Chapter 6 Infinite Series

In the previous chapter we considered integrals which were improper in the sense that the interval of integration was unbounded. In this chapter we are going to discuss a topic which is somewhat similar, the topic of infinite series. An infinite series is a sum containing an infinite number of terms. For example,

$$1 + 1 + 1 + 1 + \ldots$$
$$1 - 1 + 1 - 1 + 1 - \ldots$$
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots$$

are each examples of infinite series. Since the number of operations needed to compute such a sum is infinite, it is not clear whether or not a finite value can be associated with the sum. It is even less clear how the value of the sum can be found when it is known to be finite. We will show how it can be decided if the series can be associated with a sum and, in special cases, the value of the sum can be determined.

Infinite Series

Let $\{a_n\}$ denote a sequence of real numbers, and define a new sequence, $\{S_N\}$ by

$$S_N = \sum_{n=1}^{N} a_n, \qquad N = 1, 2, \ldots$$

Then

$$S_1 = a_1, \quad S_2 = a_1 + a_2, \quad S_3 = a_1 + a_2 + a_3, \quad etc.$$

We refer to the sequence, $\{a_n\}$, as the **sequence of terms** for the **infinite series** $S = \sum_{n=1}^{\infty} a_n$. We refer to the related sequence, $\{S_N\}$, as the **sequence of partial sums** for the infinite series. If the sequence of partial sums is a convergent sequence, then the infinite series is said to be a convergent infinite series. If the sequence of partial sums is not convergent then the infinite series is not convergent. Note that the sequence of partial sums may fail to converge due to the presence of more than a single limit point or because the sequence of partial sums is not bounded. In the latter case, we say the infinite series diverges to infinity and in the case of a sequence of partial sums having more than a single limit point, we say the infinite series simply fails to converge. In either case, we can say that the infinite series diverges.

(a) Consider the following series, known as the **harmonic series**,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Note that for any positive integer, N,

$$S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

$$S_{2N} = \sum_{n=1}^{2N} \frac{1}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{2N}.$$

Then

$$S_{2N} - S_N = \frac{1}{N+1} + \cdots + \frac{1}{2N}$$
$$\geq \frac{1}{2N} + \cdots + \frac{1}{2N} = N\left(\frac{1}{2N}\right) = \frac{1}{2}.$$

Since $S_{2N} - S_N \geq 1/2$ for every N, the sequence of partial sums is not a Cauchy sequence. It follows that the sequence of partial sums is not convergent and the infinite series is divergent.

(b) Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

The method used in (a) fails to lead to a conclusion. We will show later that this series is convergent.

(c) Consider the **geometric series**,

$$\sum_{n=0}^{\infty} a\, r^n = a + ar + ar^2 + \cdots$$

This geometric series has first term equal to a and **ratio term** equal to r. Note that

$$S_N = a + ar + ar^2 + \cdots + ar^N$$
$$\text{and} \quad rS_N = ar + ar^2 + \cdots + ar^N + ar^{N+1}.$$

Then

$$S_N - rS_N = S_N(1 - r) = a - ar^{N+1},$$

and if $r \neq 1$, we find

$$S_N = \frac{a - ar^{N+1}}{1 - r} \rightarrow \begin{cases} \dfrac{a}{1-r} & \text{if } r < 1 \\ \infty & \text{if } r > 1 \end{cases}.$$

Evidently, if $r < 1$, the sequence of partial sums is convergent to $\frac{a}{1-r}$, hence the series is also convergent and this is the value of the sum. If $r = 1$, then the series is just the sum $a + a + a + \cdots$, which is divergent. The geometric series is one of the very few infinite series for which it is possible to express S_N in terms of N and to then use this expression to determine the convergence of the series.

Tests for Convergence

The convergence or divergence of an infinite series for which no explicit formula for S_N is available must be decided by indirect tests for convergence.

Theorem 6.1 (nth term test) If S_N converges to a limit S, then a_n must converge to zero.

Proof: Note that $a_n = S_n - S_{n-1}$. Then if $\lim_{n \to \infty} S_n = S$, it is easy to see that
$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = S - S = 0.$$

Note that theorem 6.1 asserts that if a_n does not converge to zero then the sequence of partial sums does not converge and the associated infinite series is divergent. The theorem

does not say that if a_n tends to zero then the series converges.

