Chapter 5 Integration

In this chapter we introduce the notion of the Riemann integral of a bounded function on a closed bounded interval. Two equivalent definitions are given for the integral; one is stated in terms of upper and lower sums, while the other defines the integral as a limit of Riemann sums. The equivalence of these two definitions is asserted in Theorem 5.4. It is convenient to have the two definitions for the integral since certain facts about the integral follow most easily from one definition while other facts are more easily proved using the other definition.

Once the integral is defined, it then follows that in order for the integral of a given function $f(x)$ over an interval $I = [a.b]$ to exist, it is sufficient for $f(x)$ to be either continuous or monotone on $I.$ A condition which is both necessary and sufficient to imply integrability of $f(x)$ can be stated in terms of the more advanced concept of sets of measure zero.

The integral has a number of fundamental properties including linearity, additivity and positivity. In addition, there are several inequalities that apply to integrals allowing us to estimate the magnitude of an integral without actually evaluating the integral. The Cauchy-Schwarz and Minkowski inequalities are two well known inequalities for integrals. There is a mean value theorem for integrals and this leads to a variation of Taylor's theorem that involves an integral form for the remainder term.

The fundamental theorem of calculus is the most powerful tool for evaluating integrals. The fundamental theorem establishes a somewhat unanticipated link between the derivative and the integral. The integration by parts formula and the change of variables theorem extend the applicability of the fundamental theorem.

Finally, since not every combination of the so called elementary functions equals the derivative of an elementary function, it is often not possible to evaluate an integral by means of the fundamental theorem. In such cases we can resort to approximation techniques referred to as numerical quadrature schemes. The trapezoid rule and Simpson's rule are two widely used numerical integration schemes.

Partitions

Throughout this chapter, we will let I denote a closed bounded interval $[a,b].$ It will sometimes be convenient to denote the length of I by writing $|I|;$ i.e. $|I| = |[a,b]| = b - a.$

By a ***partition*** of $I,$ we mean a set $P = \{x_0, x_1, \ldots, x_N\}$ of points of the interval such that $a = x_0 < x_1 < \cdots < x_N = b.$ Then the points of the partition P determine a set of nonoverlapping intervals whose union is $I;$ i.e. for $k = 1,2,\ldots,N,$ $I_k = [x_{k-1}, x_k]$ meets I_j in at most an endpoint if $j \neq k$ and

$$I = I_1 \cup \cdots \cup I_N = \bigcup_{n=1}^{N} I_n$$

We define the ***mesh size*** for the partition $P = \{x_0, x_1, \ldots, x_N\}$ to be the length of the longest subinterval in $P;$ i.e., if we denote the mesh size of P by $\|P\|,$ then

$$\|P\| = \max_{1 \leq n \leq N} |I_n| = \max_{1 \leq n \leq N} |x_n - x_{n-1}|$$

A partition P^* is said to be a ***refinement of the partition*** P if each mesh point x_n of P is also a mesh point of $P^*.$ If P^* is a refinement of $P,$ then $\|P^*\| \leq \|P\|.$ We will denote by $\Pi[I]$ the set of all possible partitions of the interval $I.$

Sums

Suppose $f(x)$ is defined and bounded on the interval $I = [a, b]$ and let m and M denote, respectively, the greatest lower bound and least upper bound for the bounded set $Rng f = \{f(x) : x \in I\}$. Let $P = \{x_0, x_1, \ldots, x_N\}$ denote a partition for I, and for each $k = 1, 2, \ldots, N$, let m_k and M_k denote, respectively, the greatest lower bound and least upper bound for the bounded sets $J_k = \{f(x) : x \in I_k\} = \{f(x) : x_{k-1} \leq x \leq x_k\}$. Then

$$m \leq f(x) \leq M \qquad \text{for all } x, \quad a \leq x \leq b$$

and $$m_k \leq f(x) \leq M_k \qquad \text{for all } x, \text{ such that } x_{k-1} \leq x \leq x_k, \quad 1 \leq k \leq N.$$

Using these notations we can define various sums for the function $f(x)$ on the partition P.

Definition *Lower Sum*

The lower sum for $f(x)$ based on the partition P of I is denoted by $s[f : P]$. It equals

$$s[f : P] = \sum_{k=1}^{N} m_k |I_k|$$

Definition *Upper Sum*

The upper sum for $f(x)$ based on the partition P of I is denoted by $S[f : P]$. It equals

$$S[f : P] = \sum_{k=1}^{N} M_k |I_k|$$

In addition, if $\xi_1, \ldots, \xi_N$ denote points of I such that $\xi_k \in I_k$ for $1 \leq k \leq N$, then we can define

Definition *Riemann Sum*

The Riemann sum for $f(x)$ based on the partition P of I and ***evaluation points*** $\{\xi_1, \ldots, \xi_N\}$, is denoted by $RS[f : P : \xi]$. It equals

$$RS[f : P : \xi] = \sum_{k=1}^{N} f(\xi_k) |I_k|$$

Note that the upper and lower sums are not Riemann sums unless there exist points μ_k, ν_k in each I_k such that $f(\nu_k) = m_k$ and $f(\mu_k) = M_k$. We have the following properties of sums based on partitions.

Theorem ***5.1*** *Let $f(x)$ be defined and bounded on $I = [a, b]$*

a For every $P \in \Pi[I]$ and for any choice of evaluation points $\{\xi_1, \ldots, \xi_N\}$,

$$m|I| \leq s[f : P] \leq RS[f : P : \xi] \leq S[f : P] \leq M|I|$$

b If P^* is a refinement of P in $\Pi[I]$, then

$$s[f : P] \leq s[f : P^*] \qquad \text{and} \qquad S[f : P^*] \leq S[f : P]$$

c for all P and Q in $\Pi[I]$,

$$s[f : P] \leq S[f : Q]$$

Statement a) asserts that for a given function $f,$ and interval $I,$ the smallest possible lower sum equals $m|I|$ while the largest possible upper sum equals $M|I|.$ Also for any partition P of $I,$ any Riemann sum is greater than or equal to every lower sum and it is less than or equal to every upper sum. Statement b) asserts that refining a partition P of I causes lower sums to increase or stay the same while upper sums decrease or stay the same. Finally, statement c) says that no lower sum is ever greater than any upper sum.

Definition of the Integral

There are various ways in which we might define the notion of an integral of a bounded function on a bounded interval. One way is to base the definition on the notion of upper and lower sums.

Let $f(x)$ be defined and bounded on $I = [a,b].$ Then it follows from part $c)$ of the previous theorem that for any $P \in \Pi[I],$ the upper sum $S[f : P]$ is an upper bound for the set of lower sums $\{s[f : Q] : Q \in \Pi[I]\}.$ Similarly, the lower sum $s[f : P]$ is a lower bound for the set of upper sums $\{S[f : Q] : Q \in \Pi[I]\}.$ Then the completeness axiom implies that the set of lower sums has a least upper bound and the set of upper sums has a greatest lower bound. Then we can define

Definition *Lower Integral*

The ***lower integral*** for $f(x)$ on I equals $s[f] = \sup\{s[f : Q] : Q \in \Pi[I]\}$

Definition *Upper Integral*

The ***upper integral*** for $f(x)$ on I equals $S[f] = \inf\{S[f : Q] : Q \in \Pi[I]\}$

Theorem ***5.2*** *Let $f(x)$ be defined and bounded on $I = [a,b].$ Then f has both an upper integral, $S[f],$ and a lower integral, $s[f],$ on $I,$ and*

$$m|I| \leq s[f] \leq S[f] \leq M|I|$$

We now define the notion of the ***Riemann Integral*** for f on $I.$ This can be done in two ways which we will show later are equivalent.

Definition of the Integral in terms of Upper and Lower Sums

If f is defined and bounded on I then f has both lower and upper sums for every partition hence f has both an upper and a lower integral. These numbers need not be equal but for many functions $s[f]$ and $S[f]$ will have the same value.

Definition *(Riemann Integral) Let $f(x)$ be defined and bounded on $I = [a,b].$ Then f is said to be Riemann integrable on I if $s[f] = S[f].$*

In this case we use the notation $\int_a^b f(x)\,dx$ or $\int_I f$ to denote the common value $s[f] = S[f]$ which we then refer to as the ***Riemann integral*** of f over $I.$

Limit Definition of the Integral

We can also define the Riemann integral in terms of a limit of Riemann sums. For $f(x)$ defined and bounded on $I = [a,b],$ we say that the limit of $RS[f : P : \xi]$ as $\|P\|$ tends to zero exists and equals L if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $P \in \Pi[I]$ with $\|P\| \leq \delta$ and any choice of evaluation points ξ_k, it follows that $|RS[f : P : \xi] - L| \leq \varepsilon$. We write this as

$$\lim_{\|P\| \to 0} RS[f : P : \xi] = L.$$

Theorem *5.3 Suppose $f(x)$ is defined on $I = [a,b]$. If $\lim_{\|P\| \to 0} RS[f : P : \xi]$ exists then the limit value is unique. Moreover, if the limit exists then $f(x)$ must be bounded on I.*

In view of theorem 5.3, we can define the integral of f on I in terms of the limit of the Riemann sums. It remains now to show that the two ways of defining the integral are equivalent.

Theorem *5.4 Let $f(x)$ be defined and bounded on $I = [a,b]$. Then the following statements are equivalent:*

a $\lim_{\|P\| \to 0} RS[f : P : \xi]$ exists

b $s[f] = S[f] = \lim_{\|P\| \to 0} RS[f : P : \xi]$

c for each $\varepsilon > 0$ there exists $P_\varepsilon \in \Pi[I]$ such that

$$\sum_k (M_k - m_k)\,|I_k| < \varepsilon$$

d for each $\varepsilon > 0$ there exists $P_\varepsilon \in \Pi[I]$ such that $\sum_k F_k\,|I_k| < \varepsilon$ where $F_k = \sup\{|f(x) - f(y)| : x, y \in I_k\}$

Conditions for Integrability

Having defined what it means to be Riemann integrable, we would like to know if there are any functions which are Riemann integrable. The next theorem asserts that there is no shortage of such functions.

