Chapter 3 Continuity

In this chapter we begin by defining the fundamental notion of continuity for real valued functions of a single real variable. When trying to decide whether a given function is or is not continuous, it is often helpful to have more than one way of characterizing continuity and so we provide the original definition of continuity (which does not mention limits), the definition in terms of function limits and, finally, a definition in terms of sequences. Of course the definitions are equivalent but sometimes one of them is more convenient than the others in order to prove a given function is or is not continuous at a particular point.

A function that is continuous on an interval has a number of special properties. Some of the important consequences of continuity include:

- the bounded range theorem (corollary 3.7)
- the extreme value theorem (corollary 3.8)
- the intermediate value theorem
- persistence of sign

The notion of uniform continuity is also introduced. This notion will be of particular importance in connection with the discussion of integration in a later chapter. Finally we will introduce the related notions of injectivity and strict monotonicity for functions in connection with the discussion of existence and continuity of an inverse function.

Functions

A real valued function of a single real variable can be defined as a rule which assigns to each real number x in a subset D of the reals, a uniquely determined real number, $y = f(x)$. The set D is called the ***domain*** of the function and the set of values $y = f(x)$ obtained as x varies over D is called the ***range*** of f. A function can also be defined as a set of ordered pairs (x,y) of reals such that no two distinct pairs have the same first element. Then the domain, D, is the set of all first entries for pairs in the set and the range of the function is the set of all second entries. The ***graph*** of the function is the set of all the pairs, $(x,f(x))$, considered to be a set in the plane.

Example *Functions*

1(a) The function $f(x) = x^2$ having domain D equal to the closed interval $[0,2]$ assigns to each x in D, the real number x^2. The range of this function is then the set $[0,4]$. The graph of this function is the collection of points (x,x^2), $0 \le x \le 2$, in the plane. These points form a part of a parabola. Note that the domain $D = [0,2]$ is not the largest possible domain for this function but has been chosen arbitrarily for purposes of discussion in this example.
1(b) The function $f(x) = \sqrt{x-3}$ with domain D equal to the unbounded interval $[3,\infty)$ has for its range the unbounded interval $[0,\infty)$. Note that for $x < 3$, this function does not produce real values. The set $x \ge 3$ is the largest possible domain for this function.
1(c) The function defined by

$$f(x) = \begin{cases} -1 & if \quad -1 \le x < 0 \\ +1 & if \quad 0 \le x \le 1 \end{cases}$$

for x in the domain $D = [-1,1]$ has for its range the set consisting of the two points $y = 1$ and $y = -1$. The graph of this function is two unconnected horizontal line segments. Since the function is not given by a single formula that applies over the entire domain, we say the function is ***piecewise defined***.

Continuity

We begin with the first of several equivalent definitions for continuity:

> **Definition** *A function $f(x)$ with domain D is said to be* ***continuous*** *at $x_0 \in D$ if, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that $y = f(x)$ belongs to $N_\varepsilon[f(x_0)]$ whenever x belongs to $N_\delta[x_0] \cap D$.*

If f is continuous at every point x_0 in D then we say f is continuous on D, or when there is no possibility of misunderstanding, we say f is continuous. A function that is not continuous is said to be ***discontinuous***.

Example *Continuous Functions*

2(a) For fixed constants, $a \neq 0$ and b, consider the linear function $f(x) = ax + b$ with domain $D = R$. We can use the definition of continuity to show that this function is continuous at every point in its domain. To see this, let x_0 be fixed but arbitrary. Then

$$|f(x) - f(x_0)| = |(ax + b) - (ax_0 + b)| = |a||x - x_0|$$

and for any positive ε it is easy to see that $|f(x) - f(x_0)| = |a||x - x_0| < \varepsilon$ whenever $|x - x_0| < \delta = \frac{\varepsilon}{|a|}$. This is precisely the assertion that $y = f(x)$ belongs to $N_\varepsilon[f(x_0)]$ whenever x belongs to $N_\delta[x_0] \cap D$, where $\delta(\varepsilon)$ is given by $\delta = \frac{\varepsilon}{|a|}$. Since this argument did not require x_0 to have any particular value, it follows that $f(x) = ax + b$ is continuous at every x_0 in the domain $D = R$.

2(b) Consider the function $f(x) = x^2$ on $D = [-1, 2]$. Note first that for x, x_0 two distinct points in D, we can write

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0|$$

Now it is easy to see that $|x + x_0| \leq 4$ for all choices of $x, x_0 \in D$. Then to show that $f(x) = x^2$ is continuous on D, we simply write

$$|f(x) - f(x_0)| = |x + x_0||x - x_0| \leq 4\,|x - x_0|$$

and it follows immediately that for all choices of $x, x_0 \in D$, $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta = \frac{\varepsilon}{4}$.

Since we can write,

$$(x^3 - z^3) = (x - z)(x^2 + zx + z^2)$$
$$(x^4 - z^4) = (x - z)(x^3 + x^2 z + z^2 x + z^3)$$

we can prove $f(x) = x^3$ and $f(x) = x^4$ are both continuous on D by a similar argument. In fact, $f(x) = x^n$ is continuous for every $n \in N$ on every compact interval $[a, b]$ by essentially the same proof.

2(c) The following functions are continuous at every $x \in R$,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{(any polynomial)}$$

$$Sin(x), \qquad Cos(x), \qquad e^x$$

Each of the following functions is continuous at every point, $x,$ where it is defined

$$r(x) = \frac{P(x)}{Q(x)} \quad \text{for polynomials } P \text{ and } Q$$

$$Tan(x), \quad Sec(x), \quad Cot(x), \quad Csc(x)$$

$$Log(x)$$

Using the definition to prove the continuity (or discontinuity) for a function can be awkward. For that reason, we need alternative characterizations for continuity. Function limits and sequences are two notions that lead to useful alternative characterizations for continuity.

Equivalent Definitions of Continuity

We are in a position to use function limits and sequence limits to give alternative definitions for continuity.

> **Theorem** ***3.4*** *(Limit definition of continuity) The function $f(x)$ on domain D is continuous at the point $x = c$ in D if and only if $\lim_{x \to c} f(x) = f(c)$.*

Note that this theorem makes several assertions:

- the limit point c belongs to D so that $f(c)$ is defined
- the function $f(x)$ tends to a limit, $L,$ as x tends to c
- the limiting value, $L,$ and the function value, $f(c)$ are equal

If any of these assertions fails, then $f(x)$ fails to be continuous at $x = c.$ We can combine theorems 3.1 and 3.4 to obtain the following characterization of continuity.

> **Theorem** ***3.5*** *(Sequence definition of continuity) The function $f(x)$ on domain D is continuous at the point $x = c$ in D if and only if for every sequence $\{a_n\}$ in D that converges to $c,$ it follows that $\{f(a_n)\}$ converges to $f(c)$.*

Theorem 3.5 is most often used to show that a given function is discontinuous at a point. In order to do this, all that is needed is a sequence $\{a_n\}$ in D that converges to $c,$ for which the sequence of function values, $\{f(a_n)\}$ fails to converges to $f(c)$.