The next result is a simple consequence of the arithmetic with limits theorem from the chapter on sequences.

Theorem *6.2 If $\sum_n a_n$, $\sum_n b_n$ are convergent infinite series with sums A, B respectively, then*

$$\sum_n \alpha a_n + \beta b_n \text{ is also convergent to the sum } \alpha A + \beta B.$$

Positive Term Series

An infinite series for which the sequence of terms contains only positive numbers is called a positive term series. The next two results are true for series whose terms are all positive.

Theorem *6.3 (Comparison test) If $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers such that*

i) for some integer N, $a_n \leq b_n$ for all $n \geq N$

then the convergence of the series $\sum b_n$ implies the convergence of $\sum a_n$ and the divergence of the series $\sum a_n$ implies the divergence of $\sum b_n$. If the condition i) is replaced by

ii) $\displaystyle\lim_{n \to \infty} \frac{a_n}{b_n} = L < \infty$,

then the two series are either both convergent or both divergent.

The test i) is called the direct comparison test while ii) is referred to as the limit comparison test. Clearly these results are related to the squeeze theorem for sequences.

The next result is based on the similarity between improper integrals and infinite series.

Theorem *6.4 (Integral test) Suppose $a(x)$ is a continuous and monotonically decreasing function that is positive on $D = [1, \infty)$. Then the infinite series $\sum_n a_n$, where $a_n = a(n)$, and the improper integral $\int_1^\infty a(x)dx$, either both converge or both diverge.*

Proof: Let P_1 denote the partition $\{1, 2, 3, \dots\}$ for D. Then, by drawing a sketch it is easy to see that

$$\sum_{n=2}^\infty a_n = s[a, P_1] \leq \int_1^\infty a(x)dx \leq S[a, P_1] = \sum_{n=1}^\infty a_n$$

from which the result follows.

Example

(a) Consider the series of the previous example, part (b)

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

and note that $a_n = a(n)$ defines the function $a(x) = \dfrac{1}{x^2}$, on the domain, $D = [1, \infty)$. Then $a(x)$ is positive, continuous and monotone decreasing on D so the integral test applies.

3

Since,

$$\int_1^\infty \frac{1}{x^2} = \lim_{L\to\infty} \int_1^L \frac{1}{x^2} = \lim_{L\to\infty}(-\tfrac{1}{x}|_{x=1}^{x=L}) = \lim_{L\to\infty}\left(1 - \frac{1}{L}\right) = 1$$

it follows from theorem 6.4 that the series is convergent.

(b) More generally, consider

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

Then $a_n = a(n)$ becomes $a(x) = \frac{1}{x^p}$, on $D = [1,\infty)$, and

$$\int_1^\infty \frac{1}{x^p} = \lim_{L\to\infty} \int_1^L \frac{1}{x^p} = \lim_{L\to\infty}(\frac{x^{1-p}}{1-p}|_{x=1}^{x=L}) = \frac{1}{1-p} \lim_{L\to\infty}(1 - L^{1-p}).$$

In this case,

$$\lim_{L\to\infty}(1 - L^{1-p}) = \begin{cases} 1 & if \quad p > 1 \\ Div & if \quad p < 1 \end{cases}$$

and theorem 6.4 leads to the result that the series converges if $p > 1$.

(c) We can apply the integral test to the harmonic series

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

In this case $a_n = a(n)$ becomes $a(x) = \frac{1}{x}$, on $D = [1,\infty)$, and

$$\int_1^\infty \frac{1}{x} = \lim_{L\to\infty} \int_1^L \frac{1}{x} = \lim_{L\to\infty}(\ln(x)|_{x=1}^{x=L}) = \lim_{L\to\infty} \ln(L) = \infty.$$

Since the improper integral is divergent, the infinite series is divergent as well. This has already been proved by another argument.

Absolute and Conditional Convergence
Not every infinite series is a positive term series. If $\sum a_n$ is an infinite series containing both positive and negative terms, then we say that the series is ***absolutely convergent*** if the associated positive terms series $\sum |a_n|$ is convergent. Every series which is absolutely convergent can be shown to be convergent, but the converse is false. A series which is convergent but not absolutely convergent is said to be ***conditionally convergent***. A series that is conditionally convergent depends on the cancellation of positive and negative terms for its convergence while an absolutely convergent series converges simply because the terms decrease so rapidly that there is no need to rely on cancellation.

