Problems for Chapter 3.

- Let A denote a nonempty set of reals. The complement of A, denoted by $\complement A,$ or A^C is the set of all points x not in A.
- We say that x belongs to the *interior of A*, $x \in Int(A),$ if there exists a positive ε such that $N_\varepsilon(x) \subset A.$
- We say that x belongs to the *boundary of A*, $x \in \partial A,$ if for every positive ε the neighborhood $N_\varepsilon(x)$ contains points of A and points of A^c.
- We say that x is an *isolated point of* A if there exists a positive ε such that $N_\varepsilon(x)$ contains no points of A other than x.
- A set A is said to be open if all the points of A are interior points. A set A is said to be closed if A^C is open.

1. Find all the interior points, isolated points, accumulation points and boundary points for

a. $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$

b. (a,b) and $[a,b]$

c. $\mathbb{R}$ with $\mathbb{N}$ removed

d. $\mathbb{R}$ with $\mathbb{Q}$ removed

2. Give an example of:

a. A set with no accumulation points.

b. A set with infinitely many accumulation points, none of which belong to the set.

c. A set that contains some, but not all, of its accumulation points

3. Give an example of a set with the following properties or explain why no such set can exist:

a. a set with no accumulation points and no isolated points

b. a set with no interior points and no isolated points

c. a set with no boundary points and no isolated points

4. Is every interior point of A an accumulation point? Is every accumulation point of A an interior point?

5. Let x be an interior point of A and suppose $\{x_n\}$ is a sequence of points, not necessarily in $A,$ but converging to $x.$ Show that there exists an integer N such that $x_n \in A \ \forall n > N$

6. Prove the following statements

a. if G_n is open for every $n \in \mathbb{N},$ then $\bigcup_{n\in\mathbb{N}} G_n$ is open

b. F is closed if and only if F contains all its boundary points

7. Find the interior and boundary for each of the following sets.

a. $A = \left\{ \frac{1}{\sqrt{n}} \ : \ n \in N \right\}$

b. $A = \{x \in Q : \ 0 < x^2 < 2\}$

8. Use both the definition of limit and a sequence approach to establish $\lim_{x \to 2} \frac{1}{1-x} = -1$

9. Use both the definition of limit and a sequence approach to establish $\lim_{x \to 0} \frac{x^2}{|x|} = 0$

10. Use both the definition of limit and a sequence approach to establish $\lim_{x \to 1} \frac{x}{1+x} = 1/2$

11. Show that the limit: $\lim_{x \to 0} \frac{x}{|x|}$ does not exist

12. Show that the limit: $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist

13. Show that the limit: $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist

14. Show that the limit: $\lim_{x \to 0} \frac{1}{\sqrt{x}}$ does not exist

15. Prove that if $a_n \geq 0 \ \ \forall n$ and $a_n \to A,$ then $\sqrt{a_n} \to \sqrt{A}$

16. Find $\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$ or show the limit does not exist

17. Find $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$ or show the limit does not exist

18. Find $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x + 2x^2}$ or show the limit does not exist

19. Find $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ or show the limit does not exist

20. Find $\lim_{x \to 0} \sqrt{x} \ \sin\left(\frac{1}{x}\right)$ or show the limit does not exist

21. Find $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$ or show the limit does not exist

22. Find $\lim_{x \to 0} x \cos\left(\frac{1}{x^2}\right)$ or show the limit does not exist

23. Given that $x - \frac{1}{6}x^3 \leq \sin x \leq x \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)$

24. Given that $x - \frac{1}{6}x^3 \leq \sin x \leq x \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\sin x}{\sqrt{x}}\right)$

25. Given that $1 - \frac{1}{2}x^2 \leq \cos x \leq 1 \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\cos x - 1}{x}\right)$

26. Given that $1 - \frac{1}{2}x^2 \le \cos x \le 1$ *for* $x \ge 0$, find $\lim_{x \to 0} \left(\frac{\cos x - 1}{\sqrt{x}} \right)$