Continuous functions may be combined in various ways to form new functions that are also continuous. It is clear what role theorems 3.2 has in proving the following result.

> **Theorem** ***3.6*** *Suppose $f(x)$ and $g(x)$ are two real valued functions on domain $D.$ Suppose also that both f and g are continuous at $x = c$ in $D.$ Then $af(x) + bg(x)$ and $f(x)g(x)$ are continuous at $x = c$ as is the function $\frac{f(x)}{g(x)}$, provided $g(c) \neq 0.$ Moreover, if $F(x)$ is defined and continuous in some neighborhood of the point $b = f(c),$ then the composed function $F(f(x))$ is also continuous at $x = c.$*

The theorems presented so far are all concerned with determining whether a given function is continuous, either at a specified point or at all points of the domain $D.$ Now we begin to consider some of the consequences of continuity.

Consequences of Continuity

There are a number of function properties that follow from continuity. We list the most important of them here.

Theorem *3.7 A function that is continuous on a compact domain has a compact range.*

Theorem 3.7 contains more information than at first meets the eye. The following two results are contained in theorem 3.7 but are often stated separately in order to emphasize their conclusions.

Corollary *3.8 (Bounded range theorem) A function that is continuous on a compact domain is bounded*

Corollary *3.9 (Extreme value theorem) If $f(x)$ is continuous on a compact domain, D, then there exist points c and d in D such that $f(c) \leq f(x) \leq f(d)$ for all x in D.*

The corollaries do not contain any information not already asserted by the theorem but they expose the practical implications of the consequence of the theorem. In particular, corollary 3.8 asserts that a function that is continuous on a closed bounded interval is necessarily bounded (above and below). The other corollary asserts in addition that there exist points in the domain where the LUB and GLB of the range of function values are actually assumed.

Theorem *3.10 A function that is continuous on a connected domain has a connected range.*

This is the statement of the result that remains true when discussing the more general situation involving functions from R^n into R^m. In a one dimensional setting this result can be stated more simply as follows:

Theorem *3.10 For any function that is continuous on a domain, D that is an interval, the range is also an interval.*

The implication of this theorem is often expressed as the statement that the graph of a continuous function of one variable can be drawn without lifting one's pen. More precisely, it can be stated as the following result:

Corollary *3.11 (Intermediate value theorem) If $f(x)$ is continuous on a domain, D, that is an interval, then for any points $p < q$ in D and any real number S lying between the values $f(p)$ and $f(q)$, there exists an s in D such that $p \leq s \leq q$ and $f(s) = S$.*

Uniform Continuity

The previously given definition of continuity asserts that $f(x)$ is continuous at $x = x_0$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that $y = f(x)$ belongs to $N_\varepsilon[f(x_0)]$ whenever x belongs to $N_\delta[x_0] \cap D$. Note that the δ which is associated with a given ε depends not just on ε but may depend on on the point x_0 as well. In some cases, for each ε a single choice of $\delta(\varepsilon)$ serves for all x_0 in D. We say then that the function is uniformly continuous on D. The precise definition is stated as,

Definition *A function f is said to be* ***uniformly continuous*** *on D if, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for all $x_1, x_2 \in D$, $|f(x_1) - f(x_2)| < \varepsilon$ whenever*

$|x_1 - x_2| < \delta(\varepsilon)$; *i.e.,* $f(x_2)$ *belongs to* $N_\varepsilon[f(x_1)]$ *whenever* x_2 *belongs to* $N_\delta[x_1] \cap D.$

Every function that is uniformly continuous on D is necessarily continuous at every point of D but the converse is false unless D is compact (see solved problems 3.23 and 3.24).

Theorem ***3.12*** *A function that is continuous at each point of a compact domain, D, is uniformly continuous on D.*

Theorem ***3.13*** *Suppose $f(x)$ is uniformly continuous on a domain, D. Then whenever $\{a_n\}$ is a Cauchy sequence in D it follows that $\{f(a_n)\}$ is a Cauchy sequence in the range of f.*

The converse of theorem 3.13 is also true; i.e., if f maps every Cauchy sequence in D into a Cauchy sequence in $rng[f]$ then f is uniformly continuous on D. However, it is not true that if f is uniformly continuous on D and $\{f(a_n)\}$ is a Cauchy sequence in $rng[f]$ then $\{a_n\}$ must be a Cauchy sequence in D. To see this, consider the function $f(x) = x^2$ which is continuous on the compact domain $D = [-1,1]$, and therefore uniformly continuous. For any $c \in (0,1)$ let $\{a_n\}$ denote the sequence

$$a_n = \begin{cases} \frac{\varepsilon}{n} - c & \text{if } n \text{ is odd} \\ \frac{\varepsilon}{n} + c & \text{if } n \text{ is even} \end{cases}$$

where $\varepsilon > 0$ is chosen so that $\{a_n\} \subset D$. This is a sequence with two limit points so it is not a Cauchy sequence. On the other hand,

$$f(a_n) = c^2 - \frac{2\varepsilon}{n}c + \left(\frac{\varepsilon}{n}\right)^2 \to c^2$$

is convergent and thus Cauchy.

Definition *A function f is said to be* ***Lipschitz continuous*** *on D if there exists $M > 0$ such that for all $x,y \in D$, $|f(x) - f(y)| \le M\,|x - y|$.*

It is evident that any Lipschitz continuous function is necessarily uniformly continuous but the converse is false. The function $f(x) = \sqrt{x}$ on $[0,1]$ is an example of a function that is uniformly continuous but not Lipschitz continuous on $[0,1]$.

Inverse Functions

A real valued function $f(x)$ is said to be ***one to one*** or ***injective*** if

for all $x_1, x_2 \in D$, $\quad x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Recall that f is a function on D if the set of ordered pairs $\{(x,y) : x \in D,\ y = f(x)\}$ is such that no two pairs have the same first element. If f is injective, then it is also true that no two pairs have the same second element. Recall that f is a well defined function on D if any vertical line through a point of D cuts the graph of f in only one point. Similarly, f is an injective function on D if any horizontal line through a point of $f[D]$ meets the graph of f in only one point. In this case, the set of reversed pairs $\{(y,x) : x \in D,\ y = f(x)\}$ defines a function $x = g(y)$ with the property

$y = f(x)$ if and only if $x = g(y)$

We say that $x = g(y)$ is the ***inverse*** of the function $y = f(x)$, and we write $g = f^{-1}$.