Example
The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^\infty (-1)^{n+1} \frac{1}{n}$$

contains both positive and negative terms. Note that

4

$$1 - \frac{1}{2} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{} + \underbrace{\frac{1}{5} - \frac{1}{6}}_{} + \underbrace{\frac{1}{7} - \frac{1}{8}}_{} + \cdots + \underbrace{\frac{1}{2n-1} - \frac{1}{2n}}_{}$$

$$\underbrace{\frac{1}{2}} \qquad \underbrace{\frac{1}{12}} \qquad \underbrace{\frac{1}{30}} \qquad \underbrace{\frac{1}{56}} \qquad \underbrace{\frac{1}{2n(2n-1)}}$$

and since

$$\frac{1}{2n(2n-1)} \le \frac{1}{n^2} \quad if \ \ n \ge 1$$

we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} \le \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Then the series is convergent by theorem 6.3. On the other hand, the series is not absolutely convergent since we have already shown that the series whose terms are $|a_n| = \left| (-1)^{n+1} \frac{1}{n} \right| = \frac{1}{n}$, is divergent. Therefore the original series is conditionally convergent but not absolutely convergent.

Theorem *6.5 If an infinite series is absolutely convergent then it is convergent.*

The previous example illustrates that the converse of this theorem is false.

If $\{a_n\}$ is a sequence of positive numbers then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

is called an **alternating series**. The convergence of an alternating series is easily determined.

Theorem *6.6 (Alternating Series Test) Suppose the positive term sequence $\{a_n\}$ satisfies*

$$a_{n+1} < a_n \ \ for \ all \ n$$

$$\lim_{n \to \infty} a_n = 0$$

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to a finite sum S, and for each N > 0 we have $|S - S_N| \le a_{N+1}$.

Proof: Note that for all N,

$$S_{2N+2} = S_{2N} + a_{2N+1} - a_{2n+2}$$

$$S_{2N+1} = S_{2N-1} - a_{2N} + a_{2n+1}$$

Since $a_{n+1} < a_n$ for all n, it follows that the even order partial sums $\{S_{2n}\}$ are a monotone increasing sequence while the odd order partial sums are a monotone decreasing sequence. In addition, for any m and n, such that $2n - 1 > 2m$,

$$S_{2n-1} - S_{2m} = (a_{2n-1} - a_{2n}) + \cdots + (a_{2m+3} - a_{2m+2}) + a_{2m+1} > 0.$$

On the other hand, for any m and n, such that $2n - 1 < 2m$,

$$S_{2m} - S_{2n-1} = -a_{2m} + a_{2m-1} - \cdots - a_{2n+2} + a_{2n+1} - a_{2n} < 0.$$

In either case, $S_{2m} < S_{2n-1}$, which implies that the monotone increasing sequence of even order sums is bounded above be every odd order sum, while the monotone decreasing sequence of odd order sums is bounded below by every even order sum. Then the monotone sequence theorem asserts that both sequences must converge. To see that they converge to the same limit, note that

$$\lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (S_{2n+1} - S_{2n})$$
$$= \lim_{n\to\infty} a_{2n+1} = 0.$$

If we denote the limit of the sequence of partial sums by S, then

$$0 < S_{2n-1} - S < S_{2n-1} - S_{2n} = a_{2n}$$
$$0 < S - S_{2n} < S_{2n+1} - S_{2n} = a_{2n+1}$$

which implies that for all n,

$$|S - S_n| < a_{n+1}.$$

i.e., we can approximate S by S_n with an error less than a_{n+1}. ∎

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series, whose terms need not be positive, but for which $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists and equals r. This means that this series behaves ultimately like a geometric series having ratio term r and we can show that if $r < 1$, the series is convergent, (although we are not entitled to conclude that the sum equals $\frac{1}{1-r}$). Similarly, if $r > 1$, we can conclude that the series is divergent. If $r = 1$ then we can only conclude that the series does not behave like a geometric series and some other approach must be found to determine the convergence or divergence. We can state this result in a theorem.

