Chapter 2 Limits

In the next few chapters we shall investigate several concepts from calculus, all of which are based on the notion of a limit. In the normal sequence of mathematics courses that students encounter, courses like algebra, geometry and trigonometry, calculus is the first course in which the notion of limit occurs. The notion of a limit is central to the development of calculus and there are two types of limit we will discuss. The first is the notion of a sequence limit and we will then use the ideas developed from sequence limits to consider a second concept, that of function limits.

Sequences provide a convenient setting in which to introduce the notion of limit, and so, we begin this chapter developing the properties of sequences. In particular, we define what it means for a sequence to converge to a limit and we present a number of theorems which can be used to tell whether a given sequence does or does not converge. Finally, we introduce the important Cauchy criterion for convergence which allows a given sequence to be classified as convergent or not convergent without reference to the limit of the sequence.

A knowledge of sequence limits aids in the subsequent development of function limits and the second part of the chapter is devoted to seeing how sequence limits lead naturally to the definition of a function limit. Many of the properties of function limits can be related to corresponding properties of sequence limits and the question of whether a given function limit does or does not exist can be often handled using sequence limits.

In the next chapter we will use both sequence limits and function limits in defining the concept of continuity for a function of a single variable. In later chapters, the notion of a limit is fundamental is considering derivatives and integrals.

Functions and Sequences

A ***sequence*** is often defined as an ordered set of real numbers that are in one to one correspondence with the natural numbers. The correspondence is indicated by labelling the terms in the sequence in order, i.e., $a_1, a_2, \ldots$. An alternative definition of sequences can be given using the notion of a function.

A function, $f,$ can be defined as a rule which assigns to each element x in a subset $dom[f]$ of the real numbers, a corresponding element $y,$ in a subset $rng[f]$ of the real numbers. We refer to the set $dom[f]$ as the domain of the function f and the set $rng[f]$ as the range of f. We often use the notation $y = f(x)$ to denote the function $f.$ The function is said to be single valued or well defined as long as there are not two distinct y values in $rng[f]$ that correspond to the same x in $dom[f].$ This can be interpreted graphically as the assertion that if the function is "graphed" by plotting the domain on the horizontal axis, the range on the vertical axis, and then sketching a curve through the points $(x, f(x))$ for all x in the domain, a vertical line through each x in the domain cuts the graph is just one point.

In the special case that a function, $a,$ has $dom[a] = N,$ the function is called a ***sequence*** and we denote sequences by writing a_n (or $\{a_n\}$ if we want to refer to the whole sequence rather than just one term) instead of $a(n)$.

Example *Sequences*

a) The sequence $2, 4, 6, 8, \ldots$ can be described by writing $a_n = 2n,\ n = 1, 2, \ldots$ or, it can be defined recursively as $a_{n+1} = a_n + 2$.

b) The sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$ is defined by $a_n = 1 - 2^{-n},\ n = 1, 2, \ldots$

c) The sequence $1, -1, 1, -1, 1, \ldots$ can be defined by $a_n = (-1)^{n+1}$ or, recursively by

$a_{n+1} = -a_n.$
d) The definition $a_n = n^{-3}$ generates the sequence $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \ldots$ Note that in each of these examples, the domain of the sequence is N so n begins at the value 1.

Bounded, **monotone sequences**

- A sequence a_n is said to be ***bounded*** if there is a positive real number, $M,$ such that $|a_n| \leq M,\ \forall n \in N.$
- A sequence a_n is said to be ***increasing*** if $a_n \leq a_{n+1}$ for all $n \in N$
- A sequence a_n is said to be ***decreasing*** if $a_n \geq a_{n+1}$ for all $n \in N$
- A sequence a_n is said to be ***monotone*** if it is either increasing or decreasing

The sequence b), c), and d) in the example above are bounded sequences. The sequence in a) tends to $+\infty$ and, of course is not bounded. The sequence in b) is increasing while the sequence in d) is decreasing. Both of these are monotone sequences. The sequence in c) is neither increasing nor decreasing.

Definition *Convergence, Divergence*

- A sequence a_n is said to be ***convergent to limit*** L if, for every $\varepsilon > 0$ there exists $\nu \in N$ such that $|a_n - L| < \varepsilon$ for all $n > \nu;$ we can express this in words by saying that a_n lies in an $\varepsilon -$ neighborhood of L for every n sufficiently large.
- If a sequence is not convergent it is said to be ***divergent***.

Example *Convergence*

a) The sequence $\left\{\frac{1}{n}\right\}$ is convergent with limit $L = 0.$ To see this, let $\varepsilon > 0$ be given and let ν denote an integer that is larger than $\frac{1}{\varepsilon}.$ Then

$$\left|\frac{1}{n} - 0\right| < \varepsilon \qquad \text{for every } n > \nu.$$

b) Let $r < 1$ be given. Then the sequence $\{r^n\}$ is convergent with limit $L = 0.$ In this case, it is clear that for any $\varepsilon,\ 0 < \varepsilon < 1,$

$$|r^n - 0| < \varepsilon$$

$$\text{if} \qquad n\, Log\, r < Log\, \varepsilon \ \left(i.e.,\ n > \left|\frac{Log\, \varepsilon}{Log\, r}\right|\right)$$

Then $\quad |r^n - 0| < \varepsilon \quad if \quad n > \nu = \left|\frac{Log\, \varepsilon}{Log\, r}\right|.$

c) The sequence $\{a_n = (-1)^n\}$ is divergent. To see this note that $a_n = 1$ when n is even and $a_n = -1$ when n is odd. Clearly the only possible values for L are $+1$ and -1. But for $\varepsilon < \frac{1}{2},$ there is no integer ν for which it is true that $|(-1)^n - 1| < \varepsilon$ for all $n > \nu,$ nor is it true that $|(-1)^n + 1| < \varepsilon$ for all $n > \nu.$ Then a_n does not remain in any $\varepsilon -$ neighborhood of either $+1$ or -1 for all large n.

d) A sequence a_n is said to tend to $+\infty$ if, for every positive $B,$ there exists $\nu \in N$ such that $a_n > B$ for every $n > \nu;$ e.g., if $r > 1,\ a_n = r^n$ tends to $+\infty.$ A sequence a_n is said to tend to $-\infty$ if the sequence $-a_n$ tends to $+\infty.$

Subsequences and Limit Points

Let $f(x)$ be a well defined function having domain $dom[f]$ and range $rng[f]$. Any function $g(x)$ such that $dom[g]$ is a subset of $dom[f]$ and $g(x) = f(x)$ for all x in $dom[g]$, is said to be a restriction of f to the domain $dom[g] \subset dom[f]$. Now suppose $\{a_n\}$ is a sequence (a function defined on N) and let N' denote a subset of N. A sequence $\{b_n\}$ defined for $n \in N'$, is said to be a ***subsequence*** of $\{a_n\}$ if $b_n = a_n$ for $n \in N'$; i.e., $\{b_n\}$ is just a restriction of $\{a_n\}$ to the domain $N' \subset N$. In loose terms, we might say $\{b_n\}$ is a subsequence of $\{a_n\}$ if $\{b_n\}$ is obtained from $\{a_n\}$ by deleting some of the terms from $\{a_n\}$ and keeping the remaining terms in the original order. The notion of a subsequence is useful in discussing convergence for sequences.