Monotone Functions

A real valued function $f(x)$ whose domain is the interval D is said to be:

- increasing on D if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$
- decreasing on D if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$

We say that f is ***monotone*** on D if $f(x)$ is either increasing or decreasing on D. If the inequality $f(x_1) \leq f(x_2)$ is replaced by the strict inequality $f(x_1) < f(x_2)$, then we say that f is ***strictly increasing*** on D. Strictly decreasing is defined in a similar fashion and we say that f is ***strictly monotone*** if it is either strictly increasing or strictly decreasing. It is evident that a strictly monotone function must be injective. Conversely we have

Theorem *3.14 If $f(x)$ is injective and continuous on the interval I, then $f(x)$ is strictly monotone on I.*

Theorem *3.15 (Continuity of the inverse) If $f(x)$ is strictly monotone and continuous on the interval I, then $g = f^{-1}$ is strictly monotone and continuous on the interval $J = f[I]$.*

Example *Injectivity and Monotonicity*

3(a) The function $f(x) = e^x$ is continuous and strictly increasing for all real x. The range of f is the set of positive real numbers. Then theorem 3.15 implies f^{-1} is strictly monotone and increasing on the set of positive real numbers. Here $f^{-1}(x) = \ln x$.

3(b) The continuous function $f(x) = x^2$ is strictly increasing on the interval $(0,b)$ for $b > 0$. The piecewise defined function

$$g(x) = \begin{cases} x & for \quad 0 \leq x \leq 1 \\ 1 & for \quad x > 1 \end{cases}$$

is increasing but not strictly increasing on $(0,b)$ for $b > 1$. Then $f(x)$ has a continuous inverse given by $f^{-1}(x) = \sqrt{x}$, but g is not injective and has no inverse.

3(c) The function $f(x) = x^2$ is continuous on $(-3,3)$ but it is not monotone on this domain. Then theorem 3.14 implies that f is not injective. Of course this is clear since for each x, $0 < x < 3$, $f(x) = f(-x)$.

3(d) An example of an injective function which is not monotone is

$$h(x) = \begin{cases} x & for \quad 0 \leq x \leq 1 \\ 1-x & for \quad x > 1 \end{cases}$$

Of course this function is not continuous as is predicted by theorem 3.14.

Solved Problems

Continuity

Problem 3.1 Show that the function $f(x) = \frac{1}{x}$ on $D = (0,\infty)$ is continuous at each $x \in D$.

Solution: let $c \in D$ be fixed but arbitrary (note that this means $c > 0$). Fix any $\varepsilon > 0$, and note that for any x such that $x \in [c/2, 3c/2]$, i.e., for $x \in N_{c/2}(c)$, we have

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{x-c}{xc}\right| \le \frac{|x-c|}{c \cdot c/2} = \frac{2}{c^2}|x-c|.$$

Now let δ denote the smaller of the two numbers $c/2,\ \varepsilon c^2/2.$ Then $\delta < c/2$ and $\delta < \varepsilon c^2/2$ which means that

$$\left|\frac{1}{x} - \frac{1}{c}\right| \le \frac{2}{c^2}|x-c| < \varepsilon \quad \text{whenever} \quad |x-c| < \delta.$$

According to the definition of continuity, this means f is continuous at $x = c > 0.$

Problem 3.2 Show that the functions Sin(x) and Cos(x) are continuous at each $x \in D = (-\infty, \infty).$

Solution: We will need to use the following estimates for the sine and cosine functions:

$$|Sin(x)| \le |x| \quad \text{and} \quad |Sin(x)|,\ |Cos(x)| \le 1, \quad \text{for all } x \in R \qquad (1)$$

These estimates were proved in problem 2.23. We have also the following trigonometric identities that hold for all real x and y :

$$Sin(x) - Sin(y) = 2Sin\left(\frac{x-y}{2}\right) \cdot Cos\left(\frac{x+y}{2}\right), \qquad (2)$$

$$Cos(x) - Cos(y) = 2Sin\left(\frac{x+y}{2}\right) \cdot Sin\left(\frac{y-x}{2}\right), \qquad (3)$$

Using (1) in (2) and (3), leads to the estimates

$$|Sin(x) - Sin(y)| \le 2\left|\frac{x-y}{2}\right| \cdot 1 = |x-y| \qquad (4)$$

$$|Cos(x) - Cos(y)| \le 2\left|\frac{x-y}{2}\right| \cdot 1 = |x-y| \qquad (5)$$

It is now evident from (4) and (5) that for all $\varepsilon > 0,$ we can choose $\delta = \varepsilon$ and have

$$\forall y \in R, \quad f(x) \in N_\varepsilon[f(y)] \quad \text{whenever} \quad x \in N_\delta[y]$$

in either of the cases $f(x) = Sin(x)$ or $f(x) = Cos(x).$

The sine and cosine functions each satisfy an estimate of the form

$$|f(x) - f(y)| \le M|x-y| \quad \text{for all } x,y \in D, \qquad (6)$$

For these functions, the constant M is equal to 1 and $D = R.$ Any function that satisfies an estimate of the form (6) is said to be ***Lipschitz continuous*** on its domain, $D.$ From examples 3.2(a,b), we can see that all linear functions are Lipschitz continuous on $D = R,$ and $f(x) = x^2$ is Lipschitz continuous on any bounded domain, $D = [a,b].$

Problem 3.3 Use the definition of continuity to show that the function

$$f(x) = \begin{cases} -1 & for \quad -1 \le x < 0 \\ 1 & for \quad 0 \le x \le 1 \end{cases} \quad on\ \ D = [-1,1]$$

is discontinuous at $x = 0 \in D$

Solution: Since $f(0) = 1,$ for any $\varepsilon > 0,$ to say that $f(x) \in N_\varepsilon[f(0)]$ is the same as saying $|f(x) - 1| < \varepsilon.$ But for any $\delta > 0,$ the neighborhood $N_\delta[0]$ contains both positive and negative values for $x.$ For the positive x values we have $f(x) = 1,$ while for the negative x values, we have $f(x) = -1.$ Then for any $0 < \varepsilon < 1,$ there can be no $\delta > 0$ such that $f(x) \in N_\varepsilon[f(0)]$ whenever $x \in N_\delta[0].$ This proves that f is not continuous at $x = 0.$

Equivalent Definitions of Continuity

Problem 3.9 (Removable discontinuities) Let $f(x)$ be defined on $D = [-1,1]$ by

$$f(x) = \begin{cases} x & for \quad x \neq 0 \\ 1 & for \quad x = 0 \end{cases}$$

Show that f is not continuous at $x = 0$.

Solution: We can show that $\lim_{x\to 0} f(x) = 0.$ Then the function limit at $x = 0$ exists but, since $f(0) = 1,$ the function limit does not equal the function value at $x = 0.$ Then theorem 3.4 asserts that $f(x)$ is not continuous at $x = 0.$ This is an example of a ***removable discontinuity**.* This is a discontinuity that can be removed by redefining the function at the point of discontinuity. In this case, we can define $f(0)$ to equal 0 in order to make the redefined function continuous.

Problem 3.10 (Jump discontinuities) Let $f(x)$ be defined on $D = [-1,1]$ by

$$f(x) = \begin{cases} 0 & for \quad x \in [-1,0) \\ x^2+1 & for \quad x \in [0,1] \end{cases}$$

Show that f is not continuous at $x = 0$.