Theorem *6.7 (Ratio test) Suppose the sequence $\{a_n\}$ satisfies*

$$i) \quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r.$$

Then the infinite series $\sum a_n$ is absolutely convergent if $r < 1$ and is divergent if $r > 1$. If $r = 1$, the test fails.

The condition i) implies that the series behaves ultimately like a geometric series with ratio term equal to r. The same result is true if the condition i) is replaced by the condition,

$$ii) \quad \lim_{n\to\infty} |a_n|^{1/n} = r.$$

Example

(a) Consider the alternating series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{2^n}.$$

We are free to apply either theorem 6.6 or 6.7. Note that,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \left(\frac{n}{n+1} \right)^2 \frac{1}{2},$$

and since

$$\lim_{n \to \infty} \left(\frac{n}{n+1} \right)^2 = 1,$$

we have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} < 1.$$

It follows from theorem 6.7 that the series is absolutely convergent. We can also use the alternating series test which tells us that the series is convergent with $|S - S_N| \leq \dfrac{(N+1)^2}{2^{N+1}}$.

(b) If we apply theorem 6.7 to the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

we find

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n} \right)^p \to 1 \ as \ n \to \infty.$$

Then the ratio test fails to tell us anything about the convergence or divergence for this series. As we have seen in a previous example the integral test implies the series converges if $p > 1$.

Exercises
Test the following series for convergence.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{e^{-n}}{\sqrt{n+\sqrt{10}}}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n-\sqrt{10}}}{\sqrt{n+\sqrt{10}}}$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^3+\sqrt{10}}}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{n^2+4}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{n^3+4}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{\ln(n^2+2)-\ln(n^2+1)}{n^2+4}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{2^n+1}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+1}{n^2+1}$

9. $\displaystyle\sum_{n=0}^{\infty} 3^{-n}$

10. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^{-n}$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

12. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

13. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n}{\sqrt{n^2+1}}$

14. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n}{n^{1+1/n}}$

15. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\ln(n+1)}$

Power Series

For a given sequence of terms $\{a_n\}$, and a fixed real number c, define

$$F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

Then $F(x)$ is a function whose domain, D, is the set of points x, where the infinite series converges. We refer to this as a power series in x, expanded about the point $x = c$. In order to determine D, it is often possible to use the ratio test.

(a) Consider the function F_1 defined by the following power series

$$F_1(x) = \sum_{n=0}^{\infty}(n!)(x-c)^n.$$

Here we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!}\frac{(x-c)^{n+1}}{(x-c)^n}\right|$$

$$= (n+1)|x-c| \rightarrow \begin{cases} 0 & if \ x = c \\ \infty & if \ x \neq c \end{cases}$$

Then D_1 consists of just the single point, $x = c$.

(b) Consider the function F_2 defined by

$$F_2(x) = \sum_{n=0}^{\infty}\frac{1}{2^n}(x-c)^n$$

In this case,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^n}{2^{n+1}}\frac{(x-c)^{n+1}}{(x-c)^n}\right| = \frac{1}{2}|x-c| = L$$

and $L < 1$ if $|x-c| < 2$. Therefore F_2 has as its domain, $D_2 = \{|x-c| < 2.\}$. This is an open interval about $x = c$ of radius 2 or total width 4.

(c) Let F_3 be defined by

$$F_3(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Here we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n!}{(n+1)!} \frac{x^{n+1}}{x^n} \right| = \frac{|x|}{n+1} \to 0 \quad \text{for all } x.$$

Since $L = 0$ for any value of x, this series is convergent for all values of x; i.e., $D_3 = \{-\infty < x < \infty\}$.

We observe from these examples that a power series is convergent on a set of the form $D = \{|x - c| < R\}$ where

$$1/R = L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

The number $R = 1/L$ is called the **radius of convergence** of the series and, as we see, R can be 0, infinity or anything in between.