Theorem *5.5 Let $f(x)$ be defined and bounded on $I = [a,b]$. Then f is Riemann integrable if either of the following conditions holds: i) $f(x)$ is monotone on I ii) $f(x)$ is continuous on I.*

The condition that f be bounded on I is a necessary condition for integrability but it is not sufficient. The conditions of monotonicity and continuity are each sufficient to imply that f is integrable but neither condition is necessary. It is possible to state a condition that is both necessary and sufficient to imply f is Riemann integrable.

Sets of Measure Zero

A set W in R is said to have ***measure zero*** if, for all $\varepsilon > 0$ there is a countable family of open intervals $\{I_n\} = \{(a_n, b_n) : a_n < b_n\}$ such that W is contained in the union of the intervals I_n and $\sum_n |I_n| < \varepsilon$.

Any set with finitely many points is a set of measure zero. The set of all integers is an infinite set having measure zero. In fact, the set of all rational numbers is a set of measure zero. In addition, any subset of a set of measure zero is a set having measure zero. We have then the following result whose proof is beyond the scope of this course.

Theorem *5.6 (Lebesgue's theorem) Let $f(x)$ be defined and bounded on $I = [a,b]$. Then f is Riemann integrable on I if and only if the set of points of I where f is not continuous is a set of measure zero.*

Example *Integrable and Nonintegrable functions*

(a) Let $I = [-1,1]$. Then $f(x)$ is integrable on I if f is any of the following: any polynomial, $sin(x)$, $\cos(x)$, e^x etc, since all of these functions are continuous on I. However, $f(x) = 1/x$ is not integrable on I since this function is not bounded on I.

(b) Let

$$f(x) = \begin{cases} 0 & if \quad x \in [0,1) \ or \ x \in (2,3] \\ 1 & if \quad x \in [1,2]. \end{cases}$$

This function is neither continuous nor monotone on $I = [0,3]$ but f is integrable on I since the set of discontinuities consists of just the two points $x = 1$ and $x = 2$ and so has measure zero. We can also prove directly that f is integrable by using theorem 5.4.

For $n = 3,4,\ldots$ let P_n denote the partition

$$\left\{x_0 = 0, x_1 = \frac{n-1}{n}, x_2 = \frac{n+1}{n}, x_3 = \frac{2n-1}{n}, x_4 = \frac{2n+1}{n}, x_5 = 3\right\}.$$

Then

I_k	m_k	M_k
$(k=1)\ \left[0, \frac{n-1}{n}\right]$	0	0
$(k=2)\ \left[\frac{n-1}{n}, \frac{n+1}{n}\right]$	0	1
$(k=3)\ \left[\frac{n+1}{n}, \frac{2n-1}{n}\right]$	1	1
$(k=4)\ \left[\frac{2n-1}{n}, \frac{2n+1}{n}\right]$	0	1
$(k=5)\ \left[\frac{2n+1}{n}, 3\right]$	0	0

and

$$s[f,P_n] = 0|I_1| + 0|I_2| + 1|I_3| + 0|I_4| + 0|I_5| = 1 - \frac{2}{n}$$

$$S[f,P_n] = 0|I_1| + 1|I_2| + 1|I_3| + 1|I_4| + 0|I_5| = 1 + \frac{2}{n}$$

$$S[f,P_n] - s[f,P_n] = \frac{4}{n}$$

Then for any $\varepsilon > 0$ we have $S[f,P_n] - s[f,P_n] = \frac{4}{n} < \varepsilon$ for $n > 4/\varepsilon$. This proves f is integrable on I and the value of the integral is equal to

$$\lim_{n\to\infty} S[f,P_n] = \lim_{n\to\infty} s[f,P_n] = 1.$$

(c) Let $I = [0,1]$ and let $f(x)$ be the Dirichlet function which is equal to 1 on every rational number and 0 on every irrational. Let P denote any partition of I and note that every one of the subintervals in the partition will contain both rational and irrational points. Then for every k, $m_k = 0$ and $M_k = 1$ so that $S[f,P] = 1$ *and* $s[f,P] = 0$. Since this holds for **any** partition of I, it follows that $s[f] = 0$ and $S[f] = 1$ and the fact that the upper and lower integrals are unequal implies that f is not integrable on I. Note also that the set of discontinuities for this function is the set of all irrational numbers in I and this is not a set of measure zero.

Properties of the Integral

We will let $\Sigma[I]$ denote the set of functions which are defined and are Riemann integrable on the interval $I = [a,b]$; i.e., if $f(x)$ is defined and Riemann integrable on I, we will write $f \in \Sigma[I]$. In all previous discussions, we have assumed that $a < b$. In order to consider $\int_a^b f$ in the case $a \geq b$, it is convenient to **define** for any $f \in \Sigma[I]$

$$\int_a^b f = -\int_b^a f \qquad and \qquad \int_a^a f = 0.$$

Then the integral has the following basic properties:

i) $\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g$ for all $\alpha, \beta \in R$

ii) if p lies between a and b, then $\int_a^b f = \int_a^p f + \int_p^b f$

More generally, if $I_1, \ldots, I_n$ are nonoverlapping intervals whose union equals I, then

$$\int_I f = \int_{I_1} f + \cdots + \int_{I_n} f$$

iii) if $f \geq 0$ *on* I, *then* $\int_I f \geq 0$

We refer to properties i), ii) and iii) respectively, by saying the Riemann integral is ***linear, additive and positive***.

Theorem *5.8 (Sets of Measure Zero) Let $f, g \in \Sigma[I]$ and suppose the set of points $\{x \in I : f(x) \neq g(x)\}$ is a set of measure zero. Then $\int_I f = \int_I g$.*

Theorem *5.9 (Integration of composed functions) Suppose $f \in \Sigma[I]$, $f[I]$ is contained in interval, J, and g is continuous on J. Then the composed function $g[f(x)]$ is integrable on I.*

Corollary *5.10 Let $f, g \in \Sigma[I]$. Then $|f(x)|$, $f(x)\,g(x)$ and $[f(x)]^n, n \in N$, are all integrable on I*

Integral Inequalities

Theorem *5.11 Let $f, g \in \Sigma[I]$, with $|f(x)| \leq M$ for $x \in I$. Then*

i) $\left|\int_I f\right| \leq \int_I |f| \leq M|I|$

ii) $\left|\int_I fg\right| \le \left(\int_I f^2\right)^{1/2}\left(\int_I g^2\right)^{1/2}$

iii) $\left(\int_I (f+g)^2\right)^{1/2} \le \left(\int_I f^2\right)^{1/2} + \left(\int_I g^2\right)^{1/2}$

The results ii) and iii) are commonly referred to as the ***Cauchy-Schwarz and Minkowski*** inequalities, respectively.

Mean Value Theorem for Integrals

Theorem *5.12 Suppose $f(x)$ is continuous on $I = [a,b]$ and $g \in \Sigma[I]$, with $g(x) \ge 0$ for $x \in I$. Then there exists a point $c \in I$ such that*

$$\int_I fg = f(c)\int_I g$$

Integral Form of Taylor's Theorem

Theorem *5.13 Suppose $f(x)$ is continuous on $I = [a,b]$, together with all its derivatives up to the order $n+1$. Then for each $x \in I$,*

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_{n+1}$$

where

$$R_{n+1} = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

Evaluating Integrals

Theorems 5.5 and 5.6 state conditions on $f(x)$ sufficient to imply that the integral of f over I exists. For purposes of evaluating the integral, we have the next three results.

Theorem *5.14 Suppose $f \in \Sigma[I]$, $I = [a,b]$ and suppose further that $F'(x) = f(x)$ for $x \in I$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The function F is called a ***primitive or antiderivative*** for f on I if $F'(x) = f(x)$ for $x \in I$. It may not always be easy (or even possible) to determine F given f. The process of finding an antiderivative is made easier by the following results.

Theorem *5.15 (Integration by parts) Suppose $f(x)$ and $g(x)$ are continuous with continuous first derivatives on $I = [a,b]$. Then*

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

Theorem *5.16 (Change of variables formula) Suppose $\phi(x)$ is continuous with continuous first derivative on $I = [a,b]$ and that f is continuous on $J = \phi[I]$. Then*

$$\int_a^b f[\phi(t)]\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

Theorem 5.14 is one statement of the "Fundamental Theorem of Calculus". Other versions of this theorem are of interest.