Theorem *2.6 If the sequence $\{a_n\}$ converges to limit L, then every subsequence of $\{a_n\}$ must also converge to L.*

Like theorem 2.1, this theorem is most useful in proving that a sequence is not convergent. In particular, if $\{a_n\}$ contains subsequences converging to different limits then $\{a_n\}$ is not convergent. In connection with any discussion of subsequences we can define the following notion.

Definition *A number P is said to be a* ***limit point*** *for the sequence $\{a_n\}$ if $\{a_n\}$ contains a subsequence that converges to P.*

Note that if $\{a_n\}$ is convergent to L, then L is necessarily a limit point for $\{a_n\}$. On the other hand, a sequence $\{a_n\}$ may have more than one limit point but in this case the sequence cannot be convergent.

Theorem *2.7 If the sequence $\{a_n\}$ has more than one limit point, then $\{a_n\}$ diverges.*

Theorem 2.7 says essentially the same thing as theorem 2.6 but in terms of limit points instead of subsequences.

Theorem *2.8 (Bolzano-Weierstrass theorem) Every bounded sequence contains a convergent subsequence.*

An equivalent way of stating the B-W theorem is the assertion: Every bounded sequence must have a limit point. Of course both statements are equivalent to the original statement of the B-W theorem: Every bounded infinite set has an accumulation point.

Example *Subsequences and Limit Points*

a) The only limit point of the sequence $\left\{\frac{1}{n}\right\}$ is zero and this sequence converges to $L = 0$, (as do all its subsequences)

b) The sequence $\{(-1)^n\}$ is clearly bounded (choose $M = 2$) hence by theorem 2.8 it must contain a convergent subsequence. In fact, there are two convergent subsequences, $\left\{(-1)^{2n}\right\}$ and $\left\{(-1)^{2n+1}\right\}$, with the subsequence, $\left\{(-1)^{2n}\right\}$, converging to $L_1 = 1$ and the subsequence $\left\{(-1)^{2n+1}\right\}$, converging to $L_2 = -1$. Then theorem 2.6 implies that the sequence is not convergent. Alternatively, by definition, L_1 and L_2 are limit points of the original sequence, and since there are two limit points, the sequence $\{(-1)^n\}$ must diverge by theorem 2.7.

c) The sequence $a_n = n[1-(-1)^n]$ produces the terms $\{2, 0, 6, 0, 8, 0, 10, 0, \ldots\}$ and hence it is clear that $\{a_{2n} = 0\}$ is a subsequence. Obviously the subsequence converges to 0 and it is not hard to see that this is the only limit point for the original sequence $\{a_n\}$. However this sequence is not convergent since it is also clear that it is not bounded. Alternatively, $\{a_{2n-1}\}$ is also a subsequence of $\{a_n\}$ but it does not converge to the limit of the subsequence $\{a_{2n}\}$ so $\{a_n\}$ is divergent by theorem 2.6.

Properties of Sequences

Using the definition to decide if a series is convergent or divergent is usually not the most efficient way to proceed. Instead it is often easier to observe certain properties of the sequence that are sufficient to imply convergence or divergence. We will list now several important facts about sequences that can be used for this purpose.

Theorem *2.1 If a sequence $\{a_n\}$ is convergent, then it is bounded*

This theorem is most frequently used in its contrapositive form; i.e., if a sequence is not bounded then it is divergent.

Theorem *2.2 A monotone sequence is convergent if and only if it is bounded.*

Note that for a sequence that is not monotone, boundedness does not imply convergence. It is **only** for monotone sequences that boundedness implies convergence.

Theorem *2.3 (Uniqueness of limits) The limit L of a convergent sequence is unique (i.e., a convergent sequence cannot have distinct limits L and M)*

The previous three theorems only tell if a sequence is convergent or divergent. The next few theorems can be used to find the limit to which a sequence converges.

Theorem *2.4 (Squeeze play for sequences) Let $\{a_n\}$ be a convergent sequence with limit L. Suppose $\{b_n\}$ is a sequence satisfying either $L \leq b_n \leq a_n$ or, $a_n \leq b_n \leq L$ for all $n \in N$. Then $\{b_n\}$ is a convergent sequence with limit L.*

Theorem *2.5 (Arithmetic with sequences) Let $\{a_n\}$ and $\{b_n\}$be convergent sequences with limits L. and K respectively. Then*

(a) For real numbers α, β, $\{\alpha a_n + \beta b_n\}$ is a convergent sequence with limit $\alpha L + \beta K$

(b) $\{a_n b_n\}$ is a convergent sequence with limit LK

(c) $\left\{\frac{a_n}{b_n}\right\}$ is a convergent sequence with limit $\frac{L}{K}$, provided $K \neq 0$ and

$b_n \neq 0 \ \forall n.$

The Cauchy Criterion

It will be convenient to have a characterization of convergence for sequences that does not require us to know the value of the limit L in order to determine the convergence; i.e.,this definition does not refer explicitly to the limit of the sequence. We shall say that a sequence $\{a_n\}$ is a ***Cauchy sequence*** if, for every $\varepsilon > 0,$ there exists $\nu \in N,$ such that $|a_m - a_n| < \varepsilon$ for all $m, n > \nu.$ Then we have

Theorem *2.9 (Cauchy criterion) A sequence is convergent if and only if it is a Cauchy sequence.*

Example *Cauchy sequences*

a) We already know that $\left\{\frac{1}{n}\right\}$ is a convergent sequence so we should be able to show that it is a Cauchy sequence just to illustrate how the argument is constructed. Consider

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right|$$

Without loss of generality, we may suppose $m > n.$ In that case

$$|a_n - a_m| = \frac{m-n}{mn} < \frac{m}{mn} = \frac{1}{n}$$

and it follows that $\forall \varepsilon > 0$

$$|a_n - a_m| < \varepsilon \quad if \quad m > n > \frac{1}{\varepsilon}$$

which proves that $\{a_n = \frac{1}{n}\}$ is a Cauchy sequence.

b) Consider the sequence $S_n = \sum_{m=1}^{n} \frac{1}{m};$ i.e., $S_1 = 1,\ S_2 = 1 + \frac{1}{2},$ etc. We can show that this sequence is divergent by showing that it is not a Cauchy sequence. In fact, for any $n > 1,$

$$\begin{aligned} S_{2n} - S_n &= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \\ &> \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = n\left(\frac{1}{2n}\right) = \frac{1}{2} \end{aligned}$$

This shows that for any $n,$ no matter how large, the difference $S_{2n} - S_n$ is never less that $1/2$ and so it is not the case that for m and n sufficiently large, $S_m - S_n$ can be made arbitrarily small. $\{S_n\}$ is not Cauchy and therefore is not convergent.