Note that formally, we have

$$F'(x) = \sum_{n=0}^{\infty} a_n n (x - c)^{n-1}$$

and $\quad \displaystyle\int_c^x F = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1},$

and, by the ratio test, these series have the same domain of convergence as does the series for $F(x)$. Of course these are just formal remarks at this point, with no proof that these representations are valid. However, if we let

$$F_N(x) = \sum_{n=0}^{N} a_n (x - c)^n,$$

then for each integer N, $F_N(x)$ is a polynomial in x of degree N and it therefore continuous with continuous derivatives of all orders. Then we can ask,

- in what sense is it true that F_N converges to F as N tends to infinity?
- does it follow from this that F_N' converges to F' as N tends to infinity?
- does it also follow that $\int_c^x F_N$ converges to $\int_c^x F$ as N tends to infinity?
- if F_N does converge to F what properties of F_N are carried over to F?

Answering these questions will occupy the rest of the time in this course. The following examples show that the properties of the limit function are not necessarily the same as the properties of the functions in the sequence.

(a) Let

$$f_n(x) = \frac{x^2}{(1 + x^2)^n} \quad \text{for } x \in R, \quad n = 1, 2, \ldots$$

and

$$F(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n}.$$

Since $f_n(0) = 0$ *for* each n, we have $F(0) = 0$. For $x \neq 0$, the series for F is just a geometric series with $r = (1 + x^2)^{-1} < 1$, and it is easy to show that

$$F(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 + x^2 \quad for \quad x^2 > 0.$$

Then

$$F(x) = \begin{cases} 0 & if \quad x = 0 \\ 1 + x^2 & if \quad x^2 > 0 \end{cases}$$

so a convergent series of continuous functions need not have a continuous sum.

(b) Let $g_n(x) = \sin(n^2 x)/n$ for $0 \le x \le \pi$, $n = 1, 2, \ldots$. Then it is easy to show that $g_n(x)$ converges to zero as n tends to infinity for every x in $[0, \pi]; i.e,$ $g_n(x)$ converges to $g(x) = 0$. It is also clear that $g_n'(x) = n\cos(n^2 x)$. and this sequence diverges at every x where $\cos x$ is different from zero. Evidently differentiating a convergent sequence of differentiable functions need not lead to a convergent sequence much less a sequence which converges to the derivative of the limit of the original sequence.

(c) Let

$$h_n(x) = \begin{cases} (2n)^2 x & if \quad 0 \le x < \frac{1}{2n} \\ 4n - (2n)^2 x & if \quad \frac{1}{2n} < x < \frac{1}{n} \\ 0 & if \quad x > \frac{1}{n} \end{cases}$$

Then

$$\int_0^1 h_n(x)dx = 1 \quad \text{for every } n$$

but $h_n(x) \to 0$ as n tends to infinity. To see this note that for any c, $0 < c < 1$, $h_n(c) = 0$ if $\frac{1}{n} < c$; i.e., if $n > \frac{1}{c}$. . Thus the sequence of functions converges to the limit $h(x) = 0$ on $[0, 1]$ but the integrals of the $h_n's$ does not converge to the integral of $h(x) = 0$.

3. Sequences of Functions

In order to answer the questions we have about power series, it will be necessary to first consider an apparently simpler topic, convergence of sequences of functions.

Let $\{f_n(x) : n = 1, 2, \ldots\}$ denote a family of functions on a common domain, $D = [a, b]$. For each x_0 in D, consider the sequence of real numbers, $\{f_n(x_0)\}$ and let $f(x_0)$ denote the limit of this sequence as n tends to infinity. Since the value of this limit will likely depend on x_0, we denote the limit by $f(x_0)$. Then when we write

$$\lim_{n \to \infty} f_n(x) = f(x)$$

what we mean is that for each x_0 in D, the sequence of real numbers, $\{f_n(x_0)\}$ converges to a limit, denoted by $f(x_0)$. More precisely, we mean that:

Definition $\lim_{n \to \infty} f_n(x) = f(x)$ *if and only if for each x_0 in D,and for each $\varepsilon > 0$, there exists an $N > 0$ such that $|f_n(x_0) - f(x_0)| < \varepsilon$ for all $n > N$.*

Example
(a)Consider

$$f_n(x) = \frac{nx}{1 + n^2 x^2} \qquad on \qquad D = [-1, 1]$$

Then we have $f_n(0) = 0$, $\forall n$ and for $0 < |x_0| \leq 1$

$$|f_n(x_0)| = \left| \frac{nx_0}{1 + n^2 x_0^2} \right| \leq \frac{n|x_0|}{n^2 x_0^2} = \frac{1}{n|x_0|}$$

Then

$$|f_n(x_0)| \leq \frac{1}{n|x_0|} < \varepsilon \quad if \quad n > \frac{1}{\varepsilon|x_0|} = N(x_0, \varepsilon).$$

This shows that for each x_0 in $[-1, 1]$, and for each $\varepsilon > 0$, there exists $N > 0$, such that $|f_n(x_0) - 0| < \varepsilon$ for all $n > N$; i.e., $f_n(x)$ converges to the limit zero.