Solution: We can show that $f(x)$ tends to no limit as x tends to zero. Then theorem 3.4 asserts that $f(x)$ is not continuous at $x = 0.$
To show that f tends to no limit at $x = 0,$ we define a sequence of points $\{a_n\}$ in D by

$$a_n = \frac{(-1)^n}{n+1} \qquad for \ \ n \in N.$$

Then a_n converges to zero as n tends to infinity but $a_{2n-1} < 0$ while $a_{2n} > 0.$ Then the sequence of function values, $\{f(a_n)\}$ is such that the subsequence $\{f(a_{2n})\}$ converges to 1 while the subsequence $\{f(a_{2n-1})\}$ converges to zero; i.e., the sequence $\{f(a_n)\}$ fails to converge by theorem 2.7.This function, $f(x),$ is said to have a **finite jump discontinuity** at $x = 0$. We will discuss this notion further in the next problem.

Problem 3.11 (One sided limits) Let $f(x)$ be a real valued function with domain D in R and let c be an accumulation point for the set $D^+ = \{x \in D : \ x > c\}.$ Then we say that the one sided limit for $f(x)$ as x approaches c from the right, exists and equals L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - c < \delta.$ In this case we write $f(c^+) = L.$ The one sided limit for $f(x)$ as x approaches c from the left is defined similarly. If this limit exists and equals L we write $f(c^-) = L.$

Show that the function of the previous problem has one sided limits from the left and the right but these limits are not equal.

Solution: For the $f(x)$ in the previous problem, we have $f(0-) = 0$ and $f(0+) = 1.$ To see this, note that for all x in $D^- = [-1,0),$ we have $f(x) = 0.$ Then for any $\varepsilon > 0$

$$|f(x) - 0| = 0 < \varepsilon \text{ for all } x \in D^- \text{such that } 0 < |x-0| < \delta,$$

and this holds for **any** $\delta > 0.$ This proves $f(0-) = 0.$ To prove $f(0+) = 1,$ note that for all x in $D^+ = [0,1],$ we have $f(x) = 1+x^2$ hence for any $\varepsilon > 0$

$$|f(x) - 1| = x^2 < \varepsilon \ \text{ for all } \ x \in D^+ \text{ such that } 0 < x - 0 < \delta = \sqrt{\varepsilon}.$$

Since both the left hand and right hand limits at $x = 0$ exist but are not equal, we say that f has a jump discontinuity at $x = 0$. The difference $f(0^+) - f(0^-)$, equals the magnitude of the jump at $x = 0$.

Problem 3.12 (An everywhere discontinuous function) Consider the function defined by,

$$f(x) = \begin{cases} 0 & \textit{if } x \textit{ is irrational} \\ 1 & \textit{if } x \textit{ is rational} \end{cases}$$

Show that this function, called the ***Dirichlet function***, is discontinuous at every x.

Solution:Let c denote any fixed, real number. Then it follows from corollaries 1.5 and 1.6 that there exist sequences $\{a_n\}$ and $\{b_n\}$ where the $a_n's$ are all rational and the $b_n's$ all irrational and both sequences converge to c. Then the definition of f implies that $f(a_n)$ converges to the limit 1 while $f(b_n)$ converges to 0. But theorem 3.5 asserts that if f is continuous at $x = c$, then for **every** sequence that converges to c, the corresponding sequence of function values must converge to $f(c)$. Then f is not continuous at $x = c$. Since c was arbitrarily chosen, this shows that f is continuous at *no* points. It is everywhere discontinuous.

Problem 3.13 Let the function $f(x)$ be defined on $D = [0,\infty)$ as follows:

$$f(x) = \begin{cases} \frac{1}{x} & \textit{if } x > 0 \\ a & \textit{if } x = 0 \end{cases} \qquad a \in R$$

Show that f is not continuous at $x = 0$ for any choice of the constant, a.

Solution: We showed in a previous example that $1/x$ tends to $+\infty$ as x tends to 0 through positive values. This could also be accomplished by let $a_n = 1/n$ and noting then that $f(a_n) = n$. In any case, it is clear that $f(x)$ cannot tend to any real number a as x tends to zero through positive values and f is therefore not continuous at $x = 0$. This function has an infinite jump discontinuity or singularity at $x = 0$.

Problem 3.14 Let the function $f(x)$ be defined on $D = [0,\infty)$ as follows:

$$f(x) = \begin{cases} Sin\left(\frac{1}{x}\right) & \textit{if } x > 0 \\ a & \textit{if } x = 0 \end{cases} \qquad a \in R.$$

Show that f is not continuous at $x = 0$ for any choice of the constant, a.

Solution: We define the sequence

$$a_n = \frac{1}{z_n} = \frac{1}{\frac{\pi}{2} + n\pi}, \qquad n \in N.$$

Then

$$f(a_n) = Sin\left(\frac{1}{a_n}\right) = Sin(z_n) = \begin{cases} -1 & \textit{if } n \textit{ is odd} \\ 1 & \textit{if } n \textit{ is even} \end{cases}$$

from which it is evident that the sequence $\{f(a_n)\}$ has two limit points and is therefore divergent. Since f tends to no limit at $x = 0$, f is not continuous there. This discontinuity is neither a removable discontinuity nor a jump discontinuity. It is sometimes referred to as an oscillatory discontinuity.

Consequences of Continuity

Problem 3.15: Prove that a continuous function with a compact domain has a compact range.

Solution: Suppose f is continuous on the compact domain $D,$ and that the range of f is denoted by $B.$ We will suppose that B is not compact and show that this leads to a contradiction.

If B is not compact, then by the result of problem 2.19, B must contain a sequence $\{b_n\}$ that contains no subsequence converging to a point of $B.$ Since $\{b_n\}$ is in the range of $f,$ it follows that for each $n,$ there is an a_n in D such that $f(a_n) = b_n.$ Since D is compact, $\{a_n\}$ must contain a subsequence $\{a_{n'}\}$ such that $a_{n'}$ converges to a point c in $D.$ But f is continuous on D so theorem 3.5 implies that $f(a_{n'})$ converges to $f(c)$ in $B.$ But this means $\{b_{n'}\}$ is a subsequence of $\{b_n\}$ and $b_{n'}$ converges to $f(c)$ in $B.$ This contradiction shows that every sequence in B must contain a subsequence converging to a point of $B,$ which is to say, B is compact.

Problem 3.16 Prove the Bounded range theorem: A function that is continuous on a compact domain is bounded.

Solution: Suppose f is continuous on the compact domain $D,$ and that the range of f is denoted by $B.$ By the result of the previous problem, B is compact, which is to say, B is closed and bounded. But this means B is contained in a closed bounded interval, $[a,b]$, which is just the assertion that $a \le f(x) \le b$ for all x in $D;$ i.e. f is bounded on $D.$

Problem 3.17 Prove the Extreme value theorem: If $f(x)$ is continuous on a compact domain, $D,$ then there exist points c and d in D such that $f(c) \le f(x) \le f(d)$ for all x in $D.$

Solution: Suppose f is continuous on the compact domain $D.$ By the result of the previous problem, the range of f , denoted by B is contained in a closed bounded interval $[a,b]$. Then B has a greatest lower bound, $a^*,$ and a least upper bound, $b^*,$ and by theorem 1.12, these bounds both belong to $B.$ But since B is the range of $f,$ it then follows that there exist points c and d in D such that $f(c) = a^*$ and $f(d) = b^*.$ Since a^* and b^* were the greatest lower bound and least upper bound for $B,$ respectively, it follows that $f(c) \le f(x) \le f(d)$ for all x in $D.$ This theorem asserts that if f is continuous on a compact domain, then there are points in the domain where f in fact assumes its maximum and minimum values.

