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-s)^3}} e^{-\frac{x^2}{4(t-s)}} g(s)\,ds = \int_0^t \partial_x K(x,t-s)\,g(s)\,ds.$$

The functions $K(x,t),\ and\ h(x,t) = -\partial_x K(x,t)$ each satisfy

$$\partial_t u(x,t) - \partial_{xx} u(x,t) = 0, \qquad x > 0,\ t > 0,$$

and

$$\text{Lim}_{t\to 0+}\, u(x,t) = 0, \qquad for\ x \neq 0$$

This seems to violate the uniqueness of solutions to the pure initial value problem for the heat equation. However, $K(x,t),\ and\ h(x,t)$ are distributional solutions of the initial value problem and neither of them is continuous in the closed upper half plane; i.e.,

$$Lim_{x^2=t\to 0} K(x,t) = +\infty = Lim_{x^2=t\to 0} h(x,t).$$

On the other hand, we can also show that

i) $Lim_{t\to 0}\langle K(x,t),\phi(x)\rangle = \phi(0) \qquad \forall\phi \in D(R)$
ii) $Lim_{t\to 0}\langle h(x,t),\phi(x)\rangle = 2\phi'(0) \qquad \forall\phi \in D(R)$

This is equivalent to

i) $\partial_t K(x,t) - \partial_{xx} K(x,t) = 0, \quad in \quad D'(R), \qquad K(x,0) = 2\delta(x)$
ii) $\partial_t h(x,t) - \partial_{xx} h(x,t) = 0, \quad in \quad D'(R), \qquad K(x,0) = 2\delta'(x).$

To see that i) holds, write

$$\langle K(x,t),\phi(x)\rangle = \int_R \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \phi(x)\,dx = \frac{2}{\sqrt{\pi}} \int_R e^{-z^2} \phi\left(z\sqrt{4t}\right) dz$$
$$= \frac{2}{\sqrt{\pi}} \int_R e^{-z^2} [\phi\left(z\sqrt{4t}\right) - \phi(0)]\,dz + 2\phi(0)$$

This leads to the result, $Lim_{t\to 0}\langle K(x,t),\phi(x)\rangle = 2\phi(0),\ \forall\phi \in D(R).$

5. Distributions Supplement

We wish to consider two questions in the theory of distributions:

i) given $J \in D'(R),$ find $T \in D'(R),$ such that $xT(x) = J(x)$
ii) find $J \in D'(R),$ such that $J'(x) = 0$

To answer the first question we need the following lemma.

Lemma 5.1- The test function $\phi(x)$ satisfies

$$\phi(0) = 0 \text{ if and only if } \phi(x) = x\psi(x) \text{ for some test function, } \psi(x).$$

More generally, the test function $\phi(x)$ satisfies

$$\phi^{(m)}(0) = 0 \quad m = 0,1,\ldots,N-1 \text{ if and only if } \phi(x) = x^N\psi(x) \text{ for some test function, } \psi(x).$$

Proof- Clearly if $\phi(x) = x\psi(x)$, then $\phi(0) = 0$. Conversely, for any test function

$$\phi(x) = \phi(0) + \int_0^x \phi'(t)dt,$$

and if $\phi(0) = 0$, then the substitution $t = xs$ leads to

$$\phi(x) = \int_0^x \phi'(t)dt = x\int_0^1 \phi'(xs)\,ds = x\psi(x).$$

Clearly $\psi(x)$ is a test function if $\phi(x)$ is a test function. The more general statement follows by a similar argument.■

Now, using lemma 5.1, we can show

Lemma 5.2- The general solution of i) is of the form $T(x) = T_0(x) + C\delta(x)$, where T_0 is any particular solution of i) and C is an arbitrary constant.

Proof- Note that the general solution of i) is necessarily of the form $T(x) = T_0(x) + K(x)$, where T_0 is any particular solution of i) and K is a distributional solution of $xK(x) = 0$. This means that

$$0 = \langle xK(x), \psi(x)\rangle = \langle K(x), x\psi(x)\rangle \qquad \textit{for all test functions } \psi.$$