(b)Consider

$$g_n(x) = \frac{x^n}{1 + x^{2n}} \qquad on \qquad D = [0, \infty).$$

We have $g_n(0) = 0$ and $g_n(1) = \frac{1}{2}$ $\qquad \forall n$. In addition, for $0 < x_0 < 1$

$$g_n(x_0) = \frac{x_0^n}{1 + x_0^{2n}} \leq x_0^n.$$

Then for $x_0 > 1$

$$g_n(x_0) = \frac{x_0^n}{1 + x_0^{2n}} \leq \frac{x_0^n}{x_0^{2n}}$$

$$\leq \frac{1}{x_0^n} < \varepsilon \quad if \quad n > \frac{-\log \varepsilon}{\log x_0}.$$

If $0 < x_0 < 1$, then

$$g_n(x_0) = \frac{x_0^n}{1 + x_0^{2n}} \leq x_0^n < \varepsilon \quad for \quad n > \frac{\log \varepsilon}{\log x_0}$$

This shows that for each x_0 in $[0, \infty)$, and for each $\varepsilon > 0$, there exists $N(x_0) > 0$, such that $|g_n(x_0) - 0| < \varepsilon$ for all $n > N(x_0)$; i.e., $g_n(x)$ converges to the limit

$$g(x) = \begin{cases} 1/2 & if \quad x = 1 \\ 0 & otherwise \end{cases}$$

(c)Consider

$$h_n(x) = \frac{x^n}{1 + x^n} \qquad on \qquad D = [0, \infty)$$

In this case we have $\qquad h_n(0) = 0 \quad and \quad h_n(1) = \frac{1}{2} \qquad \forall n$. In addition, for $0 < x_0 < 1$,

$$h_n(x_0) = \frac{x_0^n}{1 + x_0^n} \leq x_0^n$$

and for $x_0 > 1$

$$h_n(x_0) = \frac{x_0^n}{1 + x_0^n} \leq \frac{1}{x_0^{-n} + 1}$$

Then

$$h_n(x_0) \to 0 \quad when \quad 0 \le x_0 < 1$$
$$h_n(x_0) \to 1. \quad when \quad x_0 > 1$$
$$h_n(1) = \frac{1}{2} \quad \forall n$$

hence

$$\lim_{n \to \infty} h_n(x) = \begin{cases} 0 & if \quad 0 \le x < 1 \\ 1/2 & if \quad x = 1 \\ 1 & if \quad x > 1 \end{cases}$$

(d) Consider

$$p_n(x) = \begin{cases} 1 & if \quad |x| < n \\ 0 & if \quad |x| \ge n \end{cases} \quad on \quad D = (-\infty, \infty)$$

By the definition of p_n, for any x_0, $\quad p_n(x_0) = 1 \quad if \quad n > |x_0|$;
i.e.,

$$for \ any \ x_0, \ and \ any \ \varepsilon > 0$$
$$|p_n(x_0) - 1| = 0 < \varepsilon \quad if \quad n > |x_0|.$$

Then for all x, $\quad \lim_{n \to \infty} p_n(x) = 1$.

(e) Consider the sequence,

$$q_n(x) = \frac{x^n}{2^n} \quad on \quad D = [0, 1].$$

Then for all n,

$$q_n(0) = 0 \quad and \quad q_n(x) \le \frac{1}{2^n} \quad for \quad 0 < x \le 1$$

It follows that $\lim_{n \to \infty} q_n(x) = 0$; i.e., for all $x \in [0, 1]$,

$$|q_n(x) - 0| \le 2^{-n} < \varepsilon \quad if \quad n > \frac{-\log \varepsilon}{\log 2} = N(\varepsilon).$$

Note that N here depends only on ε and not on x.