Problem 3.18 Prove theorem 3.10: A function that is continuous on a connected domain has a connected range.

Solution: Suppose f is continuous at each point c of the connected domain $D \subset R.$ Recalling that a connected subset of R must be an interval or a point, we realize that if D is a single point $c,$ then the range of f is the single point $f(c)$ which is connected, so the result is true in this degenerate case.

Now let us suppose that D is an interval of positive length and that the range of f is denoted by $B.$ If B is not connected, then B can be written as the union of sets B_1 and B_2 that are nonempty, disjoint and neither contains an accumulation point of the other. Let $D_1 = \{x \in D : f(x) \in B_1\}$ and $D_2 = \{x \in D : f(x) \in B_2\};$ i.e., $f(D_1) \subset B_1$ and $f(D_2) \subset B_2.$

Clearly D_1 and D_2 are disjoint and nonempty, but since D is connected, at least one of them must contain an accumulation point of the other. For example, suppose there is a point c in D_1 that is an accumulation point for D_2. Then there is a sequence $\{c_n\}$ in D_2 that converges to c in D_1. But f is continuous at every point of D and hence $f(c_n)$ must converge to $f(c)$. But this implies that $f(c)$ belongs to B_1 and is an accumulation point for B_2, in contradiction to the assumption that B_1 and B_2 that are nonempty, disjoint and neither contains an accumulation point of the other. Then B must be connected.

Theorem 3.10 can be stated more simply in view of the fact that a connected subset of R must be an interval or a point. We could just say that if f is continuous on an interval I, then the range of f is an interval J such that $x \in I$ if and only if $f(x) \in J$. This means that for any two points $p < q$ in I, as x increases from p to q, $f(x)$ must assume **every value** between $f(p)$ and $f(q)$. For if not then we would produce the same contradiction used above. Then for any value S that lies between $f(p)$ and $f(q)$, there must exist a value s lying between p and q such that $f(s) = S$. This is corollary 3.11, the so called ***intermediate value theorem***.

Here is a proof of corollary 3.11 based on the nested interval theorem. We have points $p < q$ in D with S lying between $f(p)$ and $f(q)$, say $f(p) < S < f(q)$. Then we must show there is an s, $p < s < q$, such that $f(s) = S$.

We define a sequence, first letting $p_1 = p$, $q_1 = q$ and $m_1 = (p_1 + q_1)/2$.

$$\text{If } f(m_1) < S \text{ then let } p_2 = m_1,\ q_2 = q_1$$

$$\text{If } f(m_1) \geq S \text{ then let } p_2 = p_1,\ q_2 = m_1$$

$$\text{Let } \quad m_2 = (p_2 + q_2)/2$$

Continue recursively,

$$\text{If } f(m_2) < S \text{ then let } p_3 = m_2,\ q_3 = q_2$$

$$\text{If } f(m_2) \geq S \text{ then let } p_3 = p_2,\ q_3 = m_2$$

$$\text{Let } \quad m_3 = (p_3 + q_3)/2$$

This generates a sequence of $p's$ and $q's$ such that

$$p = p_1 \leq p_2 \leq \cdots p_n < q_n \cdots \leq q_2 \leq q_1 = q$$

and

$$|q_n - p_n| = \frac{q-p}{2^{n-1}} \longrightarrow 0 \text{ as } n \to \infty$$

Then the nested interval theorem implies that the intersection of all the intervals $[p_n, q_n]$ is a unique point, s, with the monotone sequences $\{p_n\}$ and $\{q_n\}$ each converging to s. Since f is continuous on D, it follows that $f(p_n)$ and $f(q_n)$ each converge to $f(s)$. But the sequences have been constructed so that, $f(p_n) \leq S \leq f(q_n)$ for every n, and therefore $f(s) = S$.

Problem 3.19 Use the intermediate value theorem to prove that every polynomial of odd degree must have at least one real zero.

Solution: Suppose $P(x)$ denotes a polynomial of odd degree. Then the domain of $P(x)$ is R and the sign of $P(x)$ as x tends to $+\infty$ must be opposite to the sign of $P(x)$ as x tends to $-\infty$. Then it is always possible to find numbers $p < q$ such that 0 lies between $f(p)$ and $f(q)$. Then the intermediate value theorem implies there exists a real number s between p and q such that $P(s) = 0$.

For example, suppose $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, for n an odd positive integer. Then $P(p) < 0 < P(q)$ for p sufficiently large and negative and q sufficiently large positive. Then there must exist some s between p and q where $P(s) = 0$.

This theorem is the basis of the following algorithm for finding zeroes of a function $f(x)$ that is continuous on an interval (a,b). We select a step length $h = \dfrac{b-a}{M}$ and let $x_m = a + mh$ for $m = 1,2\ldots,M$. Then we evaluate the product, $f(x_{m-1})f(x_m)$ for $1 \le m \le M$, and any m for which this product is negative indicates a zero of the function f between x_{m-1} and x_m.

Problem 3.20 (persistence of sign) Let $f(x)$ be defined and continuous on an open interval (a,b) containing the point c. If $f(c) \ne 0$, then show that there is a neighborhood of c throughout which f has the same sign as $f(c)$.

Solution: Since (a,b) is an open interval containing the point c, there exists an $h > 0$ such that $N_h(c) \subset (a,b)$. Suppose now that $f(c) > 0$ and let $\varepsilon = \frac{1}{2}f(c) > 0$. Then by the continuity of f, there exists a δ, $0 < \delta \le h$, such that $f(x) \in N_\varepsilon(f(c))$ whenever $x \in N_\delta(c)$. But this is just the statement,

$$0 < \tfrac{1}{2}f(c) = f(c) - \varepsilon < f(x) < f(c) + \varepsilon = \tfrac{3}{2}f(c) \text{ for all x such that } c - \delta < x < c + \delta.$$

Since δ was chosen such that $0 < \delta \le h$, it follows that $(c-\delta, c+\delta)$ is contained in (a,b). This shows that f remains positive throughout a neighborhood of any point where f is positive. The proof for the negative case is similar. A related result holds when the interval I is allowed to be closed and c is an endpoint. In this case the interval on which f maintains the same sign as $f(c)$ will lie on one side of c.

Problem 3.21 (A fixed point theorem) Let $f(x)$ be defined and continuous the unit interval, $I = [0,1]$ and suppose that $0 \le f(x) \le 1$ for all x in I; i.e., the domain and range of f are both equal to I . Then show that for some c in I, we have $f(c) = c$ (we say that if f maps I continuously into itself, then f has a ***fixed point***).