Alternative Statements of the Fundamental Theorem of Calculus:
The following are three different ways of stating the fundamental theorem of calculus (FTC):

1. Suppose $f(x)$ is defined and integrable on $[a,b]$ and let $F(x) = \int_a^x f$.

- **a**. F is Lipschitz continuous on $[a,b]$
- **b**. at each point c in $[a,b]$ where f is continuous, F' exists and $F'(c) = f(c)$

proof-(a) For x,y in $[a,b]$

$$|F(x) - F(y)| = \left|\int_a^x f - \int_a^y f\right| = \left|\int_y^x f\right| \le \int_y^x |f| \le M|x-y|$$

proof (b)

$$\begin{aligned}
|D_h F(c) - f(c)| &= \left|\frac{F(c+h) - F(c)}{h} - f(c)\right| \\
&= \left|\frac{1}{h}\left\{\int_a^{c+h} f - \int_a^c f\right\} - f(c)\right| \\
&= \left|\frac{1}{h}\int_c^{c+h} f - f(c)\right| \\
&= \left|\frac{1}{h}\int_c^{c+h} f - f(c)\int_c^{c+h} 1\right| \\
&= \left|\frac{1}{h}\int_c^{c+h} [f(x) - f(c)]\right| \\
&\le \left|\frac{1}{h}\right| \max_{x\in N_h(c)} |f(x) - f(c)|\,|h| \to 0 \quad \text{as } h \to 0 \quad \blacksquare
\end{aligned}$$

2 Suppose f is continuous on $[a,b]$. Then $\Phi'(x) = f(x)$ at every $x \in [a,b]$ if and only if

$$\Phi(x) = \Phi(a) + \int_a^x f.$$

Proof- Suppose f is continuous on $[a,b]$ and $\Phi'(x) = f(x)$ at every $x \in [a,b]$. Then

$$\Phi'(x) = f(x) = F'(x) \text{ at every } x \in [a,b]$$

hence $\Phi(x) = F(x) + C$ (see problem 4.13). Since $F(a) = 0$, $\Phi(a) = C$ so

$$\Phi(x) = \Phi(a) + \int_a^x f.$$

Now suppose f is continuous on $[a,b]$ and $\Phi(x) = \Phi(a) + \int_a^x f$. Then

$$\Phi(x) = \Phi(a) + F(x)$$
$$\text{and} \quad \Phi'(x) = F'(x) = f(x) \text{ at every } x \in [a,b]$$

3 Suppose $f(x)$ is defined and integrable on $[a,b]$ and $F'(x) = f(x)$ at every $x \in [a,b]$. Then

$$F(b) - F(a) = \int_a^b f.$$

Proof- Let $P \in \Pi[a,b]$. Then for each $I_k = [x_{k-1}, x_k]$ in P, we have by the MVT for derivatives,

$$F(x_k) - F(x_{k-1}) = F'(\xi_k)(x_k - x_{k-1}) = f(\xi_k)(x_k - x_{k-1}).$$

Then

$$F(b) - F(a) = \sum_{k=1}^{N} f(\xi_k)(x_k - x_{k-1}) = RS[f,P,\xi].$$

and

$$s[f,P] \leq F(b) - F(a) \leq S[f,P].$$

Since this holds for any $P \in \Pi[a,b]$ and f is integrable, it follows that

$$\int_a^b f. = s[f] \leq F(b) - F(a) \leq S[f] = \int_a^b f..$$

These three statements differ from one another in subtle ways. The following example may illustrate the differences. Note that

$$f(x) = sgn(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

is defined and integrable on $[-1,1]$ but f is not continuous on $[-1,1]$. Then version 2 of the fundamental theorem does not apply to this f. It is also the case that there is no function $F(x)$ whose derivative equals $f(x)$ at every $x \in [-1,1]$. To see this, note that since $f(x) = sgn(x)$ is integrable on $[-1,1]$, we can define,

$$F(x) = \int_{-1}^{x} sgn(t)dt \quad \text{for } x \in [-1,1].$$

and it is not difficult to see that $F(x) = |x| - 1$. In addition, it is clear that

$$\frac{d}{dx}(|x| - 1) = sgn(x) \;\; if \;\; x \neq 0$$

but $F'(0)$ does not exist. Then version 3 of the theorem does not apply to f either. However, version 1 does apply and it asserts that $F(x) = |x| - 1$ is Lipschitz continuous and $F'(x) = sgn(x)$ for $x \neq 0$.

This example illustrates that while the anti-derivative F of an integrable function f is necessarily continuous (in fact Lipschitz continuous), its derivative need not equal f at every point in the interval. Equality between f and F' holds at every point in the interval if and only if f is continuous at every point in the interval. In this example, f was not continuous at $x = 0$ and $F'(0)$ did not exist. In general, when f is not continuous it is not everywhere equal to the derivative of any F. This might lead one to expect that for a function $F(x)$ that is differentiable at every point of an interval it must follow that $F'(x) = f(x)$ is continuous on that interval. This is not the case, as the following examples will illustrate.

Differentiable Functions with Discontinuous Derivatives

The functions

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \textit{and} \qquad g(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

are everywhere continuous and differentiable. Clearly they are differentiable at $x \neq 0$. In fact, for $x \neq 0$,

$$\frac{d}{dx}(x^2 \sin\left(\frac{1}{x}\right)) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$
$$\frac{d}{dx}(x^2 \cos\left(\frac{1}{x}\right)) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

At $x = 0$ we have:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$$
$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} h \cos\left(\frac{1}{h}\right) = 0$$

The functions

$$p(x) = 2x \sin\frac{1}{x} \qquad x \neq 0, \quad p(0) = 0,$$
$$\text{and} \qquad q(x) = 2x \cos\left(\frac{1}{x}\right) \quad x \neq 0 \quad q(0) = 0.$$

are everywhere continuous hence, by version 2 of FTC, there exist $\mathbb{C}^1$ functions $P(x)$ and $Q(x)$ such that $P' = p$ and $Q' = q$. Then we have

$$f'(x) = P'(x) - \cos\frac{1}{x}$$
$$\textit{and} \qquad g'(x) = Q'(x) + \sin\left(\frac{1}{x}\right);$$

i.e.,

$$\sin\left(\frac{1}{x}\right) = \frac{d}{dx}(g(x) - Q(x))$$
$$\cos\frac{1}{x} = \frac{d}{dx}(P(x) - f(x))$$

Then $F = g - Q$ and $G = P - f$ are everywhere differentiable functions whose derivatives are not continuous.

Improper Integrals

In defining the Riemann integral of a function $f(x)$ defined on an interval $I = [a, b]$, it is assumed that f is bounded on I and that $b - a$ is finite. If either or both of these conditions is not satisfied, then the integral of f over I is not defined. In this case we say the integral is improper. In some cases, even when the integral is improper, it is possible to assign a value to the integral.

Definition *Suppose f is defined in the interval $[a, b)$ but $f(x)$ tends to infinity as x tends to b. Suppose also that for each $\varepsilon > 0$, f is Riemann integrable on $[a, b-\varepsilon]$. Finally, suppose $\lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f = L$. If L is finite, then the improper integral $\int_a^b f$ is said to be convergent with value L. If L is infinite or the limit does not exist, then the improper integral is said to be divergent.*

A similar definition applies if f is defined in the interval $(a, b]$ but $f(x)$ tends to infinity as x tends to a. Likewise if c is an interior point of (a, b) and f is defined on $[a, c)$ and on $(c, b]$ but f tends to infinity as x tends to c, then the improper integral $\int_a^b f = \int_a^c f + \int_c^b f$ is convergent if and only if both the improper integrals on the right side of this last expression are convergent. If either one or both of the integrals is divergent then the original integral is divergent.

Example *Improper Integrals with unbounded integrand*

1. Consider the integral $\int_0^1 \frac{1}{\sqrt{x}}dx$. Since $f(x) = 1/\sqrt{x}$ tends to infinity as x tends to 0, this is an improper integral. For any $\varepsilon > 0$,

$$\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\,|_\varepsilon^1 = 2 - \sqrt{\varepsilon}$$

Then $\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = 2$, and it follows from the definition above that the integral is convergent with value 2.

2. More generally, for $0 < p < 1$ consider $\int_0^1 \frac{1}{x^p}dx$. As in the previous example, this is an improper integral as the integrand tends to infinity as x tends to 0. For any $\varepsilon > 0$,

$$\int_\varepsilon^1 \frac{1}{x^p}dx = \frac{1}{1-p}x^{1-p}|_\varepsilon^1 = \frac{1}{1-p}[1-\varepsilon^{1-p}]$$

Since $1-p > 0$, the limit $\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{x^p}dx = \frac{1}{1-p}[1-\lim_{\varepsilon\to 0}\varepsilon^{1-p}]$ exists. Then the improper integral is convergent and equals $1/(1-p)$.

3. For $\varepsilon > 0$, $\int_\varepsilon^1 \frac{1}{x}dx = \ln 1 - \ln\varepsilon = -\ln\varepsilon$, and since this tends to infinity as ε tends to zero, the improper integral $\int_0^1 \frac{1}{x}dx$ is divergent.

The second condition that must be satisfied if an integral is to be a proper integral is the condition that the interval of integration is finite. For integrals $\int_a^b f$ where $b-a$ is not finite we have,

Definition *Suppose that f is Riemann integrable on the bounded interval $[a, N]$ for every $N > a$, and that $\lim_{N\to\infty}\int_a^N f(x)dx = L$. If L is finite, then the improper integral $\int_a^\infty f(x)dx$ is convergent with value L. If L is infinite, or if the limit fails to exist, then the improper* *integral is divergent.*

A similar definition applies to the improper integral $\int_{-\infty}^{b} f(x)dx$. In addition, the improper integral $\int_{-\infty}^{\infty} f(x)dx$ can be written as

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx.$$

Then $\int_{-\infty}^{\infty} f(x)dx$ is convergent if and only if both $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{\infty} f(x)dx$ are convergent. If either or both of these integrals is divergent, then the original integral is divergent.

Finally, consider $\int_{a}^{\infty} f(x)dx$ and suppose $f(x)$ tends to infinity as x tends to a. Then

$$\int_{a}^{\infty} f(x)dx = \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx, \quad a < b < \infty,$$

and $\int_{a}^{\infty} f(x)dx$ is convergent if and only if both $\int_{a}^{b} f(x)dx$ and $\int_{b}^{\infty} f(x)dx$, are convergent. The integral $\int_{-\infty}^{a} f(x)dx$ is treated in a similar fashion.

Example *Improper Integrals on unbounded Intervals*

1. Consider the improper integral $\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx$. For any $N > 1$,

$$\int_{1}^{N} \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\,|_{1}^{N} = 2\sqrt{N} - 2.$$

Since $2\sqrt{N} - 2$ tends to infinity as N tends to infinity, the limit $\lim_{N\to\infty}\left(2\sqrt{N} - 2\right)$ fails to exist and the integral is divergent. On the other hand the improper integral $\int_{1}^{\infty} \frac{1}{x^2}dx$ is convergent since, for any $N > 1$

$$\int_{1}^{N} \frac{1}{x^2}dx = -x^{-1}|_{1}^{N} = 1 - \frac{1}{N}$$

and $1 - 1/N$ tends to the value 1 as N tends to infinity. Thus $\int_{1}^{\infty} \frac{1}{x^2}dx$ is convergent with value 1.