Pointwise and Uniform Convergence
In these examples, we were able to show that $|f_n(x) - f(x)| < \varepsilon$ for $n > N(x, \varepsilon)$. In each example but the last, the integer N depended not just on ε but on the point x in D. These examples satisfy the condition in the definition of convergence of $f_n(x)$ to $f(x)$. We say for each of these previous examples that the sequences converge **pointwise** to their limit function.

For the last example sequence $\{q_n(x)\}$, for each $\varepsilon > 0$, there exists $N > 0$ such that $\sup_{x \in D} |q_n(x) - q(x)| < \varepsilon$ for all $n > N(\varepsilon)$. It is important to notice that in this example, N depends on ε but it does not depend on $x \in D$. In such cases, we say the sequence $\{q_n(x)\}$ converges **uniformly** to the limit function, $q = 0$.

Example
(a) Consider

$$f_n(x) = (\sin x)^n \quad on \quad D = [0, \pi]$$

Then $f_n(\pi/2) = 1$ for every n and, for
$x \neq \pi/2$, $f_n(x) = (\sin x)^n = r^n$ with $r = \sin x < 1$. Then for each $x_0 \neq \pi/2$, $r_0 = \sin x_0 < 1$,

$$|f_n(x_0)| = r_0^n < \varepsilon \quad if \quad n > \frac{\log \varepsilon}{\log r_0} = N(\varepsilon, r_0)$$

and

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 1 & if \quad x = \pi/2 \\ 0 & if \quad x \neq \pi/2 \end{cases}$$

Since $N(\varepsilon, r_0)$ depends on ε and on x_0 (via r_0), we conclude that the sequence converges pointwise to its limit.

(b)Consider

$$g_n(x) = \frac{\sin (nx)}{n} \quad on \quad D = [0, \pi]$$

In this case,

$$g_n(x) = \frac{\sin (nx)}{n} \leq \frac{1}{n} \quad on \quad D$$

and

$$|g_n(x)| \leq \frac{1}{n} < \varepsilon \quad if \quad n > \frac{1}{\varepsilon} = N(\varepsilon).$$

Since N does not depend on x, we conclude that g_n converges uniformly to zero on D.

In general, to determine whether a given sequence of functions converges uniformly to a limit, it is necessary to proceed as in the most recent example to see whether N can be chosen independently of x.

For infinite series, there is a simple test which indicates uniform convergence.

Theorem *6.8 Let $\{f_n(x) : n = 1, 2, \ldots\}$ denote a family of functions on a common domain, $D = [a, b]$, and suppose there exists a sequence of positive constants $\{M_n\}$ that satisfies the following two conditions: i) for each n, $|f_n(x)| \leq M_n$ for all $x \in D$, and ii) $\sum_n M_n$ is convergent. Then $\sum_n f_n(x)$ converges uniformly on D.*

Proof To prove that $\sum_n f_n(x)$ converges it will be sufficient to prove that the sequence of partial sums, $S_N(x) = \sum_n^N f_n(x)$, is a Cauchy sequence, uniformly in x. That is, it must be proven that for every $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ (independent of x) such that $|S_N(x) - S_M(x)| < \varepsilon$ whenever, $M, N > N_\varepsilon$. To show this write

$$|S_N(x) - S_M(x)| = \left| \sum_M^N f_n(x) \right| \leq \sum_M^N |f_n(x)| \leq \sum_n^N M_n.$$

Since $\sum_n M_n$ is convergent, for every $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that $\sum_n^N M_n < \varepsilon$, and the result follows.

Corollary *Suppose the power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *has radius of convergence*
$R,\ 0 < R \leq \infty.$ *Then the series converges uniformly on the set* $\{|x-c| \leq R_0\}$ *for every*
$R_0 < R.$

Proof Note that on the set $\{|x-c| \leq R_0\},$ we have $f_n(x) = a_n(x-c)^n$ satisfying
$|f_n(x)| \leq a_n R_0^n.$ It is furthermore evident by applying the ratio test to the series $\sum_{n=0}^{\infty} a_n R_0^n$ that
the series is convergent; ie. we compute,

$$\lim_{n\to\infty} \left| \frac{a_{n+1} R_0^{n+1}}{a_n R_0^n} \right| = R_0 \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = R_0 L = R_0/R < 1,$$

and since the limit of the ratio of consecutive terms is less than one, the series converges.
Here we recall from the ratio test that the radius of convergence R is the reciprocal of the
limit, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$

