1. When you take a pill to obtain medication, the pill first goes into your stomach and the medication passes into your GI tract. From there the medication is absorbed into your bloodstream and circulated through your body before being eliminated from the blood by the kidneys and other organs. If we let \( x(t) \) denote the amount of medication in your GI tract at time \( t \), then we can model the movement of the medication out of the GI tract with the equation

\[
x'(t) = -k_1 x(t), \quad x(0) = A.
\]

This is the assertion that after taking the pill, an amount \( A \) of medication is in the GI tract and, as the drug is absorbed by the bloodstream, the amount decreases at a rate proportional to the amount currently present in the GI tract. If the amount of medication in the bloodstream at time \( t \) is denoted by \( y(t) \), then

\[
y'(t) = k_1 x(t) - k_2 y(t), \quad y(0) = 0,
\]

expresses the fact that medication is coming into the bloodstream at exactly the rate it is leaving the GI tract and it is leaving the bloodstream at a rate proportional to the amount currently present in the bloodstream. The constant here is denoted by \( k_2 \). Also, we are assuming that there is no medication in the bloodstream initially.

(a) Solve these two initial value problems for \( x(t) \) and \( y(t) \) and plot them on the same graph in the following three cases: \( k_1 < k_2, k_1 = k_2, \) and \( k_1 > k_2 \). In each case, describe what the plot tells you about the drug levels in the two systems.

(b) Consider the situation in which a pill is taken every \( T \) hours (i.e. a pill is taken at \( t = 0, T, 2T, \ldots \) etc). Write out the sequence of initial value problems that must be solved to find the drug levels in the GI system and the bloodstream for \( t \) between 0 and \( 4T \). Assume \( k_1 \neq k_2 \).

(c) Solve the initial value problems in (b) and graph \( x(t) \) and \( y(t) \) on the same axes for the following parameter values:

\[
\begin{align*}
k_1 &= 0.1, \quad k_2 = 0.2, \quad T = 1 \\
k_1 &= 0.3, \quad k_2 = 0.2, \quad T = 1 \\
k_1 &= 0.2, \quad k_2 = 0.1, \quad T = 2 \\
A &= 1 \text{ in all cases}
\end{align*}
\]

Discuss your results with respect to maintaining the drug level in the bloodstream between max and min limiting values.

2. Water flows into a lake at the rate of \( 8 \cdot 10^4 \frac{m^3}{day} \). Water evaporates from the lake at the rate of \( 2 \cdot 10^4 \frac{m^3}{day} \) and if water flows out of the lake at the rate of \( 6 \cdot 10^4 \frac{m^3}{day} \), then the volume of the lake remains constant at \( 4 \cdot 10^7 m^3 \). Pollutant flows into the lake at the rate of
1.8 \( \frac{m^3}{day} \).

(a) If we assume that the pollutant mixes immediately and completely with the water in the lake and that the lake was initially unpolluted, find the amount of pollutant in the lake as a function of time and plot your result. You can assume that no pollutant leaves the lake by evaporation.

(b) Suppose that the outflow from the lake is reduced by an amount \( \alpha \) so that the volume of the lake no longer remains constant. Find the volume of the lake as a function of \( t \) and find the amount of pollutant in the lake as a function of time and plot your result. How does this result differ from the result in part (a)? Explain why this makes sense.

(c) Suppose that the outflow from the lake is increased by an amount \( \alpha \) so that the volume of the lake now is increasing. Find the volume of the lake as a function of \( t \) and find the amount of pollutant in the lake as a function of time and plot your result. How does this result differ from the result in parts (a) and (b)? Explain why this makes sense.