27. Prove that $f(x) = |x|$ is continuous at all values of x. Does $x = 0$ require special attention?

28. Let $f(x) = \frac{\sin x}{\sqrt{x}}$. Can $f(0)$be defined in such a way that f is continuous for all x?

29. For each of the following functions, find the points where they are discontinuous or give a reason they are continuous for all x :

a. $\frac{x^3 + 1}{x^2 + 1}$

b. $\sin^2 x \cos x$

c. $\frac{\cos x - 1}{\sqrt{x}}$

d. $\frac{x - 2}{|x - 2|}$

30. The functions $f(x) = \frac{x^3 - 3x^2 + 2}{x^2 - 1}$ and $g(x) = \frac{x^2 + 4x - 5}{x^3 - 2x^2 + x}$ are each undefined at two points. Are these singularities removable?

31. If $F(x) = f(x) - g(x) + h(x)$, $G(x) = g(x) + 2h(x)$ and $H(x) = 2g(x) - h(x)$are all continuous on $\mathbb{R}$, then is it the case that $f(x)$, $g(x)$, and $h(x)$ are also all continuous on $\mathbb{R}$.

32. Give an example of functions f and g that are both discontinuous at $x = c$ but (i) $f + g$ is continuous at $x = c$ (ii) fg is continuous at $x = c$.

33. Suppose f is continuous for all x and that $f(x) = 0$ for every rational x. Show that $f(x) = 0$ for all x.

34. If f and g are continuous on $\mathbb{R}$ and $f(\frac{p}{q}) = g(\frac{p}{q})$ for all non-zero integers, p, q, then is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?.

35. Suppose $f(x) = \begin{cases} 2x & if \quad x = rational \\ x + 3 & if \quad x = irrational \end{cases}$ .Then find all the points where f is continuous.

36. Suppose f is defined on $(0, 1)$ and that $|f(x)| \le 1$ for all $x \in (0, 1)$. If $\lim_{x \to 0} f(x)$ does not exist then show there must be sequences a_n, b_n converging to 0 for which the sequences $f(a_n)$ and $f(b_n)$ converge but to different limits.

37. Suppose f is continuous at all x and let $P = \{x : f(x) > 0\}$. If $c \in P$ then show that there is an $\varepsilon > 0$ such that $N_\varepsilon(c) \subset P$.

38. Suppose f and g are continuous for all x and let $S = \{x : f(x) \ge g(x)\}$. If c is an accumulation point for S, show that $c \in S$.

39. If f and g are continuous on $[0, 1]$ and $f(x) > g(x)$ for $0 \le x \le 1$ then does there exist a $p > 0$ such that $f(x) \ge g(x) + p$ for $0 \le x \le 1$.

40. If f and g are continuous on $[0, 1]$ and $f(x) > g(x)$ for $0 < x < 1$ then there does exist a $p > 0$ such that $f(x) \ge g(x) + p$ for $0 \le x \le 1$.

41. Let $f(x) = \begin{cases} x+1 & if \quad 0 \le x \le 1,\ x \ne \frac{1}{2} \\ 0 & if \quad x = \frac{1}{2} \end{cases}$ $\qquad g(x) = \begin{cases} \frac{1}{x} & if \quad 0 < x < 1 \\ 0 & if \quad x = 0 \end{cases}$

Explain how you would prove the continuity or lack of continuity for these two functions.i.e., in each case, cite a theorem which supports your answer.

42. Let $f(x) = \dfrac{x}{x^2+1}$ and let

$A = f\{1 \le x \le 8\} \subset rng[f] \qquad M = f^{-1}\{.2 \le y < .4\} \subset dom[f]$. In answering the following questions, give reasons $($ i.e., cite theorems or give examples$)$ for your answers

a. Is A closed? Is it bounded?

b. Find the sup and the inf for A. Do these belong to the set?

c. Find the set of all y that belong to A

d. Find the set of all x that belong to M

e. Either prove that M is closed or show that it is not closed.