Consequences of Uniform Convergence

For infinite series of functions we can use theorem 6.8 to determine if the series converges
uniformly. On the other hand, we have no such simple approach to determine whether the
convergence of a sequence $\{f_n(x)\}$ of functions on a common domain D converges
uniformly or just pointwise. In general, to determine whether a given sequence of functions
converges uniformly to a limit, it is necessary to show that for each given $\varepsilon > 0,$ the
corresponding N can be chosen independently of x in $D.$ As to why it is important to know
whether or not a sequence converges uniformly, consider the following theorem.

Theorem *6.9 Let* $\{f_n(x) : n = 1, 2, \dots\}$ *denote a family of continuous functions on a common*
domain, $D = [a,b],$ *and suppose the sequence* f_n *converges uniformly on D to the limit*
f. Then f is necessarily continuous on D.

Corollary *Let* $\{f_n(x) : n = 1, 2, \dots\}$ *denote a family of continuous functions on a common*
domain, $D = [a,b],$ *and suppose the infinite series* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on D to*
the sum, $F(x).$ *Then* $F(x)$ *is necessarily continuous on D.*

With these results in hand, we can now answer each of the questions that were posed in
section 2.

Theorem *6.10 Let* $\{f_n(x) : n = 1, 2, \dots\}$ *denote a family of continuous functions on a common*
domain, $D = [a,b],$ *and suppose the sequence* f_n *converges uniformly on D to the limit*
f. Then

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f$$

Corollary *Let* $\{f_n(x) : n = 1, 2, \dots\}$ *denote a family of continuous functions on a common*
domain, $D = [a,b],$ *and suppose the infinite series* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on D to*

the sum, $F(x)$. Then

$$\sum_{n=1}^{\infty} \int_a^b f_n = \int_a^b \sum_{n=1}^{\infty} f_n = \int_a^b F$$

Theorem *6.11 Let $\{f_n(x) : n = 1, 2, \dots\}$ denote a family of continuously differentiable functions on a common domain, $D = [a,b]$, and suppose that: i) the sequence f_n converges uniformly on D to the limit f, and ii) the sequence of derivatives, f'_n converges uniformly on D to the limit g. Then f is differentiable and $f'(x) = g(x)$*

Corollary   *Let $\{f_n(x) : n = 1, 2, \dots\}$ denote a family of continuously differentiable functions on a common domain, $D = [a,b]$, and suppose that: i) the infinite series $\sum_{n=1}^{\infty} f_n$ converges uniformly on D to the sum, $F(x)$, and ii) the differentiated infinite series $\sum_{n=1}^{\infty} f'_n$ converges uniformly on D to the sum, $G(x)$. Then $F' = G$; i.e.,*

$$F'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f'_n(x) = G(x).$$

Proof of theorem 6.8- Let $\{f_n(x) : n = 1, 2, \dots\}$ denote a family of continuous functions on a common domain, $D = [a,b]$, and suppose the sequence f_n converges uniformly on D to the limit f. Let x, y denote arbitrary points in $[a,b]$ and let $\varepsilon > 0$ be given. Now write

$$|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$$
$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

Since f_n converges uniformly on D to the limit f, there exists an integer $N_\varepsilon > 0$ such that

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3},$$
$$\text{and} \quad |f(y) - f_n(y)| < \frac{\varepsilon}{3} \quad \text{for all } n > N_\varepsilon$$

Note that we have used the uniform continuity here to ensure that the same N works for both x and y. Now fix an $n > N_\varepsilon$ and use the fact that every f_n is continuous on D to conclude that there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad |x - y| < \delta.$$

Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{whenever} \quad |x - y| < \delta.$$

Exercises

Find the values of x for which the following power series converge:

1. $\displaystyle\sum_{n=1}^{\infty} n x^n$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x)^n}{2^n n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n \,(2x-2)^n}{n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n\,(x+1)^n}{2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2(x-2)^{2n}}{3^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n \,(2x-2)^n}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n \,(2x+4)^n}{n3^n}$