2. More generally, consider $\int_{1}^{\infty} \frac{1}{x^p}dx$ for $1 < p$. In this case for any $N > 1$

$$\int_{1}^{N} \frac{1}{x^p}dx = \frac{1}{1-p}(N^{1-p} - 1).$$

If $p > 1$, then $N^{1-p} = \frac{1}{N^{p-1}}$ tends to zero as N tends to infinity, and $\lim_{N\to\infty} \int_{1}^{N} \frac{1}{x^p}dx = \frac{1}{p-1}$. Then the improper integral is convergent with value $\frac{1}{p-1}$. Clearly the integral $\int_{1}^{\infty} \frac{1}{x^p}dx$ is divergent if $0 < p < 1$.

3. The improper integral $\int_{1}^{\infty} \frac{1}{x}dx$ is divergent. To see this, note that

$$\int_{1}^{N} \frac{1}{x^p}dx = \ln N - 0.$$

Since $\ln N$ tends to infinity as N tends to infinity, the limit $\lim_{N\to\infty} \int_{1}^{N} \frac{1}{x}dx$ fails to exist.

Tests for Convergence of Improper Integrals

It is not always possible to determine the convergence or divergence of an improper integral from the definition. In particular, if it is not possible to find an antiderivative for the integrand then the definition cannot be applied directly. In such cases there are comparison principles that may be useful in deciding if an improper integral is convergent.

Theorem *5.17 Suppose functions f and g are defined and non-negative on the interval* $[a,b)$, *where b may be infinite. Suppose also that f and g are Riemann integrable on* $[a,\lambda]$ *for all* $\lambda,\ a < \lambda < b$ *and that* $f(x) \leq g(x)$ *for all* $x,\ a \leq x \leq b$. *Then* $\int_a^b f$ *is convergent if* $\int_a^b g$ *is convergent, and* $\int_a^b g$ *is divergent if* $\int_a^b f$ *is divergent.*

Sometimes a pointwise comparison is not available. In such cases a limit version of the comparison result may apply.

Theorem *5.18 Suppose functions f and g are defined and non-negative on the interval* $[a,b)$, *where b may be infinite. Suppose also that f and g are Riemann integrable on* $[a,\lambda]$ *for all* $\lambda,\ a < \lambda < b$ *and that*

$$\lim_{x\to b}\frac{f(x)}{g(x)} = L \quad \text{where } 0 < L < \infty.$$

Then either $\int_a^b f$ *and* $\int_a^b g$ *both converge or both diverge. If* $L = 0$, *then* $\int_a^b f$ *is convergent if* $\int_a^b g$ *is convergent. If* $L = \infty$ *then* $\int_a^b g$ *diverges is* $\int_a^b f$ *diverges.*

Examples

1. Consider the improper integral:

$$\int_1^\infty \frac{1}{1+x^3}\,dx.$$

Here there is no evident antiderivative but the following comparison is clear

$$\frac{1}{1+x^3} \leq \frac{1}{1+x^2} \quad \text{for } 1 \leq x < \infty.$$

Then

$$\int_1^\infty \frac{1}{1+x^3}\,dx \leq \int_1^\infty \frac{1}{1+x^2}\,dx = \arctan x|_1^\infty = \pi/4$$

and theorem 5.17 implies that the original improper integral is convergent. Presumably the value could be approximated by a numerical integration scheme.

2. Consider the improper integral:

$$\int_1^\infty \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx.$$

Again there is no evident antiderivative but it is not difficult to see that

$$\frac{1}{2\sqrt{x}} \leq \frac{1}{\sqrt{x}+\sqrt[3]{x}} \quad \text{for } 1 \leq x < \infty.$$

Since

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

is divergent, it follows from theorem 5.17 that the original improper integral is divergent.

3. Consider the improper integral:

$$\int_0^1 \frac{1}{\sqrt{x}\,\ln x}\,dx.$$

We can use L'Hopital's rule to show that

$$\lim_{x\to 0} \sqrt{x}\,\ln x = 0$$

and since $\lim_{x\to 1} \sqrt{x}\,\ln x = 0,$ this integrand becomes undefined at both endpoints of the interval of integration. Then for some point $a,\ 0 < a < 1,$ we separate the integral into two parts,

$$\int_0^1 \frac{1}{\sqrt{x}\,\ln x}\,dx = \int_0^a \frac{1}{\sqrt{x}\,\ln x}\,dx + \int_a^1 \frac{1}{\sqrt{x}\,\ln x}\,dx.$$

For the first of these two pieces, we apply theorem 5.18 with

$$f(x) = \frac{1}{\sqrt{x}\,\ln x} \text{ and } g(x) = \frac{1}{x^{1/2-\varepsilon}} \quad 0 < \varepsilon < 1/2.$$

Another application of L'Hopital's rule shows that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 0.$$

Since $L = 0$ and $\int_0^a g$ is convergent, it follows from theorem 5.18 that $\int_0^a f$ is convergent. For the second piece, we apply the same theorem, this time with

$$f(x) = \frac{1}{\sqrt{x}\,\ln x} \text{ and } g(x) = \frac{1}{x-1}.$$

This time L'Hopital's rule shows that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 1,$$

and since $\int_a^1 g$ is divergent, $\int_a^1 f$ diverges as well. The original improper integral was split into two pieces and since one of these pieces is divergent, the original integral is divergent as well.

It is not always possible to devise a comparison argument that will settle the convergence question for an improper integral. In that case there are numerous more specialized results which may be useful. The following theorem is one such example.

Theorem *5.19 The improper integral* $\int_a^\infty f(x)g(x)dx$ *is convergent if the following conditions hold:*

$$i)\ f \text{ is continuous and } F(x) = \int_a^x f \text{ is bounded for } x > a$$

$$ii)\ g \text{ is continuously differentiable with } g'(x) < 0$$

$$iii)\ g(x) \to 0 \text{ as } x \to \infty$$

Examples

1. Consider the integral:

$$I = \int_0^\infty \frac{\sin x}{x}\,dx.$$

The integrand is bounded at $x = 0$, but the integral is still improper as the interval of integration is infinite. Theorems 5.17, 5.18 do not apply since the integrand is not non-negative. However, $g(x) = 1/x$ with $g'(x) < 0$ and $g(x) \to 0$ as $x \to \infty$. In addition $f(x) = \sin x$ is continuous and $F(x) = \int_0^x \sin t\, dt = 1 - \cos x$ is bounded for all $x > 0$. Then theorem 5.19 applies the integral I is convergent. It is interesting to note that I is convergent, although the integrals, $\int_1^\infty \frac{1}{x} dx$ and $\int_0^\infty |\frac{\sin x}{x}| dx$ are both divergent. To see that $\int_0^\infty |\frac{\sin x}{x}| dx$ is divergent, note that for each integer n,

$$\left| \frac{\sin x}{x} \right| \geq \frac{|\sin x|}{(n+1)\pi} \quad \text{for } n\pi < x < (n+1)\pi.$$

Then

$$\int_{n\pi}^{(n+1)\pi} |\frac{\sin x}{x}| dx \geq \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| dx$$
$$\geq \frac{2}{\pi} \frac{1}{n+1}.$$

It follows that for each L, $n\pi < L < (n+1)\pi$, we have

$$\int_0^L |\frac{\sin x}{x}| dx \geq \frac{2}{\pi} \sum_{n=0}^{N-1} \frac{1}{n+1}.$$

But this sum tends to infinity as N tends to infinity, hence the integral $\int_0^\infty |\frac{\sin x}{x}| dx$ is divergent even though $\int_0^\infty \frac{\sin x}{x} dx$ is convergent. When an improper integral, $\int_a^b |f|$ converges we say $\int_a^b f$ is an absolutely convergent improper integral. In this case $\int_0^\infty \frac{\sin x}{x} dx$ is convergent but not absolutely convergent. What this means is that the convergence of the integral I relies on the cancellation that occurs as $\sin x$ alternates between positive and negative values. When the absolute value is inserted, no cancellation can occur and the integral diverges.

2. Consider the integral:

$$I = \int_1^\infty \frac{\sin x}{x^2} dx.$$

Since the integrand is not non-negative we cannot apply a comparison theorem directly. However, it follows from theorem 5.11i that

$$\left| \int_1^\infty \frac{\sin x}{x^2} dx \right| \leq \int_1^\infty \left| \frac{\sin x}{x^2} \right| dx \leq \int_1^\infty \frac{1}{x^2} dx < \infty.$$

Then the original integral is absolutely convergent.

3. Consider the integrals

$$I_p = \int_0^\infty \sin(x^p) dx \quad \text{for } p = 1, 2$$

When $p = 1$,

$$\int_0^N \sin x\, dx = 1 - \cos N$$

and since this tends to no limit as $N \to \infty$, I_1 is divergent. When $p = 2$, the situation is different. The change of variable, $x = \sqrt{t}$ with $dx = \frac{1}{2\sqrt{t}} dt$, leads to

$$I_2 = \int_0^\infty \sin(x^2) dx = \int_0^\infty \sin t \, \frac{1}{2\sqrt{t}} dt.$$

Now we can apply theorem 5.19 with $f(t) = \sin t$ and $g(t) = \frac{1}{2\sqrt{t}}$ in order to conclude that I_2 is convergent.

It is interesting to note that in order for $\int_a^\infty f(x)dx$ to converge, it is not necessary for $f(x)$ to tend to zero as x tends to infinity. In fact, $f(x)$ may even grow without bound as x tends to infinity. Consider the monotone function defined as

$$f(x) = \begin{cases} n \text{ if } n < x < n + n^{-3} \\ 0 \quad\quad \text{otherwise} \end{cases}$$

The graph of this function is a sequence of square pulses located at integer values of x. The pulse at $x = n$ is n units high and n^{-3} units in width so the area under the pulse is n^{-2}. Then

$$\int_1^\infty f(x)dx = \sum_{n=1}^\infty \frac{1}{n^2}$$

and this series can be shown to converge. Then the improper integral $\int_1^\infty f(x)dx$ is convergent even though $f(x)$ grows without bound as x tends to infinity.