Solution: If $f(x)$ is continuous on I, then so too is the function $g(x) = f(x) - x$. In addition, $g(0) = f(0) \ge 0$ and $g(1) = f(1) - 1 \le 0$, (since $f(0)$ and $f(1)$ must lie in $I = [0,1]$). If either $g(0) = 0$ or $g(1) = 0$ then the fixed point is an endpoint of I. If neither $g(0)$ nor $g(1)$ is zero, then we have $g(1) < 0 < g(0)$ and it follows from the intermediate value theorem that there exists a point c in I such that $g(c) = 0$; ie., such that $f(c) = c$.

Uniform Continuity

Problem 3.22 Give an example of a function $f(x)$ that is continuous on $D = (0,\infty)$ and a sequence $\{a_n\}$ in D such that $\{a_n\}$ is a Cauchy sequence but $\{f(a_n)\}$ is not Cauchy.

Solution: In problem 3.1, the function $f(x) = \dfrac{1}{x}$ was shown to be continuous at each point in $(0,\infty)$. The sequence $a_n = \dfrac{1}{n}$ is contained in D and a_n converges to 0 as n tends to infinity. ($x = 0$ is an accumulation point of D but does not belong to D). Since the sequence is convergent, it is Cauchy. On the other hand, the sequence of function values, $f(a_n) = n$ is not bounded, hence it is not Cauchy.

Problem 3.23 Prove theorem 3.13, that for $f(x)$ uniformly continuous on D, $\{f(a_n)\}$ is necessarily a Cauchy sequence whenever $\{a_n\}$ in D is Cauchy.

Solution: Suppose f is uniformly continuous on D and $\{a_n\}$ is a Cauchy sequence in D. By the uniform continuity of $f,$ we know that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x,y \in D,$

$$|f(x) - f(y)| < \varepsilon \qquad whenever \qquad |x - y| < \delta.$$

Since $\{a_n\}$ is a Cauchy sequence in $D,$ we know that there exists $M \in N$ such that

$$|a_m - a_n| < \delta \qquad \text{whenever} \qquad m,n > M.$$

Then it follows that for all $m,n > M,$ we have $|f(a_m) - f(a_n)| < \varepsilon,$ which is to say $\{f(a_n)\}$ is a Cauchy sequence.

Problem 3.24 Show that $f(x) = Sin(\frac{1}{x})$ is not uniformly continuous on $D = (0,\infty).$

Solution: Let $a_n = \frac{2}{n\pi}$, $n \in N.$ Then $\{a_n\}$ is a Cauchy sequence in D that is converging to $0.$ However, the sequence of function values

$$f(a_n) = \begin{cases} \pm 1 & if \quad n \ is \ odd \\ 0 & if \quad n \ is \ even \end{cases}$$

has more than a single limit point and is therefore not a Cauchy sequence. Then theorem 3.13 implies that f is not uniformly continuous on $D.$

Problem 3.25 Show that:

(a) $f(x) = \frac{1}{x}$ is not uniformly continuous on $D_1 = (0,1)$

(b) $f(x) = \frac{1}{x}$ is uniformly continuous on $D_2 = [1,\infty)$

(c) $g(x) = x^2$ is uniformly continuous on $D_1 = (0,1)$

(d) $g(x) = x^2$ is not uniformly continuous on $D_2 = [1,\infty)$

Solution:

a) Consider the sequence of points $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$ in $D_1 = (0,1)$ and note that

$$|f(x_n) - f(y_n)| = 1$$

$$\text{but} \qquad |x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}.$$

This means there are points in D_1 which are as close to one another as we like but the corresponding function values are separated by a constant distance of $1.$

b) For $x,\ y$ in $D_2,$ $\qquad |f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{xy}|x - y|.$

Note that if $x,\ y \in D_2 = [1,\infty),$ then $xy \geq 1,$ hence

$$|f(x) - f(y)| \leq |x - y| \qquad \text{for all } x,y \in D_2$$

Then f is Lipschitz continuous on D_2 so it is uniformly continuous there.

c) For $x,\ y$ in $D_1,$ $\qquad |g(x) - g(y)| = |x^2 - y^2| = |x + y||x - y|.$

When $x,\ y \in D_1 = (0,1),$ it follows that, $|x + y| \leq 2,$ hence

$$|f(x) - f(y)| \leq 2|x - y| \qquad \text{for all } x,y \in D_1$$

This shows that g is Lipschitz continuous on D_1 so $g(x)$ is uniformly continuous there.

d) Consider the sequence of points $x_n = n + \frac{1}{2n}$ and $y_n = n - \frac{1}{2n}$ in D_2 and note that

$$|g(x_n) - g(y_n)| = \left|\left(n + \frac{1}{2n}\right)^2 - \left(n - \frac{1}{2n}\right)^2\right| = 2 \qquad \forall n.$$

But for all n, $|x_n - y_n| = \frac{1}{n}$, and this means there are points in D_2 which are as close to one another as we like but the corresponding function values are separated by a constant distance of 2.

Does a) imply that $f(x)$ is not continuous on D_1? Not at all. Note that in order for x and y to get closer together, they **both** must move closer to $x = 0$. If we fix one of the points, say we fix $y = c$, we can make $f(x)$ close to $f(y)$ by moving x close to $y = c$. This is the continuity of $f(x)$ on D. which we showed in problem 3.1. However, as the fixed point, $y = c$, is fixed closer to $x = 0$ the moveable point, x, has to be moved even closer to $y = c$ in order to obtain closeness of $f(x)$ to $f(y)$. This shows the lack of uniform continuity of $f(x)$ on D_1.

Problem 3.26 Prove that a function which is continuous on a compact domain is uniformly continuous.

Solution: Suppose $f(x)$ is continuous on the closed bounded interval $I = [a,b]$. If f is not uniformly continuous on I, then there exists an $\varepsilon > 0$ and sequences $\{a_n\}$, $\{b_n\}$ in I such that

$$\text{for each } n, \qquad |a_n - b_n| < \frac{1}{n} \qquad \textit{and} \qquad |f(a_n) - f(b_n)| > \varepsilon. \qquad (1)$$

Since I is compact, $\{a_n\}$ must contain a subsequence $\{a_{n'}\}$ that converges to a in I. Then

$$|b_{n'} - a| \leq |b_{n'} - a_{n'}| + |a_{n'} - a| \leq \frac{1}{n'} + |a_{n'} - a|$$

and it follows that the subsequence $\{b_{n'}\}$ of $\{b_n\}$ must also converge to a. Since f is continuous on I, both of the sequences $\{f(a_{n'})\}$ and $\{f(b_{n'})\}$ must converge to the limit $f(a)$ which contradicts (1). Then if f is continuous on a compact domain I, it follows that f must in fact be uniformly continuous on I.

Note that this result implies $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$. On the other hand, f is not Lipschitz continuous on $[0,1]$. To see this, consider x,y such that $0 < x < 1$ and $y = 0$. Then

$$|f(x) - f(y)| = \sqrt{x} \leq M|x - y| = Mx$$

is equivalent to $x \geq \frac{1}{M^2}$. Since this is obviously not true for all x in $[0,1]$, it follows that f is not Lipschitz continuous on $[0,1]$.