43. 1. Suppose A is an infinite subset of the reals and that p is the LUB for A but p does not belong to A. Tell whether the following statements are true or false and give coherent reasons for your answer.

a. p is an accumulation point for A

b. there is a "largest value" x in A such that $x \le p$.

44. State the compact range (extreme value, intermediate value) theorem. Give an example where **one** of the hypotheses is not satisfied and the conclusion then fails to hold.

45. State the persistance of sign theorem Explain the use of this theorem to prove the following result:: If f and g are continuous on $\mathbb{R}$ and $f(x) = g(x)$ for each rational x, then $f(x) = g(x)$ for all real x

46.

a. State a condition on $f(x)$, and D that implies that f is uniformly continuous on D

b. State a condition on $f(x)$, that implies that f is uniformly continuous on (a,b)

c. Is $f(x) = \dfrac{1}{1+x^2}$ uniformly continuous on $(0,\infty)$?

47. Use the intermediate value theorem to tell how many real zeroes exist for the function $f(x) = \sin x - \cos x$ as well as to determine the approximate location of these zeroes.

48. Suppose f and g are continuous on $(-\infty, \infty)$ and that $f(x) = g(x)$ at every rational value for x. Use the persistence of sign result to show that f and g must be equal at every real value.

49. Given that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(0,1)$, is f uniformly continuous on $(0,1)$? Given that the function $g(x) = \sqrt{x}$ is continuous on $(0,1)$, is g uniformly continuous on $(0,1)$?

50. Let A denote the set of all the rational numbers between 0 and 1.

- **a**. Is this a closed set?
- **b**. Is it an open set?
- **c**. What are the accumulation points (boundary points, interior points) for A?

51. For each of the following statements, first, tell whether the following are true or false, then state a theorem which shows a statement is true or give a counter example that shows it is false.

- **a**. If $f(x)$ is continuous and injective on $[a,b]$ and $f(a) < f(b)$, then for all $x,y \in [a,b]$, $x < y$ if and only if $f(x) < f(y)$.

- **b**. There exist sequences $\{a_n\}$ which are bounded but which contain no convergent subsequences.

- **c**. For every real value, x, there is a sequence of rational numbers that converges to x

- **d**. If $\{a_n\}$ in D converges to $c \in D$, and $\{f(a_n)\}$ converges to $f(c)$, then f must be continuous at $x = c$.

- **e**. If F is continuous on D where $F(x) = f(x)g(x)$, then f and g are continuous on D.

52. Use the intermediate value theorem to tell how many real zeroes exist for the function $f(x) = \sin x - \cos x$ as well as to determine the approximate location of these zeroes.

53. Use the intermediate value theorem to tell how many positive zeroes exist for the function $f(x) = \sin x - \frac{1}{x}$ as well as to determine the approximate location of these zeroes.

54. Tell whether the following statements are true or false and cite a theorem to justify your answer:

- **a**. If $f(x)$ is continuous and monotone on $[a,b]$ then $\forall\, x,y \in [a,b]$, $x < y$ implies $f(x) \neq f(y)$.

- **b**. If $\{a_n\}$ is a monotone sequence that does not converge then $\left\{\frac{1}{a_n}\right\}$ must tend to zero as n tends to infinity..

c. If $f(x)$ is continuous on $(-\infty, \infty)$ and $f(x) = 0$ if x is rational, then $f(x) = 0$ at every real x.

d. If A is an infinite subset of the reals and $p = \sup A$ but p is not in A then there is no largest x in A such that $x \leq p$.

e. If F is continuous on D where $F(x) = 5f(x) + 4g(x)$, then f and g are continuous on D.

55. Let $f(x) = \frac{\sin x}{\sqrt{x}}$. Can $f(0)$ be defined in such a way that f is uniformly continuous on $[0, 1]$? Is f uniformly continuous on $[0, \infty)$?