Solved Problems

Partitions and Sums

Problem 5.1 Suppose $m \le f(x) \le M$ for all $x \in I = [a,b]$. Then show that for every $P \in \Pi[I]$ and each set of evaluation points $\{\xi_k\}$, we have

$$m|I| \le s[f : P] \le RS[f : P : \xi_k] \le S[f : P] \le M|I|$$

Solution: Let $P = \{x_0, x_1, \dots, x_N\}$ and let m_k, M_k denote, respectively, the GLB and the LUB for $f(x)$ on the interval, $I_k = [x_{k-1}, x_k]$, $1 \le k \le N$. Then for each k, we have $m \le m_k \le f(\xi_k) \le M_k \le M$, and it follows that

$$m\sum_k |I_k| \le \sum_k m_k|I_k| \le \sum_k f(\xi_k)|I_k| \le \sum_k M_k|I_k| \le M\sum_k |I_k|.$$

But this is just what we were to prove.

Problem 5.2 Suppose $f(x)$ is defined and bounded on $I = [a,b]$. Let $P = \{x_0, x_1, \dots, x_N\}$ denote a partition of I and let P' denote a refinement of P. Then show that

$$s[f : P] \le s[f : P'] \qquad and \qquad S[f : P'] \le S[f : P].$$

Solution: Suppose first that P' is obtained from P by adding a single point, z, between x_0 and x_1. Then let M_{11}, M_{12} denote, respectively, the LUB for $f(x)$ on the subintervals $[x_0, z]$ and $[z, x_1]$. Then clearly $M_{11} \le M_1$, and $M_{12} \le M_1$, and hence

$$M_{11}(z - x_0) + M_{12}(x_1 - z) \le M_1(z - x_0) + M_1(x_1 - z) = M_1(x_1 - x_0) = M_1|I_1|$$

If we add $\sum_{k=2}^N M_k|I_k|$ to both sides of this last inequality, we obtain $S[f : P'] \le S[f : P]$. A similar argument leads to $s[f : P] \le s[f : P']$. Repeating this argument several times then provides a proof in the case that more than one point is added to P to obtain P'.

Problem 5.3 Suppose $f(x)$ is defined and bounded on $I = [a,b]$. Then show that for arbitrary $P, Q \in \Pi[I]$, we have $s[f : P] \le S[f : Q]$.

Solution: Let P' denote the partition obtained by combining the points of P with those of Q. Then P' is a refinement of both P and Q and it follows by the results in the previous problems that

$$s[f:P] \le s[f:P'] \le S[f:P'] \le S[f:Q].$$

Problem 5.4 For $I = [0,1]$ let P_n denote the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ for $n > 1.$ For $f(x) = x^2,$ compute $s[f,P_n]$ and $S[f,P_n]$ and then show that

$$s[f] = \sup s[f,P_n] = \inf S[f,P_n] = S[f]$$

Solution: Since $f(x) = x^2$ is increasing on $[0,1],$ it follows that $m_k = \min\{f(x) : x \in I_k = [x_{k-1}, x_k]\}$ occurs at the left endpoint of the interval so that $m_k = f(x_{k-1}) = x_{k-1}^2.$ Similarly, $M_k = f(x_k) = x_k^2.$ That is,

$$m_k = \left(\frac{k-1}{n}\right)^2 \qquad and \qquad M_k = \left(\frac{k}{n}\right)^2 for k = 1,2,\ldots,n.$$

Since $|I_k| = \frac{1}{n}$ for every k (i.e., the partition is uniform) we have

$$s[f:P_n] = \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\left(0 + 1^2 + \cdots + (n-1)^2\right)$$

$$S[f:P_n] = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\left(1^2 + \cdots + (n-1)^2 + n^2\right).$$

In problem 1.5, we showed that for any positive integer M,

$$1^2 + \cdots + (M-1)^2 + M^2 = \frac{1}{3}M\left(M + \frac{1}{2}\right)(M+1).$$

Using this result with $M = n-1$ in $s[f:P_n],$ and with $M = n$ in $S[f:P_n],$ leads to

$$s[f:P_n] = \frac{1}{3n^3}(n-1)\left(n - \frac{1}{2}\right)n = \frac{1}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right) < \frac{1}{3}$$

$$S[f:P_n] = \frac{1}{3n^3}n\left(n + \frac{1}{2}\right)(n+1) = \frac{1}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right) > \frac{1}{3}.$$

It follows that

$$\frac{1}{3} = \sup\{s[f:P_n] : n \in N\} \le \sup\{s[f:P] : P \in \Pi[I]\} = s[f]$$

$$\frac{1}{3} = \inf\{S[f:P_n] : n \in N\} \ge \inf\{S[f:P] : P \in \Pi[I]\} = S[f].$$

i.e.,

$$\frac{1}{3} \le s[f] \le S[f] \le \frac{1}{3}.$$

But then all the inequalities in these last expressions are in fact equalities and this is what we were to prove.

Definition of the Integral

Problem 5.5 Suppose that $f(x)$ is defined and bounded on $I = [a,b].$ Then show that $s[f] = S[f]$ if and only if for all $\varepsilon > 0$ there exists a $P_\varepsilon \in \Pi[I]$ such that $S[f:P_\varepsilon] - s[f:P_\varepsilon] < \varepsilon$

Solution: Suppose first that $s[f] = S[f]$ and let $\varepsilon > 0$ be given. Since,

$$s[f] = \sup\{s[f : P] : P \in \Pi[I]\}$$
$$\text{and} \qquad S[f] = \inf\{S[f : P] : P \in \Pi[I]\}$$

it follows from the definitions of sup and inf that there exist partitions $P, Q \in \Pi[I]$ such that

$$s[f] - \frac{\varepsilon}{2} < s[f : P]$$
$$\text{and} \qquad S[f] + \frac{\varepsilon}{2} > S[f : Q].$$

Let P_ε denote the partition obtained by combining the points of P with those of Q. Then P_ε is a refinement of both P and Q and then it follows from the last two inequalities together with the result of problem 5.3 that

$$s[f] - \frac{\varepsilon}{2} < s[f : P] \leq s[f : P_\varepsilon] \leq S[f : P_\varepsilon] \leq S[f : Q] < S[f] + \frac{\varepsilon}{2}.$$

But $s[f] = S[f]$ and thus $S[f : P_\varepsilon] - s[f : P_\varepsilon] < \varepsilon.$

Now suppose that for all $\varepsilon > 0$ there exists a $P_\varepsilon \in \Pi[I]$ such that $S[f : P_\varepsilon] - s[f : P_\varepsilon] < \varepsilon.$ We know that for all partitions $P \in \Pi[I]$,

$$s[f : P] \leq s[f] \qquad \text{and} \qquad S[f : P] \geq S[f]$$

hence

$$S[f] - s[f] \leq S[f : P] - s[f : P].$$

In particular, for the partition $P = P_\varepsilon$ this last expression becomes

$$S[f] - s[f] \leq S[f : P_\varepsilon] - s[f : P_\varepsilon] < \varepsilon.$$

Since this holds for all $\varepsilon > 0,$ it follows that $S[f] \leq s[f].$ But it is also true that $S[f] \geq s[f],$ and these two results together imply that $S[f] = s[f].$

Problem 5.6 Suppose $f(x)$ is monotone on $I = [a,b].$ Then prove that f is Riemann integrable on $I.$

Solution: Suppose $f(x)$ is monotone increasing on $I.$ If we can show that for all $\varepsilon > 0$ there exists a $P_\varepsilon \in \Pi[I]$ such that $S[f : P_\varepsilon] - s[f : P_\varepsilon] < \varepsilon,$ then by the previous problem (and theorem 5.4), it will follow that f is integrable on $I.$

Let $\varepsilon > 0$ be given, and choose a positive integer n such that $n\varepsilon > (b-a)(f(b) - f(a)).$ Then choose P_e to be the partition of I consisting of the points $x_k = a + (b-a)\frac{k}{n},\ 0 \leq k \leq n.$ Since f is monotone increasing, it follows that for each k,

$$m_k = \min\{f(x) : x_{k-1} \leq x \leq x_k\} = f(x_{k-1})$$
$$M_k = \max\{f(x) : x_{k-1} \leq x \leq x_k\} = f(x_k),$$

hence

$$S[f : P_\varepsilon] - s[f : P_\varepsilon] = \sum_{k=1}^{n}(M_k - m_k)|I_k| = \sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))|I_k|.$$

But $|I_k| = \dfrac{b-a}{n}$ for every k and so,

$$
\begin{aligned}
S[f:P_\varepsilon] - s[f:P_\varepsilon] &= \frac{b-a}{n}\sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) \\
&= \frac{b-a}{n}(f(x_n) - f(x_0)) \\
&= \frac{b-a}{n}(f(b) - f(a)) < \varepsilon.
\end{aligned}
$$

This proves the result in the case that f is monotone increasing. The proof when f is monotone decreasing is similar.

Problem 5.7 Suppose $f(x)$ is continuous on $I = [a,b]$. Then prove that f is Riemann integrable on I.

Solution: Suppose $f(x)$ is continuous on the interval I. Since I is compact, f is then uniformly continuous on I and it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| \le \frac{\varepsilon}{b-a} \quad \text{whenever} \quad |x-y| \le \delta.$$

Then, given $\varepsilon > 0$, let P_ε denote the uniform partition used in the previous problem, with n chosen such that $n\delta > b-a$. Then for each k, $1 \le k \le n$, we have

$$|I_k| = x_k - x_{k-1} = (b-a)/n < \delta$$

and

$$M_k - m_k = \max\{f(x) - f(y) : x,y \in I_k\} \le \frac{\varepsilon}{b-a}.$$

But then it follows that

$$
\begin{aligned}
S[f:P_\varepsilon] - s[f:P_\varepsilon] &= \frac{b-a}{n}\sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) \\
&\le \frac{b-a}{n}\, n\, \frac{\varepsilon}{b-a}. = \varepsilon
\end{aligned}
$$

and this implies via theorem 5.4 that f is integrable on I.