Inverse Functions

Problem 3.27 Prove that if f is continuous and one to one on $I = [a,b]$, then f must be strictly monotone on I.

Solution; Note first that since f is one to one on I, $f(a) \neq f(b)$. Suppose then that $f(b) > f(a)$. We will show then that f is strictly increasing on I; i.e., if we choose an x, $a < x < b$, then we will show that $f(a) < f(x) < f(b)$.

To see this, suppose we had $f(x) < f(a) < f(b)$. Then applying the intermediate value theorem to the interval $[x,b]$ leads to the existence of a point c, $x < c < b$, such that $f(c) = f(a)$. But this contradicts the assumption that f is one to one and we conclude that we cannot have $f(x) < f(a) < f(b)$ with $a < x < b$. Similarly, if we had $f(a) < f(b) < f(x)$ with

$a < x < b$, we would obtain the same contradiction and it follows that $f(a) < f(x) < f(b)$ for every x such that $a < x < b$.

Now choose y, $a < x < y < b$, and proceed as above to show first that $f(a) < f(y) < f(b)$. Then one final application of this same argument leads to the result, $f(x) < f(y)$ whenever $x < y$. This establishes that f is strictly increasing. The same sort of argument can be used to show f is strictly decreasing in the case that $f(b) < f(a)$.

Problem 3.28 Prove that if f is continuous and strictly monotone on $I = [a,b]$, then $g = f^{-1}$ must be continuous on $J = [c,d]$.

Solution; Suppose that f is strictly increasing on I. The proof in the case where f is strictly decreasing is similar.

Note first that if f is continuous and strictly increasing on $I = [a,b]$, then $g = f^{-1}$ is defined and has as its domain, $J = [c,d]$ where $c = f(a)$ and $d = f(b)$. In order to prove $g(y)$ is continuous, we shall suppose that $g(y)$ is discontinuous at some point y_0 in J and show that this leads to a contradiction. If $g(y)$ is discontinuous at some point y_0 in J, it means that there exists a sequence $\{y_n\}$ in J such that y_n converges to y_0 but $g(y_n)$ does not converge to $g(y_0)$. Let $g(y_0)$ be denoted by X_0. Then $f(X_0) = y_0$.

Let $x_n = g(y_n)$ and note that since $x_n \in I = [a,b]$ for every n, this bounded sequence must contain a subsequence, $\{x_{n'}\}$ converging to a limit, x_0 in I. Since $g(y_n)$ does not converge to $g(y_0)$, it must be that $x_0 \neq g(y_0) = X_0$.

Now the continuity of f on I implies that $f(x_{n'})$ is convergent to $f(x_0)$. But $f(x_{n'}) = y_{n'}$ and since this is a subsequence of $\{y_n\}$, it must be that $y_{n'}$ converges to y_0. Therefore, $f(x_0) = y_0$. On the other hand, we have $f(X_0) = y_0$ and $x_0 \neq X_0$. This is in contradiction to the assumption that f is strictly monotone (hence one to one) so our assumption that $g(y)$ can be discontinuous at some point in J leads to a contradiction, proving the result.

Exercises

1. Use the definition of function limit to find the following limits and prove they exist.

a. $\lim_{x\to 2} \dfrac{1}{1-x}$

b. $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2+2x-8}$

c. $\lim_{x\to 1} \dfrac{x}{1+x}$

d. $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

2. Find the following function limits

a. $\lim_{x\to\infty} \dfrac{x}{1+x^2}$

b. $\lim_{x\to\infty} \cos x$

c. $\lim_{x\to\infty} \dfrac{\cos x}{1+x^2}$

d. $\lim_{x\to 0+} \sqrt{x+4}$

e. $\lim_{x\to 0+} \cos\left(\frac{1}{x}\right)$

3. For each of the following functions, find the points where they are discontinuous or give a reason they are continuous for all x :

a. $\dfrac{x^3+1}{x^2+1}$

b. $\sin^2 x \cos x$

c. $\dfrac{\cos x - 1}{\sqrt{x}}$

d. $\dfrac{x-2}{|x-2|}$

4. The functions $f(x) = \dfrac{x^3 - 3x^2 + 2}{x^2 - 1}$ and $g(x) = \dfrac{x^2 + 4x - 5}{x^3 - 2x^2 + x}$ are each undefined at two points. Are these singularities removable?

5. If $F(x) = f(x) - g(x) + h(x)$, $G(x) = g(x) + 2h(x)$ and $H(x) = 2g(x) - h(x)$are all continuous on $\mathbb{R}$, then is it the case that f, g, and h are also all continuous on $\mathbb{R}$?

6. Give an example of functions f and g that are both discontinuous at $x = c$ but (i) $f+g$ is continuous at $x = c$ (ii) fg is continuous at $x = c$.

7. Suppose f is continuous for all x and that $f(x) = 0$ for every rational x. Show that $f(x) = 0$ for all x.

8. If f and g are continuous on $\mathbb{R}$ and $f(\frac{p}{q}) = g(\frac{p}{q})$ for all non-zero integers, p, q, then is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?.

9. Suppose $f(x) = \begin{cases} 2x & if \quad x = rational \\ x+3 & if \quad x = irrational \end{cases}$. Then find all the points where f is continuous.

10. Suppose f is defined on $(0,1)$ and that $|f(x)| \le 1$ for all $x \in (0,1)$. If $\lim\limits_{x\to 0} f(x)$ does not exist then show there must be sequences a_n, b_n converging to 0 for which the sequences $f(a_n)$ and $f(b_n)$ converge but to different limits.

11. Suppose f is continuous at all x and let $P = \{x : f(x) > 0\}$. If $c \in P$ then show that there is an $\varepsilon > 0$ such that $N_\varepsilon(c) \subset P$.

12. Suppose f and g are continuous for all x and let $S = \{x : f(x) \ge g(x)\}$. If c is an accumulation point for S, show that $c \in S$.

13. If f and g are continuous on $[0,1]$ and $f(x) > g(x)$ for $0 \le x \le 1$ then does there exist a $p > 0$ such that $f(x) \ge g(x) + p$ for $0 \le x \le 1$.

14. If f and g are continuous on $[0,1]$ and $f(x) > g(x)$ for $0 < x < 1$ then there does exist a $p > 0$ such that $f(x) \ge g(x) + p$ for $0 \le x \le 1$.

15. Let $f(x) = \begin{cases} x+1 & if \quad 0 \le x \le 1,\ x \ne \frac{1}{2} \\ 0 & if \quad x = \frac{1}{2} \end{cases}$ $\qquad g(x) = \begin{cases} \frac{1}{x} & if \quad 0 < x < 1 \\ 0 & if \quad x = 0 \end{cases}$

Explain how you would prove the continuity or lack of continuity for these two functions; i.e., in each case, cite a theorem which supports your answer.