Now it follows from lemma 1 that $0 = \langle K(x), \phi(x)\rangle$ for all test functions such that $\phi(0) = 0$. Now any test function $\phi(x)$ can be written in the form $\phi(x) = \phi(0)\,\phi_1(x) + \phi_0(x)$, where ϕ_0, ϕ_1 denote test functions such that $\phi_0(0) = 0$ and $\phi_1(0) = 1$; e.g.,

$$\phi(x) = \phi(0)\,\phi_1(x) + [\phi(x) - \phi(0)\,\phi_1(x)].$$

Then for arbitrary test function $\phi(x)$ we can write

$$\langle K(x), \phi(x)\rangle = \langle K(x), \phi(0)\,\phi_1(x) + \phi_0(x)\rangle = \phi(0)\langle K, \phi_1\rangle + \langle K, \phi_0\rangle = C\phi(0)$$

which is to say, $K = C\delta$. More generally, similar arguments show that the general distributional solution of $x^N T(x) = J(x)$ is of the form

$$T(x) = T_0(x) + \sum_{n=0}^{N-1} C_n\,\delta^{(n)}(x). \ ■$$

As an application of this result, consider the problem of finding the Fourier transform of the tempered function $f(x) = Sgn(x)$.
Note that

$$T_F\left[\tfrac{d}{dx}Sgn(x)\right] = i\alpha\, T_F[Sgn(x)] = i\alpha\, F(\alpha).$$

But $\quad \frac{d}{dx}Sgn(x) = 2\,\delta(x) \quad \textit{and} \quad T_F[2\,\delta(x)] = \frac{1}{\pi}$

hence

$$i\alpha F(\alpha) = \frac{1}{\pi}.$$

Then it follows from lemma 2 that $F(\alpha) = -\frac{i}{\pi\alpha} + C\delta(\alpha)$. Now it follows from previous observations about the Fourier transform, that since $f(x)$ is an odd real valued function of x, then $F(\alpha)$ is an odd and purely imaginary valued function of α. Then the fact that $\delta(\alpha)$ is an even function of α implies that $C = 0$.

Similarly, the Fourier transform of the Heaviside step function can be obtained by a similar computation. We have

$$T_F\left[\tfrac{d}{dx}H(x)\right] = i\alpha T_F[H(x)] = i\alpha h(\alpha).$$

Also

$$\tfrac{d}{dx}H(x) = \delta(x) \quad and \quad T_F[\delta(x)] = \frac{1}{2\pi}$$

Then

$$i\alpha h(\alpha) = \frac{1}{2\pi},$$

and lemma 2 implies $h(\alpha) = -\frac{i}{2\pi\alpha} + C\delta(\alpha)$. In order to determine the constant C, note that

$$H(x) + H(-x) = 1, \quad hence \quad T_F[H(x)] + T_F[H(-x)] = \delta(\alpha).$$

But

$$T_F[H(x)] + T_F[H(-x)] = h(\alpha) + h(-\alpha) = 2C\delta(\alpha)$$

which implies $C = 1/2 \; and \; h(\alpha) = \frac{1}{2}\delta(\alpha) - \frac{i}{2\pi\alpha}$.

In order to answer the question raised in ii), we need

Lemma 5.3- The test function $\phi(x)$ satisfies

$$\phi(x) = \psi'(x) \quad \text{for some test function, } \psi(x)$$

if and only if $\int_R \phi(x)\,dx = 0.$

Proof- If $\phi(x) = \psi'(x)$ for some test function, $\psi(x)$ then

$$\int_R \psi'(x)\,dx = \psi(\infty) - \psi(-\infty) = 0.$$

Conversely if $\int_R \phi(x)\,dx = 0$, then

$$\psi(x) = \int_{-\infty}^{x} \phi(s)\,ds \qquad satisfies \qquad \psi'(x) = \phi(x).$$

In addition, ψ has compact support since $\int_R \phi(x)\,dx = 0$ and ϕ has compact support. Then ψ is a test function.■

Note that any test function can be written in the form $\phi(x) = A\phi_1(x) + \phi_0(x)$ where

$$A = \int_R \phi(x)\,dx, \qquad \int_R \phi_0(x)\,dx = 0, \qquad \textit{and} \qquad \int_R \phi_1(x)\,dx = 1.$$

e.g., $\phi = A\phi_1 + (\phi - A\phi_1)$.

Now, using this result we can show

Lemma 5.4- The distributional solution of $u'(x) = 0$ is the regular distribution generated by the locally integrable function $u(x) = C$.

Proof- If $u'(x) = 0$ then

$$\left\langle u', \psi \right\rangle = -\left\langle u, \psi' \right\rangle = 0 \qquad \text{for all test fucntions } \psi.$$

i.e.,

$$\left\langle u, \phi \right\rangle = 0 \qquad \text{for all test functions } \phi = \psi'$$

By the previous result then

$$\begin{aligned}\left\langle u, \phi \right\rangle &= \langle u, A\phi_1(x) + \phi_0(x) \rangle \\ &= A < u, \phi_1 > + < u, \phi_0 > \\ &= C \int_R \phi(x)\,dx \qquad \textit{where} \quad C = \ < u, \phi_1 >\end{aligned}$$

This shows that

$$\left\langle u, \phi \right\rangle = \int_R C\phi(x)\,dx \qquad \forall \phi \in D \ \blacksquare$$