Solved Problems: Sequence Limits

Sequence Limits

Problem 2.1 (*A constant sequence*) Show that the sequence $\{1, 1, 1, \ldots\}$ converges and find the limit.

Solution:This sequence has $a_n = 1$ for every n and is therefore called a constant sequence. Since $|a_n - 1| = 0$ for every $n,$ it is clear that for every $\varepsilon > 0$ and $\nu = 1$ we have $|a_n - 1| < \varepsilon$ for every $n > \nu$. Then by the definition of convergence, this sequence converges to $L = 1$.

Problem 2.2 (*An eventually constant sequence*) Show that the sequence $\{1, 2, 3, 4, 4, 4, 4, \ldots\}$ converges and find the limit.

Solution:This sequence has $a_n = 4$ for every $n > 4$. This is an example of what we call an eventually constant sequence. Since $|a_n - 4| = 0$ for every $n > 4$, it is clear that for every $\varepsilon > 0$ and $\nu = 4$ we have $|a_n - 1| < \varepsilon$ for every $n > \nu$. Then by the definition of convergence, this sequence converges to $L = 4$.

Problem 2.3 Find the limit of the sequence $\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$.

Solution: After some thought it can be seen that this is the sequence of terms, $a_n = 1 - \left(\frac{1}{2}\right)^n$, $n = 1, 2, \ldots$ and by computing several terms for large values of n, it seems that a_n is converging to $L = 1$. To prove that a_n converges to $L = 1$, we write

$$|a_n - 1| = \left|1 - \left(\tfrac{1}{2}\right)^n - 1\right| = \left|\left(\tfrac{1}{2}\right)^n\right| = 2^{-n}.$$

Now we can prove by induction that, $2^{-n} < \dfrac{1}{n}$, for all $n \in N$. Then it follows that for every $\varepsilon > 0$, we have $|a_n - 1| < \varepsilon$ for all $n > \dfrac{1}{\varepsilon}$

Problem 2.4 (*Tending to infinity*) If $r > 1$ then show that $a_n = r^n$ tends to $+\infty$.

Solution: We have to show that for any $B > 0$, there exists $\nu \in N$ such that $r^n > B$ for all $n > \nu$. If we choose $\nu = \dfrac{Log B}{Log r}$ then $n > \nu$ implies $n Log r > Log B$ which leads to $r^n > B$.

Problem 2.5 (*A divergent sequence*) Prove that the sequence $\{0, 2, 0, 4, 0, 8, 0, 16, 0. \ldots\}$ is divergent.

Solution: Since the terms of this sequence satisfy $a_{2n} = 2^n$, it is evident that the sequence is not bounded. Then by theorem 2.1 the sequence is not convergent.

Problem 2.6 Find the limit of the sequence

$$b_n = \frac{1}{\sqrt{n^2 + 1}}.$$

Solution: For each $n \in N$, it is clear that $0 < n^2 < n^2 + 1$. Then

$$0 < \frac{1}{\sqrt{n^2 + 1}} < \frac{1}{n}.$$

Since $a_n = \dfrac{1}{n}$ is a convergent sequence with limit equal to zero, it follows from the squeeze play theorem that $\{b_n\}$ is convergent to $L = 0$.

Problem 2.7 Find the limit of the sequence $c_n = \dfrac{n}{3n + 1}$.

Solution: For each $n \in N$,

$$\frac{n}{3n + 1} = \frac{1}{3 + \frac{1}{n}} = \frac{a_n}{b_n},$$

and we can show that $\{a_n\}$ and $\{b_n\}$ are both convergent with limits $L_a = 1$ and $L_b = 3$, respectively. Since b_n is different from 0 for all n, it follows from the arithmetic for sequences theorem, part (c), that $\{c_n\}$ converges to $\frac{1}{3}$.

Problem 2.8 Find the limit of the sequence $c_n = \dfrac{3n^2+5n+4}{n^2+3n+2}$.

Solution: For each $n \in N$,

$$c_n = \frac{3n^2+5n+4}{n^2+3n+2} = \frac{3+\frac{5}{n}+\frac{4}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}} = \frac{a_n}{b_n}.$$

It is not hard to see that a_n converges to 3 and b_n converges to 1 and using the same argument used in the previous problem, then we obtain the result that c_n converges to 3. Alternatively, we could write

$$\begin{aligned} c_n &= \frac{3n^2+5n+4}{n^2+3n+2} \\ &= \frac{3n^2+3n+2n+4}{(n+1)(n+2)} \\ &= \frac{3n(n+1)}{(n+1)(n+2)} + \frac{2n+4}{(n+1)(n+2)} \end{aligned}$$

and

$$\begin{aligned} c_n &= \frac{3n}{n+2} + \frac{2}{n+1} \\ &= \frac{3}{1+\frac{2}{n}} + \frac{2}{n+1} \\ &= 3p_n + 2q_n. \end{aligned}$$

Now p_n converges to 3 and q_n converges to 0 so it follows from the arithmetic for sequences theorem, part (a), that $\{c_n\}$ converges to 3.

Properties of Sequences

Problem 2.9 Prove theorem 2.1, that convergence implies boundedness. Prove that for monotone sequences, boundedness implies convergence.

Solution: Suppose $\{a_n\}$ converges to limit $L.$ Then for $\varepsilon = 1$ there is $\nu \in N$ such that $|a_n - L| < 1$ for $n > \nu.$ This implies that $|a_n| < L+1$ for $n > \nu.$ Now let M denote the largest of the $\nu+1$ numbers, $|a_1|, |a_2|, |a_3|, \dots, |a_\nu|, L+1.$ Then it is clear that $|a_n| \le M$ for all $n \in N.$ This implies that all convergent sequences must be bounded.

It is only for monotone sequences that boundedness implies convergence. Suppose $\{a_n\}$ is monotone increasing and bounded. In particular, this means

$$a_1 \le a_2 \le \cdots \le B$$

for some $B.$ Then the set of sequence values $S = \{a_1, a_2, \dots\}$ is bounded above and therefore has a least upper bound, say we call it $L.$ Then for any $\varepsilon > 0,$ $L-\varepsilon$ is not an upper bound for $S,$ which means there is some $a_M \in S$ such that $L-\varepsilon < a_M \le L.$ But $\{a_n\}$ is monotone increasing so $L-\varepsilon < a_n \le L$ for every $n \ge M$ which is to say,

$$\text{for every } \varepsilon > 0,\ \exists M \in \mathbb{N} \text{ such that } |a_n - L| \le \varepsilon \text{ for every } n \ge M.$$

The proof in the case that the sequence is monotone decreasing is completely similar.

Problem 2.10 Prove theorem 2.3, on the uniqueness of limits.

