Problems Chapter 4 Differentiation

1. Use the definition to find the derivative of the following functions:

- **a.** x^4
- **b.** $\frac{1}{x^2}$
- **c.** $\frac{1}{\sqrt{x}}$
- **d.** $\cos x$

2. Use the rules for derivatives to find the derivative of the following functions:

- **a.** $\frac{x}{1+x^2}$
- **b.** $\sqrt{x^2 - 3x + 1}$
- **c.** $\tan(x^2) \qquad |x| < \pi/2$
- **d.** $\cos^{-1}(x)$

3. Which of the following functions is differentiable at $x = 0$? Which of the following functions is differentiable for $x \neq 0$? Where the derivative exists, is it continuous?

- **a.** $x|x|$
- **b.** $|x+1| + |x-1|$
- **c.** $x\cos\left(\frac{1}{x}\right)$
- **d.** $x^2\cos\left(\frac{1}{x}\right)$
- **e.** $f(x) = \begin{cases} x^2 & if \quad x \in Q \\ 0 & if \quad x \notin Q \end{cases}$

4. Let $f(x) = \begin{cases} x^2 & if \quad x \geq 0 \\ \alpha x & if \quad x < 0 \end{cases}$

- **a.** for which values of α is f continuous at $x = 0$?
- **b.** for which values of α is f differentiable at $x = 0$?
- **c.** When f is differentiable at $x = 0$, does $f''(0)$ exist?

5. . Find all the points where $f(x) = \sqrt{1 - \cos x}$ is not differentiable. Explain why the derivative fails to exist.

6. Let $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \leq 0$.

- **a.** For what values of $p \in R$ is the function $F(x) = x^p H(x)$ continuous at $x = 0$?
- **b.** For what values of $p \in R$ is the function $F(x) = x^p H(x)$ differentiable at $x = 0$?
- **c.** Compute $F'(x)$ for each p and at each x where the derivative exists.

7. Find a function $f(x)$ such that

$$f'(-1) = f'(0) = f'(+1) = 0$$
$$f''(-1) > 0 \qquad f''(0) < 0 \qquad f''(+1) > 0$$

8. Give an example of a function $f(x)$ that is continuous on $[-1, 1]$ such that:

a. f has a maximum at some $c \in (-1, 1)$ but $f'(c) \neq 0$

b. $f'(c) = 0$ at some $c \in (-1, 1)$ but c is neither the max nor the min for f on $[-1, 1]$.

c. f' is zero at both the max and the min for f on $[-1, 1]$

9. Suppose f is continuous and differentiable on $[-1, 1]$ and that $f'(x)$ is continuous on $[-1, 1]$ as well. Show that f is Lipschitz continuous.

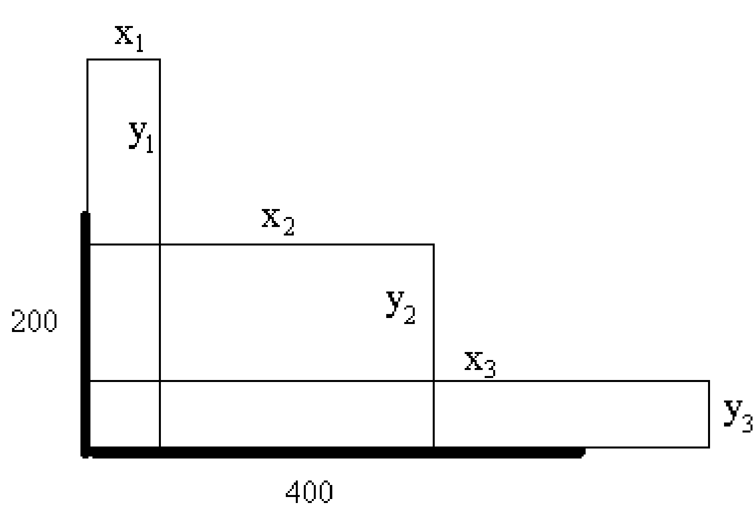
10. A function $F(x)$ is said to be periodic with period L if $F(x + L) = F(x)$ for all x. Suppose F is periodic and continuous on R. Then show that F is bounded and uniformly continuous on R.

11. Using 580 feet of fence wire, build a rectangular pen of maximum area by making use of the fixed walls of length 200 feet and 400 feet, respectively as shown.below. Note that there are 3 different configurations that make use of the fixed walls. In each of these configurations, the area is equal to $x_j y_j$ but in each case, x_j and y_j satisfy a different condition (e.g. in the second case the condition is, $x_2 + y_2 = 580$).

a. In each of the three configurations:express y in terms of x; what are the max and min values for x and y in each configuration? express the area in terms of x

b. determine the maximum area that can be achieved and explain how the Max-min theorem must be used in finding the maximum area in this problem.

c. plot the graph of $A(x)$ versus x and show where the max occurs.



12. Find all real values of a for which the function $f(x) = x^3 + ax^2 + 3x + 15$ is strictly increasing on $(0,1)$.

13. Give an example of a function $f(x)$ which is differentiable with a differentiable derivative $f'(x)$ but whose second derivative $f''(x)$ is discontinuous.

14. Suppose f is differentiable at every x. Prove $g(x) = f(x)^2$ is differentiable at every x.

15. Suppose f is differentiable at every x. Prove $g(x) = f(x-1)f(x+1)$ is differentiable at every x.

16. Find all the critical points for $f(x) = x^x$

17. Suppose $f \in C[a,b] \cap D(a,b)$ and $|f'(x)| \leq 2$ on $[a,b]$. Prove that f is uniformly continuous on $[a,b]$. Give an example of a function that is uniformly continuous on $[a,b]$ but its derivative is not bounded on $[a,b]$.