Problem 5.8 Suppose $f(x)$ is Riemann integrable on $I = [a,b]$. Then prove that f is bounded on I.

Solution: Suppose $f(x)$ is Riemann integrable on I. Then for any choice of evaluation points $\{\xi_k\}$, the limit of the Riemann sums, $RS[f,P,\xi_k]$ as $\|P\|$ tends to zero exists; denote the value of this limit by L. We will show that if f is not bounded on I then a contradiction arises.

Fix $\varepsilon = 1$ and, for an arbitrary $\delta > 0$, let P denote a partition of I with $\|P\| < \delta$. If f is not bounded on I, then there exist (at least) one subinterval, I_{k^*}, in the partition P such that for any $M > 0$, there exists a point $\xi_{k^*} \in I_{k^*}$ with $f(\xi_{k^*}) > M$. In particular, choose $M = \dfrac{L+1}{\delta}$. Let the remaining evaluation points be arbitrarily chosen. Then the Riemann sum $RS[f,P,\xi_k]$ satisfies

$$RS[f,P,\xi_k] = \sum_{k \ne k^*} f(\xi_k)|I_k| + f(\xi_{k^*})|I_{k^*}| > \sum_{k \ne k^*} f(\xi_k)|I_k| + (L+1).$$

But then $|RS[f,P,\xi_k] - L| > 1$ for all partitions of I with $\|P\| < \delta$, which implies that the limit of $RS[f,P,\xi_k]$ as $\|P\|$ tends to zero does not exist. This contradiction with the assumption that f is Riemann integrable on I shows that being Riemann integrable on I and being unbounded on I are incompatible properties for f.

Problem 5.9

a) Show that any finite set of points $\{x_1,\ldots,x_M\}$ is a set of measure zero.

b) Show that any countable set of points $\{x_k : k \in N\}$ is a set of measure zero.

Solution: Let $\varepsilon > 0$ be fixed but arbitrary. We have to find a set of intervals whose union contains all the points $\{x_1,\ldots,x_M\}$ but the sum of whose lengths is not more than ε. Consider the intervals $I_k = \left(x_k - \frac{\varepsilon}{2M}, x_k + \frac{\varepsilon}{2M}\right),\ k = 1,\ldots,M.$ Then the intervals $\{I_k\}$ cover the points $\{x_1,\ldots,x_M\}$ but $|I_k| = \frac{\varepsilon}{M}$ for every k and,

$$\sum_{k=1}^{M}|I_k| = M \bullet \frac{\varepsilon}{M} = \varepsilon.$$

This proves (a).

A set of points $\{x_k : k \in N\}$ is countable if it can be put in one to one correspondence with the positive integers. For such a set, we define the set of intervals,

$$I_k = \left(x_k - \frac{\varepsilon}{2^{k+1}}, x_k + \frac{\varepsilon}{2^{k+1}}\right),\ k \in N.$$

Then the union of these intervals cover the points and, $|I_k| = \frac{\varepsilon}{2^k}$ for every k, hence

$$\sum_{k=1}^{M}|I_k| = \varepsilon\sum_{k=1}^{M}2^{-k} = \varepsilon.$$

This proves (b).

Clearly the set of all positive integers is a countable set and therefore has measure zero. The set of rational numbers in the interval $(0,1)$ is another example of a countable set. To see this, list the rationals in $(0,1)$ in the following order,

$$\begin{array}{ccccc}
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{3} & \frac{2}{5} & \frac{2}{7} & \cdots & \\
\frac{3}{4} & \frac{3}{5} & \frac{3}{7} & \cdots & \\
\frac{4}{5} & \frac{4}{7} & \frac{4}{9} & \frac{4}{11} & \cdots \\
\frac{5}{6} & \frac{5}{7} & \frac{5}{8} & \frac{5}{9} & \cdots \\
\frac{6}{7} & \frac{6}{11} & \frac{6}{13} & \frac{6}{17} & \cdots \\
\vdots & & & &
\end{array}$$

i.e., we have listed all the rationals (reduced to lowest terms) having numerator 1, followed by those having numerator 2, etc. Then every rational between 0 and 1 is on this list and we can now ”count” the rationals by ordering them as follows:
$1/2, 1/3, 2/3, 3/4, 2/5, 1/4, 1/5, 2/7, 3/5, \ldots;$ the counting path is a "snake" that passes through each rational in the array above. In this way, each rational is counted once which is to say, the rational numbers are countable and are therefore a set of measure zero.

Problem 5.10 Tell whether each of the following functions is integrable on $I = [0,1]$

a) A function $f(x)$ having a finite number of finite jump discontinuities in I

b) The function $g(x)$ that equals 1 on each rational in $(0,1)$ and is zero otherwise

c) The function

$$h(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x \text{ is irrational} \\ \frac{1}{n} \ \ if \ \ x = \frac{m}{n} & (\text{reduced to lowest terms}) \end{cases}$$

Solution: The set of discontinuities for a function $f(x)$ having a finite number of finite jump discontinuities in I is the finite set of points where the function has a finite jump. Since a finite set has measure zero, this function is integrable. The set of discontinuities for the function $g(x)$ that equals 1 on each rational in $(0,1)$ and is zero otherwise is the whole interval $(0,1)$; i.e., at each point c in the interval there is a sequence of rationals that converge to c as well as a sequence of irrationals that converge to c. Then $g(x) = 1$ at each x in the rational sequence while $g(x) = 0$ at each x in the irrational sequence so $\lim_{x \to c} g(x)$ does not exist and g is not continuous at any point c in $(0,1)$.

The function $h(x)$ is discontinuous at each rational point. To see this, suppose $x = \frac{m}{n} \in (0,1)$ with m and n having no common factors. Then $h(x) = \frac{1}{n}$. Fix $\varepsilon > 0$ such that $0 < \varepsilon < \frac{1}{n}$. The density of the irrationals implies that for every $\delta > 0$ there are irrational points y such that $|x - y| < \delta$, and at all such points, we have $h(y) = 0$. This leads to $|h(x) - h(y)| = \frac{1}{n} > \varepsilon$ for $|x - y| < \delta$; i.e., h is not continuous at $x.= \frac{m}{n} \in (0,1)$ with m and n having no common factors.. On the other hand, we can show that h is continuous at each irrational point in $(0,1)$. To see this, let x_0 be an irrational number in $(0,1)$ and fix an $\varepsilon > 0$. Note that there can be only finitely many integers n such that $n \leq \frac{1}{\varepsilon}$, hence there can be only finitely many $x.= \frac{m}{n} \in (0,1)$ with m and n having no common factors such that $h(x) = \frac{1}{n} \geq \varepsilon$. If we choose $\delta > 0$ sufficiently small that none of these finitely many points belong to $N_\delta(x_0)$, then

$$|h(x_0 - h(y))| = |h(y)| < \varepsilon \qquad for \ all \ y \in N_\delta(x_0).$$

Then h is continuous at each irrational point in I and we see that the set of points of I where h is discontinuous is just the rational points in I. Since the set of discontinuities has measure zero, this function $h(x)$ is Riemann integrable on I.

Properties of the Integral

Problem 5.11 Suppose that if f and g are Riemann integrable on $I = [a,b]$. Then show that

a) if $f(x) \geq 0$ for all $x \in I$, then $\int_I f \geq 0$

b) if p lies between a and b then $\int_a^p f + \int_p^b f = \int_a^b f$

c) for every real constant α, αf is integrable and $\int_I \alpha f = \alpha \int_I f$

d) $\int_I (f+g) = \int_I f + \int_I g$

Solution:(a) Let $P \in \Pi[I]$. Then $f(x) \geq 0$ for all $x \in I$ implies that for each subinterval I_k in the partition, $m_k = GLB\{f(x) : x \in I_k\} \geq 0$. This implies in turn that $s[f : P] \geq 0$, and since this holds for every partition $P \in \Pi[I]$, it follows that

$$\int_I f = s[f] = LUB\{s[f : P] : P \in \Pi[I]\} \geq 0.$$

Note that this result implies that

for f,g Riemann integrable on I and $f(x) \geq g(x)$ for all $x \in I,$ then $\int_I f \geq \int_I g.$

(b) For p between a and $b,$ and any $P \in \Pi[I],$ p must belong to one of the subinterval I_k in $P.$ Then let P' denote the refinement of the partition P obtained by replacing $I_k = [x_{k-1}, x_k]$ by the two subintervals, $[x_{k-1}, p],$ and $[p, x_k].$ If $p = x_k$ for some $k,$ we just take $P' = P.$ In either case, we have

$$s[f,P] \leq s[f,P'] \qquad and \qquad S[f,P] \geq S[f,P']$$

We also have

$$\begin{aligned} s[f,P'] &= s[f, \{x_0, x_1, \ldots, p, \ldots x_M\}] \\ &= s[f, \{x_0, x_1, \ldots, p\}] + s[f, \{p, \ldots x_M\}] \\ &\leq \int_a^p f + \int_p^b f \end{aligned}$$

and

$$\begin{aligned} S[f,P'] &= S[f, \{x_0, x_1, \ldots, p, \ldots x_M\}] \\ &= S[f, \{x_0, x_1, \ldots, p\}] + S[f, \{p, \ldots x_M\}] \\ &\geq \int_a^p f + \int_p^b f. \end{aligned}$$

Then

$$s[f,P] \leq s[f,P'] \leq \int_a^p f + \int_p^b f \qquad and \qquad S[f,P] \geq S[f,P'] \geq \int_a^p f + \int_p^b f.$$

hence

$$LUB\{s[f,P] : P \in \Pi[I]\} = \int_a^b f \leq s[f,P'] \leq \int_a^p f + \int_p^b f$$

$$GLB\{S[f,P] : P \in \Pi[I]\} = \int_a^b f \geq S[f,P'] \geq \int_a^p f + \int_p^b f.$$

But these imply

$$\int_a^b f \leq \int_a^p f + \int_p^b f \leq \int_a^b f$$

i.e., $\int_a^b f = \int_a^p f + \int_p^b f,$ which proves (b).