Solution: Suppose $\{a_n\}$ converges to limit L and also converges to limit $M.$ Choose $\varepsilon > 0$ sufficiently small that $N_\varepsilon(L) = \{x : |x-L| < \varepsilon\}$ and $N_\varepsilon(M)$ are disjoint. Then the convergence assumptions imply there exists $\nu \in N$ such that $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$ for $n > \nu$. But this is the statement that for $n > \nu$, a_n belongs to the disjoint sets $N_\varepsilon(L)$ and $N_\varepsilon(M)$. Since this is clearly not possible, the assumption that there are two limits must be false.

Problem 2.11 Prove theorem 2.4, the squeeze play theorem for sequences.

Solution: Suppose $\{a_n\}$ converges to limit L and that $\{b_n\}$ is such that $L \le b_n \le a_n$ for all n. Now for every $\varepsilon > 0$ there exists $\nu \in N$ such that $|a_n - L| < \varepsilon$ for all $n > \nu$. But $L \le b_n \le a_n$ implies that $|b_n - L| \le |a_n - L|$ and it follows that $|b_n - L| < \varepsilon$ for all $n > \nu$; i.e., $\{b_n\}$ converges to L.

Problem 2.12 Prove the sequence $a_n = 2^{1/n}$ converges to the limit, 1.

Solution: For each $n \in N$ we have

$$2^{1/n} > 1 \qquad (1)$$

Then for each $n \in N$ there exists a unique number $c_n > 0$ such that

$$2^{1/n} = 1 + c_n \qquad (2)$$

Then $2 = (1 + c_n)^n$ and by the result known as Bernoulli's inequality (proved in problem 1.6) we have then

$$2 = (1 + c_n)^n \ge 1 + nc_n \quad \text{for all } n \in N \qquad (3)$$

It is evident from (3) that $c_n \le \frac{1}{n}$, and this, together with (1) and (2) leads to

$$1 \le 2^{1/n} = 1 + c_n \le 1 + \frac{1}{n} \qquad \text{for all } n \in N.$$

Since $1 + \frac{1}{n}$ converges to 1, we can apply the squeeze play theorem to conclude that $2^{1/n}$ must also converge to 1. Note that introducing the auxiliary sequence $\{c_n\}$ made it easier to apply the squeeze play theorem.

Problem 2.13 Prove the sequence $a_n = n^{1/n}$ converges to the limit, 1.

Solution: For each $n \in N$,

$$n^{1/n} \ge 1 \qquad (1)$$

Then there exist real numbers c_n with $c_1 = 0$ and $c_n > 0$ for $n > 1$ such that for every $n \in N$

$$n^{1/n} = 1 + c_n \qquad (2)$$

Then the binomial theorem implies

$$n = (1 + c_n)^n = 1 + nc_n + \frac{n(n-1)}{2} c_n^2 + \cdots$$

hence

$$n \ge 1 + \tfrac{1}{2} n(n-1) c_n^2 \qquad \text{for all } n \in N \qquad (3)$$

Now it follows that $n - 1 \ge \frac{1}{2} n(n-1) c_n^2$ and

$$c_n^2 \le \frac{2}{n} \qquad \text{for } n > 1 \qquad (4)$$

Now (4) together with (1) and (2) imply the result.

Problem 2.14 Show that the sequence $e_n = (1 + \frac{1}{n})^n$ converges to a limit, $e,$ whose value is between 2 and 3.

Solution:We can compute the first few terms of this sequence $e_1 = 2,$
$e_2 = 2.25, e_3 = 2.37,\ e_4 = 2.441\ldots$ which suggests that this is an increasing sequence. If we can prove that the sequence is increasing and prove also that it is bounded above by 3, then the result will follow from corollary 2.2.

The binomial theorem implies

$$e_n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\,\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\,\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots 2 \bullet 1}{n!}\,\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \qquad (1)$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

Evidently e_n consists of a sum of $n+1$ terms. In the same way, we can show that

$$e_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \cdots$$

$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right).$$

Then $$e_{n+1} \geq e_n + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right),$$

and it follows that $$2 \leq e_1 \leq e_2 \leq \cdots \leq e_n \leq e_{n+1} \leq \cdots \qquad (2)$$

i.e., $\{e_n\}$ is an increasing sequence. To show the sequence is bounded, note first that

$$\left(1 - \frac{k}{n}\right) \leq 1 \text{ for } k = 1, 2, \ldots, n \qquad (3)$$

and, by mathematical induction,

$$k! \geq 2^{k-1} \qquad \text{for } k = 1, 2, \ldots \qquad (4)$$

Using (3) and (4) in (1) leads to

$$2 \leq e_n \leq 1 + 1 + 2^{-1} + 2^{-2} + \cdots + 2^{-n+1}.$$

But $$1 + 2^{-1} + 2^{-2} + \cdots + 2^{-n+1} = \frac{1 - 2^{-n}}{1 - \frac{1}{2}} = 2 - 2^{-n+1}$$

and

$$2 \leq e_n \leq 3 - 2^{-n+1} \leq 3, \text{ for every } n \in N.$$

Then $\{e_n\}$ is an increasing sequence, bounded above by 3 so it follows from corollary 2.2 that the sequence converges to a limit we will denote by $e.$ Since $2 \leq e_n \leq 3$ for every $n \in N,$ it follows that $2 \leq e \leq 3.$

Cauchy Sequences

Problem 2.15 Prove that every convergent sequence is a Cauchy sequence.

Solution: Suppose $\{a_n\}$ is a convergent sequence with limit L. Then for every $\varepsilon > 0$ there exists $\nu \in N$ such that $|a_n - L| < \frac{\varepsilon}{2}$ for all $n > \nu$. Then

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ for all } m,n > \nu.$$

But this is the assertion that $\{a_n\}$ is a Cauchy sequence.

Problem 2.16 Prove that every Cauchy sequence is bounded.

Solution: Suppose $\{a_n\}$ is a Cauchy sequence. Then there is $\nu \in N$ such that $|a_n - a_m| < 1$ for all $m,n > \nu$. This leads to the result,

$$|a_n| < |a_\nu| + 1 \qquad \text{for all } n > \nu.$$

Let M denote the largest of the numbers $|a_1|, |a_2|, \ldots, |a_\nu|, |a_\nu| + 1$. Then $|a_n| < M$ for all $n \in N$, which is to say, the sequence is bounded.

Problem 2.17 Prove that every Cauchy sequence is convergent

Solution: Suppose $\{a_n\}$ is a Cauchy sequence. By the previous result, the sequence is bounded and by the Bolzano-Weierstrass theorem then the sequence contains a convergent subsequence, $\{a_{n'}\}$, $n' \in N'$, for N' a subset of N. Let the limit of the subsequence be denoted by L and we will now show that a_n must also converge to L.