18. Suppose $f \in C[a,b] \cap D(a,b)$ and f has an absolute max at an interior point $c \in (a,b)$. Does this imply $f'(c) = 0$? If $f'(c) = 0$ at an interior point $c \in (a,b)$, does this imply that $f(x)$ has a max or a min at $x = c$?

19. If $x(t) = a \cos t$ and $y(t) = b \sin t$, find the extreme values for $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ and locate the points on the path where they occur.

20. Consider the function, $f(x) = 1 - (x-1)^{2/3}$ on $[0,2]$. Does Rolle's theorem apply in this case? Explain.

21. Show that between any two zeroes of $e^x \sin x = 1$, there is at least one real zero of $e^x \cos x = -1$.

22. Show that if $0 < a < b$, then $(1 - a/b) < \ln(b/a) < b/a - 1$.

23. Prove that $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$. Then prove that $0 \leq \sin x \leq 2x/\pi$ for $0 \leq x \leq \pi/2$.

24. If $\alpha\beta > 0$ then show that $x^3 + \alpha x^2 + \beta = 0$ has one real root.

25. If $f'(x) > c > 0$ for all $x \geq 0$, show that $\lim_{x\to\infty} f(x) = \infty$

26. If $f \in C^2(R)$ and $f(x) = 0$ has 3 real roots, show that there is some $z \in R$ where $f''(z) = 0$

27. Suppose $f \in C[a,b] \cap D(a,b)$ Let S denote the set of slopes of all possible secant lines for $y = f(x)$ for $x \in (a,b)$ and let D denote the set of all possible values for $f'(x)$ for $x \in (a,b)$. Show that $S \subset D$ but D need not equal S.

28. Show that the conclusions of the mean value theorem may fail if we drop the condition that f is differentiable at **every** point of (a,b). What if we allow f to be discontinuous at a or b?

29. Show that if $f \in C[a,b]$ and f has first and second derivatives at each point of (a,b), then there exists a $c \in (a,b)$ such that
$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(c)(b-a)^2.$

30. If $f\left(\frac{x+y}{2}\right) = \frac{f(x)-f(y)}{x-y}$ for $x \neq y$, what can you say about $f(x)$?

31. Use what you know about geometric series and Taylor series to obtain the result,
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \xi^6$ for $0 < \xi < x$.

32. Use the result of the previous problem to obtain,
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{\xi^7}{7}$ for $0 < \xi < x$.

33. Use the result of problem 31 to obtain
$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7\xi^6$ for $0 < \xi < x$.

34. We say $f(x) = O(x^p)$ as $x \to 0$, if $f(x)x^{-p}$ tends to a finite limit as $x \to 0$. In this case we say $f(x)$ vanishes to order p at $x = 0$. Find the order for $f(x) = \sin^2 2x$ and $g(x) = 1 - \cos 3x$.

35. Find $O(x^p)$ for $f(x) = x^3 - \sin^3 x$ and $g(x) = x - \ln(1+x) - 1 + \cos x$.

36. Find real numbers A *and* α such that

a. $\text{Lim}_{x\to 0} \frac{1-\cos x}{Ax^\alpha} = 1$

b. $\text{Lim}_{x\to 0} \frac{\sin x - x}{Ax^\alpha} = 1$

37. Can we use L'Hopital's rule be used to determine the limit of the sequence, $a_n = n^3 e^{-n^2}$.

38. Can L'Hopital's rule be used to find $\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$

39. Suppose f is such that for some $C > 0$, $|f(x) - f(y)| \leq C|x-y|^2$. Prove that f is a constant function.

40. Attack or defend the statement "if f is differentiable at all x and $f'(a) = 0$ at some point $x = a$, then f is not injective.

41. Show $f(x) = x^3 + x^2 + 8$ has an inverse and find the derivative of f^{-1}.

42. Suppose $f, g \in D(R)$ and $f(0) = g(0)$. Show that if $f' \leq g'$ on R then $f(x) \leq g(x)$ for all $x \geq 0$.

43. Suppose $f \in D(R)$ and $f(0) = 0$. Show that if $1 \leq f'(x) \leq 2$ on R then $x \leq f(x) \leq 2x$ for all $x \geq 0$.

44. Suppose $f \in D(R)$ and $|f'(x)| < 1 \ \forall x$ For $s_0 \in R$ define $s_n = f(s_{n-1})$ for $n = 1, 2, \ldots$ Show that $\{s_n\}$ is a Cauchy sequence.

45. Evaluate the following limit: $\lim\limits_{x \to 0}\left[\frac{1}{\sin x} - \frac{1}{x}\right]$

46. Evaluate the following limit: $\lim\limits_{x \to 0}[\cos x]^{\frac{1}{x^2}}$

47. Evaluate the following limit: $\lim\limits_{x \to 0}\left[\frac{e^{2x} - \cos x}{\sin x}\right]$

48. Evaluate the following limit: $\lim\limits_{x \to \infty}[x + e^x]^{\frac{1}{x}}$

49. Evaluate the following limit: $\lim\limits_{x \to \infty}\left[1 + \frac{1}{x}\right]^{2x}$

50. Show that if $f(x) = \begin{cases} e^{-1/x} & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}$ then $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$