(c) and (d) can be proved together by noting that for a Riemann sum on any partition $P \in \Pi[I],$

$$RS[\alpha f, P, \xi] = \sum_k \alpha f(\xi_k)|I_k| = \alpha \sum_k f(\xi_k)|I_k| = \alpha RS[f, P, \xi]$$

and

$$\begin{aligned} RS[f+g,P,\xi] &= \sum_k (f(\xi_k) + g(\xi_k))\,|I_k| \\ &= \sum_k (f(\xi_k)|I_k| + \sum_k g(\xi_k))\,|I_k| \\ &= RS[f,P,\xi] + RS[g,P,\xi]. \end{aligned}$$

Since this is true for any partition, it follows that

$$\lim_{\|P\|\to 0} RS[\alpha f,P,\xi] = \alpha \lim_{\|P\|\to 0} RS[f,P,\xi];$$

which proves (c) and

$$\lim_{\|P\|\to 0} RS[f+g,P,\xi] = \lim_{\|P\|\to 0} RS[f,P,\xi] + \lim_{\|P\|\to 0} RS[g,P,\xi];$$

proving (d).

Problem 5.12 Suppose that f and g Riemann integrable on $I = [a,b]$. Then show that, $|f(x)|,\ f^2(x), f(x)\,g(x), f^n(x)$ are all integrable on I.

Solution: Let

$$f_+(x) = \begin{cases} f(x) & \text{if} \quad f(x) \ge 0 \\ 0 & \text{if} \quad f(x) < 0 \end{cases} \quad \textit{and} \quad f_-(x) = \begin{cases} -f(x) & \text{if} \quad f(x) \le 0 \\ 0 & \text{if} \quad f(x) > 0 \end{cases}.$$

Since f is integrable and $f(x) = f_+(x) - f_-(x)$, it is not difficult to see that f_+ and f_- must also be integrable on I. Now it is also the case that $|f(x)| = f_+(x) + f_-(x)$ and then it follows from (d) of the previous problem that $|f|$ is integrable.

To see that f^2 is integrable, let $P \in \Pi[I]$ and note that

$$S[f^2,P] = \sum_k M_k^2\,|I_k| \qquad \textit{and} \qquad s[f^2,P] = \sum_k m_k^2\,|I_k|$$

so

$$\begin{aligned} S[f^2,P] - s[f^2,P] &= \sum_k (M_k^2 - m_k^2)\,|I_k| \\ &= \sum_k (M_k + m_k)(M_k - m_k)\,|I_k| \\ &\le 2M \sum_k (M_k - m_k)\,|I_k|. \end{aligned}$$

Since f is integrable on I, it follows that for any $\varepsilon > 0$, P can be chosen such that $S[f,P] - s[f,P] < \frac{\varepsilon}{2M}$. Then $S[f^2,P] - s[f^2,P] < \varepsilon$ and we see that f^2 is integrable on I.

To see that fg is integrable it is enough to note that $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$. Since previous results imply that for f and g integrable we have that $f+g$, $(f+g)^2$, f^2 and g^2 are all integrable, we see that fg must also be integrable. It follows that for every integer n, f^n is integrable since $f^3 = ff^2$ etc.

Problem 5.13 Suppose that f is Riemann integrable on $I = [a,b]$ and $|f(x)| \le M$ for x in I. Then show that

$$\left|\int_I f\right| \le \int_I |f| \le M|I|$$

Solution: We first note that $-|f(x)| \le f(x) \le |f(x)|$, and then use the fact, proved in the previous problem, that $|f(x)|$ is integrable since f is integrable. Then it is a consequence of property (a) of problem 5.11 that

$$-\int_I |f(x)| \le \int_I f(x) \le \int_I |f(x)|$$

which is the same as $\left|\int_I f\right| \le \int_I |f|$. In the same way, $|f(x)| \le M$ implies $\int_I |f| \le M|I|$.

Problem 5.14 Suppose $f(x)$ is continuous on $I = [a, b]$ and $g \in \Sigma[I]$, with $g(x) \ge 0$ for $x \in I$. Then show that:

a) there exists a point $c \in I$ such that

$$\int_I f = f(c)|I|$$

b) there exists a point $c \in I$ such that

$$\int_I fg = f(c)\int_I g$$

Solution: Since f is continuous on I, we can write $m \le f(x) \le M$ for all x in I where m and M denote the minimum and maximum values assumed by f on I. Then

$$\int_I m = m(b-a) \le \int_I f \le \int_I M = M(b-a)$$

and

$$m \le \frac{\int_I f}{b-a} \le M.$$

The extreme value theorem implies there exist points p and q in I such that $f(p) = m$ and $f(q) = M$ so

$$f(p) \le \frac{\int_I f}{b-a} \le f(q).$$

and then the intermediate value theorem implies there exists a point c in I such that

$$f(c) = \frac{\int_I f}{b-a}.$$

This proves (a). Note that $f(c)$ can be viewed as the average value of f on I since $f(c)(b-a)$ has the same value as $\int_I f$.

To prove (b) write $mg(x) \le f(x)g(x) \le Mg(x)$ for all x in I and proceed as before to get

$$m\int_I g(x) \le \int_I f(x)g(x) \le M\int_I g(x).$$

If $g(x)$ is identically zero on I, then b holds for any choice of c in I. For a nontrivial g, since, $g(x) \ge 0$, we have $\int_I g > 0$ and then

$$m \le \frac{\int_I fg}{\int_I g} \le M.$$

Then we complete the proof as in the proof of (a).

Problem 5.15 Suppose that f is integrable on $[a,b]$ and that g is continuous on the range of f, i.e., on the interval $[c,d] = f[a,b]$. Then show that $g(f(x))$ is integrable on $[a,b]$.

Solution: We have to show that for each $\varepsilon > 0$ there is a partition $P \in \Pi[a,b]$ such that

$$S[g \circ f,P] - s[g \circ f,P] < \varepsilon \qquad (1)$$

Let $\varepsilon > 0$ be fixed and let $\varepsilon' = \frac{\varepsilon}{b-a+2K}$ where $K = \sup_{c \le s \le d}|g(s)|$. Since g is continuous on the compact interval, $[c,d]$, g is bounded (i.e., K is finite) and g is uniformly continuous. This implies that there exists a $\delta > 0$ such that

$$\forall s,t \in [c,d] \qquad |s-t| < \delta \qquad \text{implies} \quad |g(s)-g(t)| < \varepsilon' \qquad (2)$$

By choosing δ smaller, if necessary, we make $\delta < \varepsilon'$.

Next, we observe that since f is integrable on $[a,b]$, there exists a partition $P \in \Pi[a,b]$ such that

$$S[f,P] - s[f,P] = \sum_{i=1}^{N}(M_i - m_i)|I_i| < \delta^2 \qquad (3)$$

where M_i, m_i denote, respectively, the upper and lower bounds for f on the subinterval I_i. Note that this is the difference between the upper and lower sums for f, not $g \circ f$, and that the difference is less than δ^2.

Now we will separate the partition P into two parts; let $A = \{i : 1 \le i \le N,\ M_i - m_i < \delta\}$ and let $B = \{i : 1 \le i \le N,\ M_i - m_i \ge \delta\}$. Then for $i \in A$ we have

$$|f(x) - f(y)| \le M_i - m_i < \delta \quad \text{for all} \quad x,y \in I_i$$

so

$$|g(s) - g(t)| = |g(f(x)) - g(f(y))| \le \varepsilon' \qquad \text{by } (2).$$

That is, $G_i - g_i \le \varepsilon'$ for all $i \in A$, where G_i, g_i denote the upper and lower bounds for $g \circ f(x)$ for $x \in I_i$. It follows from this that

$$\sum_{i\in A}(G_i - g_i)|I_i| \le \varepsilon' \sum_{i \in A}|I_i| \le \varepsilon'(b-a) \qquad (4)$$

On the other hand, for $i \in B$, we have $M_i - m_i \ge \delta$ which, combined with (3), leads to

$$\sum_{i\in B}|I_i| = \frac{1}{\delta}\sum_{i\in B}(M_i - m_i)|I_i| \le \frac{1}{\delta}\{S[f,P] - s[f,P]\} < \delta < \varepsilon'.$$

Finally, since $K = \sup_{c \le s \le d}|g(s)|$, we have

$$\sum_{i\in B}(G_i - g_i)|I_i| \le 2K\sum_{i\in B}|I_i| \le 2K\varepsilon' \qquad (5)$$

and then (4) and (5) together imply,

$$\begin{aligned} S[g \circ f,P] - s[g \circ f,P] &= \sum_{i\in A}(G_i - g_i)|I_i| + \sum_{i\in B}(G_i - g_i)|I_i| \\ &\le \varepsilon'(b-a) + 2K\varepsilon' < \varepsilon \end{aligned}$$

which is (1).■

Note that this theorem could be used to prove that $|f(x)|$ and $f(x)^n$ is integrable if $f(x)$ is integrable since the absolute value and power functions are continuous. Note also that it is not sufficient for g and f to be only integrable. For example let $g(x) = 0$ if $x = 0$ and $g(x) = 1$ if $x \in (0,1]$. Let $f(x)$ be the function in part (c) of problem 5.10. Then $g \circ f(x) = 0$ if x is irrational and $g \circ f(x) = 1$ if x is rational which is the example we gave previously of a function that is not Riemann integrable.