Since $\{a_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there is $\nu \in N$ such that

$$|a_n - a_m| < \frac{\varepsilon}{2} \text{ for all } m,n > \nu. \qquad (1)$$

In addition, since the subsequence is convergent to L, there exists $\mu \in N'$ such that $\mu > \nu$ and

$$|a_m - L| < \frac{\varepsilon}{2} \text{ for all } m \in N' \textit{ with } m > \mu. \qquad (2)$$

Finally, since $\mu > \nu$, it follows from (1) that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $m,n > \mu$. Then for $n > \mu$,

$$|a_n - L| = |a_n - a_m + a_m - L| \leq |a_n - a_m| + |a_m - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{for } m > \mu.$$

But this implies that $\{a_n\}$ is convergent to the limit L.

Problem 2.18 A sequence $\{a_n\}$ satisfying

$$|a_{n+2} - a_{n+1}| \leq C|a_{n+1} - a_n| \qquad \text{for all } n \in N$$

for some constant C, $0 < C < 1$, is said to be a ***contraction***. Prove that every contraction is a Cauchy sequence.

Solution: Suppose $\{a_n\}$ is a contraction. Then

$$|a_{n+2} - a_{n+1}| \leq C|a_{n+1} - a_n| \leq C^2|a_n - a_{n-1}|$$
$$\leq C^3|a_{n-1} - a_{n-2}| \leq \cdots \leq C^n|a_2 - a_1|.$$

For integers m,n, $m > n$, the triangle inequality implies

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \qquad (2)$$

Then, combining (1) and (2), leads to

$$|a_m - a_n| \leq [C^{m-2} + C^{m-3} + \cdots + C^{n-1}]|a_2 - a_1| = \left[C^{n-1} \sum_{k=0}^{m-n-1} C^k \right] |a_2 - a_1| \qquad (3)$$

For $0 < C < 1$, $\displaystyle\sum_{k=0}^{N} C^k = \frac{1 - C^{N+1}}{1 - C}$,

and combining this with (3) gives the result that for all integers $m, n,\ m > n$,

$$|a_m - a_n| \leq \left[C^{n-1} \frac{1 - C^{N+1}}{1 - C} \right] |a_2 - a_1| \leq \frac{C^{n-1}}{1 - C} |a_2 - a_1| \qquad (4)$$

Since $0 < C < 1,\ C^{n-1}$ tends to zero as n tends to infinity. Then for any $\varepsilon > 0$, there exists an integer v such that

$$\frac{C^{n-1}}{1 - C} |a_2 - a_1| < \varepsilon \qquad \text{for } n > v$$

and this, in turn, implies $|a_m - a_n| \leq \varepsilon$ for $m > n > v$. Thus every contraction is a Cauchy sequence. Since a Cauchy sequence is convergent, we can let m tend to infinity in (4) to get

$$|L - a_n| \leq \frac{C^{n-1}}{1 - C} |a_2 - a_1| \quad \text{for all } n \in N,$$

where we have denoted the limit of the sequence by L.

More Sequence Properties

Problem 2.19 Let A denote a set of real numbers. Then prove the following statements are equivalent:

(a) A is closed and bounded

(b) Every sequence in A contains a subsequence that converges to a point in A

Solution: Proof that (a) implies (b): We suppose A is closed and bounded. Then any sequence in A is bounded and, by the Bolzano-Weierstrass theorem, the sequence contains a convergent subsequence. The limit point of this subsequence is an accumulation point for A, and since A is closed, this limit point belongs to A.

Proof that (b) implies (a): The assertion that (b) implies (a) is equivalent to the assertion that ”not (a)” implies ”not (b)” and this is the statement we are going to prove. The assertion ”not (a)”, holds if A is not closed, not bounded, or both.

Suppose that A is not closed. Then there exists a point $p \notin A$ such that p is an accumulation point for A. If p is an accumulation point of A, then there must exist a sequence, $\{a_n\}$, of points of A that converges to p. Then by theorem 2.6, every subsequence of $\{a_n\}$ must also converge to p. But then theorem 2.7 implies that none of the subsequences of $\{a_n\}$ converges to a point of A. This proves that (b) is false when A is not closed.

Next, suppose A is not bounded; in particular, suppose A has no upper bound. Then there exists a sequence of points p_n in A such that $p_1 \in A$ and $p_{n+1} > p_n + 1$. This definition ensures that for all $m, n \in N$ with $m > n$, we have $p_m - p_n > 1$. Then no subsequence of the sequence $\{p_n\}$ is a Cauchy sequence. This proves that (b) is false if A is not bounded and completes the proof that (b) must be false if (a) is false. Then the statements (a) and (b) are

equivalent.
A set of real numbers that is closed and bounded is sometimes referred to as a compact set. Then (b) is an alternative way of defining compactness.

Problem 2.20 Prove the following assertions:

(a) Let $a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ for $n \in N$
and show that a_n converges to a limit $L,\ 1 < L < 2.$

(b) For positive real number, $B,$ define the following sequence recursively,

$$a_1 = B, \qquad a_{n+1} = \frac{a_n^2 + B}{2a_n}, \text{ for } n = 1, 2, \ldots$$

Then a_n converges to $\sqrt{B}$;

Note that this is an algorithm for computing the square root of a number recursively.

Solution: (a) It is clear from the definition that $a_{n+1} \geq a_n,$ so this is an increasing sequence. In addition,

$$\begin{aligned} a_n &= 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} \\ &\leq 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^{n-1}}; \end{aligned}$$

i.e.,

$$a_n \leq \frac{1 - 2^{-n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} < 2 \text{ for all } n \in N.$$

Since $\{a_n\}$ is increasing and bounded above, it is convergent to some limit $L < 2.$ In fact, the sequence converges to $e - 1.$

(b) Let $b = \sqrt{B}$ and note that

$$(a_n - b)^2 = a_n^2 + b^2 - 2b\,a_n \geq 0$$

which leads to

$$a_{n+1} = \frac{a_n^2 + b^2}{2a_n} = \frac{a_n^2 + B}{2a_n} \geq b \quad \text{for all } n;$$

i.e., the sequence $\{a_n\}$ is bounded below. Note further that, this result implies that $a_n^2 + b^2 \leq 2a_n^2,$ which in turn leads to

$$a_{n+1} = \frac{a_n^2 + b^2}{2a_n} \leq a_n;$$

i.e., the sequence is decreasing. Since the sequence is decreasing and bounded below, it is convergent to a limit we will denote by $\beta.$ Since we know the sequence converges, theorem 2.5 permits us to conclude that

$$a_{n+1} - \frac{a_n^2 + b^2}{2a_n} = 0 \text{ for each n,}$$

and, letting n tend to infinity, $\beta - \dfrac{\beta^2 + b^2}{2\beta} = 0.$

Now it is evident that $\beta = b = \sqrt{B}$.