16. Let $f(x) = \dfrac{x}{x^2+1}$ and let $A = f\{1 \le x \le 8\} \subset rng[f]$ $M = f^{-1}\{.2 \le y < .4\} \subset dom[f]$. In answering the following questions, give reasons (i.e., cite theorems or give examples) for your answers

a. Is A closed? Is it bounded?

b. Find the sup and the inf for A. Do these belong to the set?

c. Find the set of all y that belong to A

d. Find the set of all x that belong to M

e. Either prove that M is closed or show that it is not closed.

17. 1. Suppose A is an infinite subset of the reals and that p is the LUB for A but p does not belong to A. Tell whether the following statements are true or false and give coherent reasons for your answer.

a. p is an accumulation point for A

b. there is a "largest value" x in A such that $x \le p$.

18. State the compact range (extreme value, intermediate value) theorem. Give an example where **one** of the hypotheses is not satisfied and the conclusion then fails to hold.

19. State the persistence of sign theorem Explain the use of this theorem to prove the following result:: If f and g are continuous on $\mathbb{R}$ and $f(x) = g(x)$ for each rational x, then $f(x) = g(x)$ for all real x

a. State a condition on $f(x)$, and D that implies that f is uniformly continuous on D

b. State a condition on $f(x)$, that implies that f is uniformly continuous on (a,b)

c. Is $f(x) = \dfrac{1}{1+x^2}$ uniformly continuous on $(0,\infty)$?

20. Use the intermediate value theorem to tell how many real zeroes exist for the function $f(x) = \sin x - \cos x$ as well as to determine the approximate location of these zeroes.

21. Suppose f and g are continuous on $(-\infty,\infty)$ and that $f(x) = g(x)$ at every rational value for x. Use the persistence of sign result to show that f and g must be equal at every real value.

22. Given that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(0,1)$, is f uniformly continuous on $(0,1)$? Given that the function $g(x) = \sqrt{x}$ is continuous on $(0,1)$, is g uniformly continuous on $(0,1)$?

23. Let A denote the set of all the rational numbers between 0 and 1.

a. Is this a closed set?
b. Is it an open set?
c. What are the accumulation points (boundary points, interior points) for A?

24. For each of the following statements, first, tell whether the following are true or false, then state a theorem which shows a statement is true or give a counter example that shows it is false.

a. If $f(x)$ is continuous and injective on $[a,b]$ and $f(a) < f(b)$, then for all $x,y \in [a,b]$, $x < y$ if and only if $f(x) < f(y)$.
b. There exist sequences $\{a_n\}$ which are bounded but which contain no convergent subsequences.
c. For every real value, x, there is a sequence of rational numbers that converges to x.
d. If $\{a_n\}$ in D converges to $c \in D$, and $\{f(a_n)\}$ converges to $f(c)$, then f must be continuous at $x = c$.
e. If F is continuous on D where $F(x) = f(x)g(x)$, then f and g are continuous on D.

25. Use the intermediate value theorem to tell how many real zeroes exist for the function $f(x) = \sin x - \cos x$ as well as to determine the approximate location of these zeroes.

26. Use the intermediate value theorem to tell how many positive zeroes exist for the function $f(x) = \sin x - \frac{1}{x}$ as well as to determine the approximate location of these zeroes.

27. Tell whether the following statements are true or false and cite a theorem to justify your answer:

a. If $f(x)$ is continuous and monotone on $[a,b]$ then $\forall\, x,y \in [a,b]$, $x < y$ implies $f(x) \neq f(y)$.

b. If $\{a_n\}$ is a monotone sequence that does not converge then $\left\{\frac{1}{a_n}\right\}$ must tend to zero as n tends to infinity..

c. If $f(x)$ is continuous on $(-\infty,\infty)$ and $f(x) = 0$ if x is rational, then $f(x) = 0$ at every real x.

d. If A is an infinite subset of the reals and $p = \sup A$ but p is not in A then there is no largest x in A such that $x \leq p$.

e. If F is continuous on D where $F(x) = 5f(x) + 4g(x)$, then f and g are continuous on D.

28. Let $f(x) = \frac{\sin x}{\sqrt{x}}$. Can $f(0)$ be defined in such a way that f is uniformly continuous on $[0,1]$? Is f uniformly continuous on $[0,\infty)$?

29. Suppose $f(x)$ is continuous on $\mathbb{R}$. Show that $S = \{x \in \mathbb{R} : f(x) < 0\}$ is an open set.

30. A continuous function on a compact domain has a compact range. Give examples to show that if either (a) the domain is not closed or (b) the domain is not bounded, then the range is not compact.

31. Can you use the compact range theorem to show that $f(x) = 1/x$ on domain $[0,1]$ is not continuous?

32. Suppose $f(x)$ is continuous at $x = a$ and $f(a) < 0$. Show $\exists\ \delta > 0$ such that $f(x) < 0$ $\forall x \in N_\delta(a)$.

33. Suppose: $f(x)$ is continuous on $dom[f]$ and $\{x_n\} \subset dom[f]$ is such that $x_n \to \alpha$ and $f(x_n) < 0$ for each n. Show that: $f(\alpha) \leq 0$. Hint: Suppose $f(\alpha) > 0$ and show this leads to a contradiction.

34. Suppose f is continuous on $[0,2]$ and $f(0) = f(2)$. Show there exist points x and y in $[0,2]$ such that $|x-y| = 1$ and $f(x) = f(y)$. Hint: Consider the function $g(x) = f(x+1) - f(x)$ *for* $x \in [0,1]$

35. These functions are each undefined at two points. Are these singularities removable?

$$f(x) = \frac{x^3+4x-5}{x^3-2x^2+x}$$
$$g(x) = \frac{x^3-3x^2+2}{x^2-1}$$

36. If

$$F(x) = f(x) - g(x) + h(x)$$
$$G(x) = g(x) + 2h(x)$$
$$H(x) = 2g(x) - h(x)$$

are all three continuous on the whole real line, then is it the case that f, g, h are also continuous on the real line?

37. If f and g are continuous on $\mathbb{R}$ and $f(\frac{m}{n}) = g(\frac{m}{n})$ for all non-zero integers m, n, then is it true that $f(x) = g(x)\ \forall x \in \mathbb{R}$?

38. If f and g are continuous on $[0,1]$ and $f(x) > g(x)$ for $0 \leq x \leq 1$, then does there exist a $p > 0$ such that $f(x) \geq g(x) + p$ for $0 \leq x \leq 1$?

39. Suppose f is continuous on $[0,1]$ and that $f(x) = 0$ for all $x \in \mathbb{Q} \cap [0,1]$. Then show that $f(x) = 0$ for all $x \in [0,1]$.

40. Let $S = \{x : x^2 \leq 4\}$. Show that if c is an accumulation point for S, then c belongs to S.

41. Let $T = \{x : \sin x > 0\}$. Show that every point of T is an interior point. What are the boundary points of T?

42. Show that $g(x) = f^2(x)$ is continuous at every point where $f(x)$ is continuous.

43. Prove that $f(x) = |x|$ is continuous at all values of x. Does $x = 0$ require special attention?

44. Let $f(x) = \frac{\sin x}{\sqrt{x}}$. Can $f(0)$be defined in such a way that f is continuous for all x?