Problem 5.16 Let

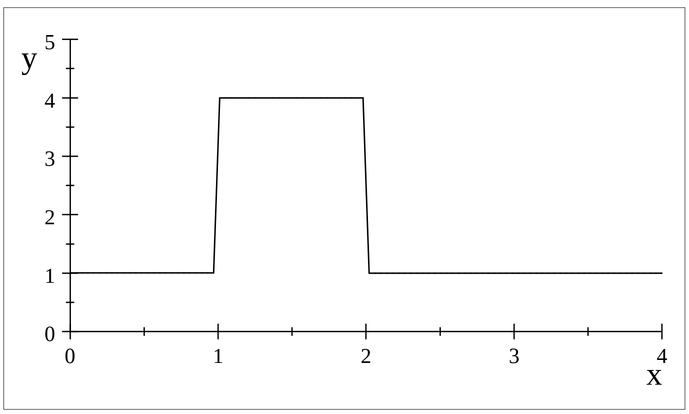
$$f(x) = \begin{cases} 1 & if \quad 0 < x < 1 \quad or \quad 2 < x < 3 \\ 4 & if \quad 1 < x < 2 \end{cases}$$

and

$$F(x) = \int_0^x f.$$

a) obtain an explicit formula for $F(x)$
b) sketch a graph of F and tell where F is differentiable
c) compute the derivative of F at each $x \in (0,3)$ where it exists

Solution: The graph of f is as follows:



For $x \in (0,1)$ we have

$$F(x) = \int_0^x 1 = x.$$

For $x \in (1,2)$ we have

$$F(x) = \int_0^x f = \int_0^1 1 + \int_1^x 4 = 1 + 4x - 4 = 4x - 3.$$

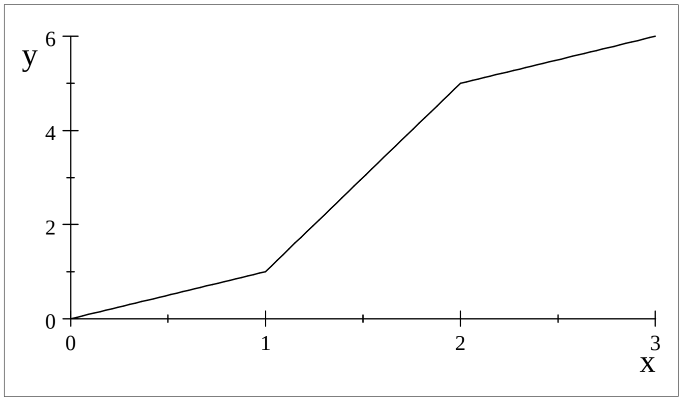
For $x \in (2,3)$ we have

$$F(x) = \int_0^x f = \int_0^1 1 + \int_1^2 4 + \int_2^x 1 = 1 + 4 + x - 2 = x + 3.$$

Then

$$F(x) = \begin{cases} x & if \quad x \in (0,1) \\ 4x - 3 & if \quad x \in (1,2) \\ x + 3 & if \quad x \in (2,3) \end{cases}$$

The graph of F is as follows:



We can see that $F'(x)$ exists at each x except $x = 1$ and $x = 2$. Then

$$F'(x) = \begin{cases} 1 & if \;\; x \in (0,1) \\ 4 & if \;\; x \in (1,2) \\ 1 & if \;\; x \in (2,3) \end{cases} = f(x)$$

Problem 5.17 (*Integration by parts*) Suppose $f(x)$ and $g(x)$ are continuous with continuous first derivatives on $I = [a,b]$. Then show that

$$\int_a^b f(x)g'(x)\,dx = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'(x)g(x)\,dx.$$

Solution: We have by the product rule that $(f(x)g(x))' = f'(x)\,g(x) + f(x)g'(x)$. Then it follows from the result of problem 5.16 that

$$\int_a^b (f(x)g(x))' = (f(x)g(x))|_{x=a}^{x=b} = f(b)\,g(b) - f(a)\,g(a)$$

while

$$\int_a^b (f(x)g(x))' = \int_a^b [f'(x)\,g(x) + f(x)g'(x)].$$

Then

$$\int_a^b [f'(x)\,g(x) + f(x)g'(x)].= f(b)\,g(b) - f(a)\,g(a)$$

which is the result to be proved.

Problem 5.18 (*Change of variables formula*) Suppose $\phi(x)$ is continuous with continuous first derivative on $I = [a,b]$ and that *f* is continuous on $J = \phi[I]$. Then show that

$$\int_a^b f[\phi(t)]\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

Solution: Here we are going to use the chain rule for derivatives in reverse. Let $G(t) = F(\phi(t))$ where $F' = f$. The chain rule for differentiation implies $G'(t) = F'[\phi(t)]\,\phi'(t) = f[\phi(t)]\,\phi'(t)$. Then, using the result of problem 5.16,

$$\int_a^b G'(t) = G(b) - G(a) = F(\phi(b)) - F(\phi(a)).$$

But

$$\int_a^b G'(t) = \int_a^b f[\phi(t)]\,\phi'(t)\,dt$$

and

$$F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

Then it follows that $\int_a^b f[\phi(t)]\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$

Exercises

1. Let $f(x)$ be defined on $[0,6]$ as follows:

$$f(x) = \begin{cases} x & if \quad x = 1,2,3,4,5 \\ 0 & otherwise \end{cases}$$

Using the partition $P_n = \left\{0, 1-\frac{1}{n}, 1+\frac{1}{n}, 2-\frac{1}{n}, 2+\frac{1}{n}, 3-\frac{1}{n}, \ldots, 5-\frac{1}{n}, 5\right\}$, compute $S[f,P_n]$ and $s[f,P_n]$. Show that $S[f,P_n]$ and $s[f,P_n]$ tend to the same limit as n tends to infinity.

2. Let $f(x)$ be defined on $[0,6]$ as follows:

$$f(x) = \begin{cases} 1 & if \quad x \in [1,2) \\ 2 & if \quad x \in [2,4) \\ 3 & if \quad x \in [4,6] \end{cases}$$

Using the partition $P_n = \left\{0, 2-\frac{1}{n}, 2+\frac{1}{n}, 4-\frac{1}{n}, 4+\frac{1}{n}, 6-\frac{1}{n}, \ldots, 6\right\}$, compute $S[f,P_n]$ and $s[f,P_n]$. Show that $S[f,P_n]$ and $s[f,P_n]$ tend to the same limit as n tends to infinity.

3. State three separate conditions on $f(x)$, each of which is sufficient to imply $f(x)$ is integrable on $I = [0,4]$.

4. Let $f(x) = \sqrt{x+4}$ on $I = [0,6]$ and let $P = \{0,2,4,6\}$ and $P' = \{0,1,2,3,4,5,6\}$.

a. a) Compute $s[f,P]$, $S[f,P]$, $s[f,P']$, and $S[f,P']$.

b. Sketch the graph of f on I and show why $s[f,P] < s[f,P']$ and $S[f,P'] < S[f,P]$.

c. As the partition mesh size tends to 0, what happens to $S[f,P] - s[f,P]$? In part c, it is not enough to just **say** what happens, you have to explain **why** it happens.

5. Suppose $f(x) \in \mathbb{C}^1[0,\infty)$ and $f'(x) \geq \alpha > 0$ for all $x \geq 0$. Show that $f(x)$ is not bounded on its domain. Hint: use the MVT for derivatives. Use an example to show that f may still be bounded if $f'(x) \geq 0$.

6. Suppose f is integrable on $I = [a,b]$ and let $F(x) = \int_x^b f$. a) Is $F(x)$ Lipschitz continuous on I? b) Is $G(x) = \int_a^{\cos x} f$ Lipschitz on I?

7. Suppose $f,g \in \mathbb{C}[a,b]$ and that $\int_a^b f = \int_a^b g$. a) prove there is a $c \in [a,b]$ such that $f(c) = g(c)$. b) Is it the case that $f(x) = g(x)$ for all x in $[a,b]$?

8. Suppose f is integrable on $I = [a,b]$ and $m \leq f(x) \leq M$ for $x \in I$.

a. make a sketch to show why $m(b-a) \leq \int_a^b f \leq M(b-a)$

b. prove that the estimate from part a) holds.

c. Explain why, if f is continuous on $[a,b]$ then there exists a c in I such that

$$f(c) = \frac{1}{b-a}\int_a^b f$$

but if f is only integrable, the result may not hold.

9. Suppose f is continuous but not constant on $[a,b]$. Show that $\int_a^b f^2 > 0$.

10. For f integrable on $[a,b]$ let $F(x) = \int_{2x}^b f$

a. Show that F is uniformly continuous on $[a,b]$.

b. Show that at each x in $[a,b]$ where f is continuous, $\lim_{h\to 0} D_h F(x)$ exists.

c. Find $F'(x)$.

11. Calculate $\lim_{h\to 0}\frac{1}{h}\int_3^{3+h}\sin(t^2+1)dt$

12. Let $G(x) = \int_{\cos x}^2 e^{t^2}dt$. Find $G'(x)$ at all x where the derivative exists.

13. Find a function $f \in C[1,\infty)$ such that $f(x) \neq 0$ and $[f(x)]^2 = 2\int_1^x f$ for $x \in [1,\infty)$.

14. Under what conditions on f and F is it true that $\int_a^b f = F(b) - F(a)$? Give examples where version 3 of the fundamental theorem applies and where it does not apply.

15. Under what conditions on f is it true that $\frac{d}{dx}\int_a^x f = f(x)$? Give an example of an f where the result holds and another example where it does not hold.

16. Determine the convergence or divergence of the following improper integrals:

a. $\int_0^1 \ln x\, dx$

b. $\int_0^1 x\ \ln x\, dx$

c. $\int_2^\infty \frac{\ln x}{x}\ dx$

d. $\int_2^\infty \frac{1}{x(\ln x)^2}\ dx$

e. $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx$

f. $\int_0^2 \ln(\ln x)dx$

g. $\int_1^3 \frac{1}{\ln(\ln x)}dx$

h. $\int_{-\infty}^\infty e^{-x}dx$