Exercises Sequences

1. Consider the sequence of prime numbers, $1, 2, 3, 5, 7, 11, \ldots$ Is this really a sequence? How do you define a_n?

2. What is the next term in the sequence $3, 1, 5, 1, 7, \ldots$ Give a definition for a_n.

3. Find an N such that $|a_n - L| \leq 10^{-3}$ for $n > N$

a. $a_n = \dfrac{2}{\sqrt{n+1}}$

b. $a_n = 1 - \dfrac{1}{n^3}$

c. $a_n = 2 + 2^{-n}$

d. $a_n = \dfrac{n+1}{2n+3}$

4. Prove convergence/divergence for $a_n = \dfrac{2n^2 + 5n - 6}{n^3}$

5. Prove convergence/divergence for $a_n = \dfrac{3n+5}{6n+11}$.

6. Prove convergence/divergence for $a_n = \dfrac{n\sqrt{n+2} + 1}{n^2 + 4}$

7. Prove convergence/divergence for $a_n = \sqrt{n+1} - \sqrt{n}$

8. Prove convergence/divergence for $a_n = \sqrt{n}\,(\sqrt{n+1} - \sqrt{n})$

9. Suppose a_n assumes only integer values. Under what conditions does this sequence converge?

10. Show that the sequences a_n and $b_n = a_{n+10^6}$ either both converge or both diverge.

11. Let $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$. List the first few terms of this sequence. Prove that the sequence converges to $\left(1 + \sqrt{5}\right)/2$.

12. A subsequence $\{a_{n_k}\}$ is obtained from a sequence $\{a_n\}$ by deleting some of the terms a_n, and retaining the others in their original order. Explain why this implies that $n_k \geq k$ for every k.

13. Which statements are true? Explain your answer.

a. If $\{a_n\}$ is unbounded then either $\lim_n a_n = \infty$ or else $\lim_n a_n = -\infty$

b. If $\{a_n\}$ is unbounded then $\lim_n |a_n| = \infty$

c. If $\{a_n\}$ and $\{b_n\}$ are both bounded then so is $\{a_n + b_n\}$

d. If $\{a_n\}$ and $\{b_n\}$ are both unbounded then so is $\{a_n + b_n\}$

e. If $\{a_n\}$ and $\{b_n\}$ are both bounded then so is $\{a_n b_n\}$

f. If $\{a_n\}$ and $\{b_n\}$ are both unbounded then so is $\{a_n b_n\}$

14. Which statements are true? Explain your answer.

a. If $\{a_n\}$ and $\{b_n\}$ are both divergent then so is $\{a_n + b_n\}$

b. If $\{a_n\}$and $\{b_n\}$are both divergent then so is $\{a_nb_n\}$
c. If $\{a_n\}$and $\{a_n + b_n\}$are both convergent then so is $\{b_n\}$
d. If $\{a_n\}$and $\{a_nb_n\}$are both convergent then so is $\{b_n\}$
e. If $\{a_n\}$ is convergent then so is $\{a_n^2\}$
f. If $\{a_n\}$ is convergent then so is $\{1/a_n\}$
g. If $\{a_n^2\}$ is convergent then so is $\{a_n\}$

15. Either give an example of a sequence with the following property or else state a theorem that shows why no such example is possible.

a. a sequence that is monotone increasing but is not bounded

b. a sequence that converges to 6 but contains infinitely many terms that are not equal to 6 as well as infinitely many terms that are equal to 6

c. an increasing sequence that is bounded but is not convergent

d. a sequence that converges to 6 but no term of the sequence actually equals 6.

e. a sequence that converges to 6 but contains a subsequence converging to 0.

f. a convergent sequence with all negative terms whose limit is not negative

g. an unbounded increasing sequence containing a convergent subsequence

h. a convergent sequence whose terms are all irrational but whose limit is rational.

16. How are the notions of accumulation point of a set and limit point of a sequence related? How does this relate to the two formulations of the Bolzano-Weierstrass theorem?

17. Prove: If the Cauchy sequence $\{a_n\}$ contains a subsequence $\{a_{n_k}\}$ which converges to limit $L,$ then the original sequence must also converge to $L.$

18. Show that $1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$ for $a \neq 1$ and any positive integer $n.$
Find $\lim\limits_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1.$What is the limit if $|a| \geq 1$?

19. Let $\{s_n\}$ be such that $|s_{n+1} - s_n| \leq 2^{-n}$ for all $n \in N.$ Prove that this is a Cauchy sequence. Is this result true under the condition $|s_{n+1} - s_n| \leq \frac{1}{n}$?

20. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1.$Find the first few terms of this sequence. Use induction to show that $s_n > \frac{1}{2}$ for all $n.$Show that this sequence is nonincreasing. Prove that the sequence converges and find its limit.

21. Let $s_1 = 1$ and $s_{n+1} = \left(1 - \dfrac{1}{4n^2}\right)s_n$ for $n \geq 1.$Determine if the sequence converges and, if it does, find the limit.

22. For each of the following sequences state a theorem which establishes the convergence/divergence:

a. $a_n = n^{1/3}$

b. $a_n = \dfrac{n^2+3}{n+2}$

c. $a_n = (2 + 10^{-n})(1 + (-1)^n)$

d. $a_n = \dfrac{1}{n^2+3n+2}$

e. $a_n = 1 + 2^{-n}$

f. $a_n = \sqrt{n+1}$

g. $a_n = \sum_{k=1}^{n} \dfrac{1}{k}$ (hint: show that $a_{2n} - a_n$ does not tend to 0 as $n \to \infty$)

h. $\{a_n\} = \left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \dots\right\}$

23. Let
$a_1 = 0.1,\ a_2 = 0.101,\ a_3 = 0.101001,\ a_4 = 0.1010010001,\ a_5 = 0.101001000100001, \dots$
Show that this is a sequence of rational numbers that converges to a limit L. Is the limit L rational?

24. Which statements are true?:

a. a sequence is convergent if and only if all its subsequences are convergent.

b. a sequence is bounded if and only if all its subsequences are bounded.

c. a sequence is monotone if and only if all its subsequences are monotone.

d. a sequence is divergent if and only if all its subsequences are divergent.

25. The sequence $\{a_n\}$ has the property, $\forall \varepsilon > 0,\ \exists N_\varepsilon$ such that $|a_{n+1} - a_n| < \varepsilon$ when $n > N_\varepsilon$. Is the sequence necessarily a Cauchy sequence?

26. Prove that a monotone decreasing sequence is convergent if and only if it is bounded.

27. Prove that $a_n = 2^{1/n}$ is convergent.

28. Prove convergence/divergence for

a. $a_n = \dfrac{2n^2 + 5n - 6}{n^3}$

b. $a_n = \dfrac{n\sqrt{n+1} + 1}{n^2 + 4}$

29. Let $\{a_n\}$ denote a bounded but divergent sequence. Prove, by stating theorems from these notes, that the sequence must contain at least two subsequences which converge to different limits.

30. For each of the following sequences, use theorems rather than the definition of convergence to establish their convergence/divergence:

a. $a_n = \log n$

b. $a_n = \dfrac{n^2+3}{n^3+2}$

c. $a_n = \dfrac{n}{n+1}[1 + (-1^n)]$

d. $a_n = \left\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots\right\}$

31. Show that if, $a_n = \dfrac{n+1}{n}$ then $|a_n - a_m| \leq 10^{-3}$ when $m > n > 10^3$. Is $\{a_n\}$ a Cauchy sequence?

32. Tell which of the following statements is true and explain your answer:

- **a.** a sequence $\{a_n\}$ is convergent if and only if every subsequence of $\{a_n\}$ converges
- **b.** a sequence $\{a_n\}$ is bounded if and only if every subsequence of $\{a_n\}$ is bounded
- **c.** a sequence $\{a_n\}$ is monotone if and only if every subsequence of $\{a_n\}$ is monotone
- **d.** a sequence $\{a_n\}$ is divergent if and only if every subsequence of $\{a_n\}$ is divergent

33 Suppose $\{a_n\}$ is a sequence of positive numbers. What can you conclude about $\{a_n\}$ if:

- **a.** $\dfrac{a_{n+1}}{a_n} < 1$ for all n
- **b.** $\dfrac{a_{n+1}}{a_n} > 1$ for all n
- **c.** $\lim_{n\to\infty} \dfrac{a_n}{n} = \infty$
- **d.** $\lim_{n\to\infty} \dfrac{1 + \frac{1}{n}}{a_n} = \infty$

Function Limits

In the previous sections we considered limits of sequences. Now we introduce the related notion of a function limit. Let $f(x)$ denote a real valued function with domain D in R and let c denote an accumulation point for D.

Definition *The limit of* $f(x)$ *as* x *approaches* c *exists and equals* L *if, for every* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that* $|f(x) - L| < \varepsilon$ *whenever* $0 < |x - c| < \delta$. *We indicate this by writing* $\lim_{x\to c} f(x) = L$

Note that c is an accumulation point for D but need not belong to D. Therefore, we allow x to approach c but it is not necessary that x ever equal c for the limit to exist.

Example *Function Limits*

(a) Consider the function $f(x) = \sqrt{x+1}$ on the domain $D = (0, 10)$. Then $c = 0$ is an accumulation point for D but does not belong to D. We wish to show that $\lim_{x\to 0} f(x)$ exists and equals 1. To do this, we have to show that $|f(x) - L| = \left|\sqrt{x+1} - 1\right| < \varepsilon$ whenever $|x - 0| < \delta(\varepsilon)$ and, we have to determine the dependence of δ on ε. We write first,

$$\left(\sqrt{x+1} - 1\right)\left(\sqrt{x+1} + 1\right) = (x+1) - 1 = x$$

which suggests

$$\left|\sqrt{x+1} - 1\right| = \left|(\sqrt{x+1} - 1)\tfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right| = \frac{|x-0|}{\left|\sqrt{x+1} + 1\right|}.$$

Moreover,

$$\frac{1}{\sqrt{11}+1} \le \frac{1}{\left|\sqrt{x+1}+1\right|} \le \frac{1}{1+1} = \frac{1}{2} \quad \text{for all } x \in (0,10).$$

Then

$$\left|\sqrt{x+1}-1\right| \le \frac{1}{2}|x-0| < \varepsilon \quad \text{whenever} \quad |x-0| < \delta(\varepsilon) = 2\varepsilon.$$

This proves that the limit, $\lim_{x\to 0}\sqrt{x+1}$, exists and equals 1. Note that $c = 0$ does not belong to D. Note also that the value of $L = \frac{1}{2}$ is what you obtain when the value zero is substituted into the function $f(x)$. Aside from the fact that $c = 0$ is not in the domain of $f(x)$, this is not a valid way of evaluating the limit of $f(x)$ in general. It is valid if it is known that c is in the domain and that $f(x)$ is continuous at c. We will explain this further when discussing continuity in the next chapter.

(b) Consider the function

$$f(x) = \frac{81-x^2}{x-9} \quad \text{on } D = \{1 > |x-9| > 0\}.$$

i.e, $D = (8,9) \cup (9,10)$ and the point $c = 9$ is not in D. In order to evaluate

$$\lim_{x\to 9} f(x) = \lim_{x\to 0}\frac{81-x^2}{x-9},$$

we write

$$\frac{81-x^2}{x-9} = \frac{(9-x)(9+x)}{x-9} = -(9+x).$$

This suggest that $L = -18$ and that

$$\left|\frac{81-x^2}{x-9} - (-18)\right| = |9+x-18| = |x-9|.$$

This last string of equalities implies that

$$\left|\frac{81-x^2}{x-9} - (-18)\right| = |f(x)-L| < \varepsilon$$

whenever

$$|x-9| = |x-c| < \delta(\varepsilon) = \varepsilon.$$

As in the previous example, it is not acceptable to evaluate this limit by simply substituting $x = 9$ into the simplified formula for $f(x)$, even though this does produce the correct result. We will see shortly that if c belongs to the domain of f, and if f is continuous at $x = c$, then it is the case that $\lim_{x\to c} f(x)$ exists and equals $f(c)$. In example (b), we do not know at this point if f is continuous at $x = 9$ nor do we know at the beginning of the example that the limit exists. Therefore to jump to the conclusion that the limit is equal to $f(9)$ is incorrect.

We can also define the notion of the limit of $f(x)$ as x tends to infinity.

Definition *The limit of $f(x)$ as x tends to infinity exists and equals L if, for every $\varepsilon > 0$ there exists $B > 0$ such that $|f(x)-L| < \varepsilon$ whenever $x > B$. We write then, $\lim_{x\to\infty} f(x) = L$. The definition of the limit of $f(x)$ as x tends to negative infinity is defined similarly.*

Function limits and Sequence limits

The notion of function limit is closely related to the notion of the limit of a sequence. The sequence $\{a_n\}$ converges to the limit L if a_n gets close to L as n gets large (i.e. n gets close to ∞), while $\lim_{x \to c} f(x) = L$ if $f(x)$ gets close to L as x gets close to c. The following theorem relates the two notions.

Theorem *2.10 Suppose $f(x)$ is a real valued function with domain D in R and let c denote an accumulation point for D. Then the following assertions are equivalent:*

(a) $\lim_{x \to c} f(x) = L$.

(b) if $\{a_n\}$ is a sequence in D that converges to c, then $\{f(a_n)\}$ is a sequence that converges to L

This theorem is often most useful for proving that certain function limits fail to exist. To do so requires us to find a sequence in D converging to c while the sequence of function values fails to converge to L.

Example *Function limits and sequence limits*

(a) Consider the function

$$f(x) = \begin{cases} -1 & if \quad -1 \leq x < 0 \\ +1 & if \quad 0 \leq x \leq 1 \end{cases}$$

with $D = [-1, 1]$. We can use theorem 2.10 to show that $f(x)$ tends to no limit as x tends to 0 for this function. For this purpose, note that the sequence

$$a_n = (-1^n)\frac{1}{n}, \qquad n \in N$$

tends to $c = 0$ as n tends to infinity. However, the sequence of function values

$$f(a_n) = \begin{cases} -1 & if \quad n \text{ is odd} \\ +1 & if \quad n \text{ is even} \end{cases}$$

Since this sequence has two limit points, it is divergent. Then we have found a sequence a_n such that assertion b of the theorem is false and since the two assertions are equivalent, it follows that assertion a must also then fail.

(b) Consider the function $f(x) = \frac{1}{x}$ with $D = (0, \infty)$. Then $c = 0$ is an accumulation point of D and we can use theorem 2.10 to show that $f(x)$ tends to no limit as x tends to $c = 0$. In this case, the sequence $a_n = \frac{1}{n}$ is a sequence in D that tends to 0 as n tends to infinity but the sequence of function values $f(a_n) = n$ is an unbounded, and hence divergent, sequence. Then it follows from the theorem that no limit exists as x tends to 0.

Properties of Function Limits

We have results for function limits that are analogous to theorem 2.4 for sequence limits.

Theorem *2.11 (Arithmetic with function limits) Suppose $f(x)$ and $g(x)$ are two real valued functions on domain D. Suppose also that c is an accumulation point for D and that*

$$\lim_{x \to c} f(x) = L, \qquad \lim_{x \to c} g(x) = K$$

$$\textit{Then} \quad \textit{(i)} \lim_{x \to c} [a f(x) + b g(x)] = aL + bK \quad \textit{for all } a, b \in R$$

$$\textit{(ii).} \lim_{x \to c} f(x) g(x) = LK$$

$$\textit{(iii)} \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K} \quad \textit{if} \quad K \neq 0.$$

We also have a result for function limits that is analogous to theorem 2.5, the squeeze play theorem for sequence limits.

Theorem *2.12 Suppose $f(x)$ and $g(x)$ are two real valued functions on domain D. Suppose also that c is an accumulation point for D and that*

$$\lim_{x \to c} f(x) = L = \lim_{x \to c} g(x).$$

Then if $h(x)$ is also defined on D with $f(x) \leq h(x) \leq g(x)$ for all x in D, it follows that $\lim_{x \to c} h(x) = L.$

Solved Problems: Function Limits

Problem 2.21 Show that $\lim_{x \to 1} \frac{x+1}{x^2+2} = \frac{2}{3}$

Solution: We have to show that for a given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$\left| \frac{x+1}{x^2+2} - \frac{2}{3} \right| < \varepsilon \quad \text{whenever} \quad |x-1| < \delta;$$

i.e., $f(x) \in N_\varepsilon\left[\frac{2}{3}\right]$ whenever $x \in N_\delta[1]$. To see how δ should be chosen, we write

$$\left| \frac{x+1}{x^2+2} - \frac{2}{3} \right| = \left| \frac{3(x+1) - 2(x^2+1)}{3(x^2+2)} \right| = \left| \frac{(2x-1)(x-1)}{3(x^2+2)} \right| \qquad (1)$$

For $|x-1| < 1$, (i.e., $0 < x < 2$) it is easy to see that $|2x-1| \leq 3$ and $|3(x^2+2)| \geq 6$ so it follows from (1) that

$$\left| \frac{(2x-1)(x-1)}{3(x^2+2)} \right| \leq \frac{3}{6} |x-1| = \frac{1}{2} |x-1|$$

hence

$$\left| \frac{x+1}{x^2+2} - \frac{2}{3} \right| \leq \frac{1}{2} |x-1| < \varepsilon \qquad \text{if} \qquad |x-1| < 2\varepsilon = \delta.$$

Of course we already imposed the condition that $|x-1| < 1$ so in order to have **both** $\delta \leq 1$ and $\delta \leq 2\varepsilon$, we have to choose δ to equal the smaller of the two numbers, 1 and 2ε. Note that we were able to estimate the quotient by finding an upper bound for the numerator and a lower bound for the denominator for x in D.

Problem 2.22 Show that

$$(a)\lim_{x\to\infty}\frac{1}{1+x^2} = 0 \qquad (b)\lim_{x\to\infty}\frac{x^2}{1+x^2} = 1$$

Solution: We will show first that $\frac{1}{x}$ tends to zero as x tends to infinity. For any fixed $\varepsilon > 0$ choose $B > \frac{1}{\varepsilon}$. Then it is easy to see that

$$\left|\frac{1}{x}\right| < \varepsilon \quad \text{whenever } x > B = \frac{1}{\varepsilon}.$$

Now to show (a), note that

$$\frac{1}{1+x^2} < \frac{1}{x^2} < \frac{1}{x} \quad \text{for } x > B.$$

Then theorem 2.12 implies that $\frac{1}{x^2}$ and $\frac{1}{1+x^2}$ both must tend to zero as x tends to infinity. This includes the result (a). To show that (b) holds, write

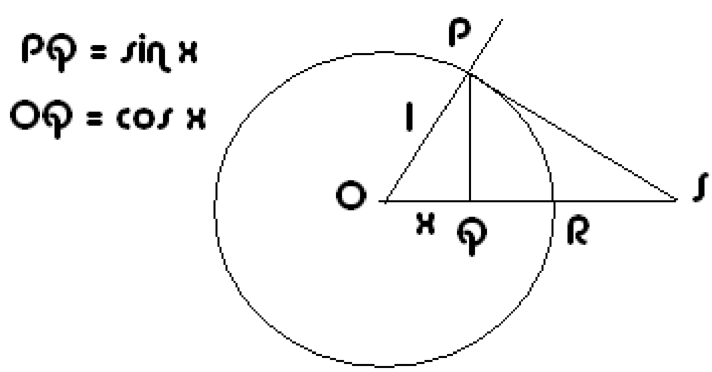
$$\frac{x^2}{1+x^2} = \frac{1}{1+\frac{1}{x^2}}$$

As a result of what was shown in part (a), we see that the denominator of this expression tends to 1 as x tends to infinity. Then by part (iii) of theorem 2.11 we have the result (b).

Problem 2.23 Show that

$$(a)\lim_{x\to 0} Sin(x) = 0 \qquad (b)\lim_{x\to 0} Cos(x) = 1$$

Solution: In the following figure we see a circle of radius 1 and center O with angle POS denoted by x. If PQ is perpendicular to the radius OR, then it follows that the length of PQ equals $Sin(x)$ while the lengths of OQ and QR equal $Cos(x)$ and $1 - Cos(x)$, respectively.



For any positive x, the length of the straight line chord, PR is clearly less than the length of the circular arc PR, and since the radius of the circle equals 1, the arc PR just equals the angle x in radian measure. If we denote the length of the chord by $|PR|$ then $|PR| < x$. Then, using the Pythagorean theorem,

$$|PQ|^2 + |QR|^2 = |PR|^2 < x^2;$$

i.e.,

$$Sin(x)^2 + (1 - Cos(x))^2 < x^2.$$

This last inequality implies that for $x \geq 0$,

$$0 \leq Sin(x) \leq x \qquad and \qquad 0 \leq 1 - Cos(x) \leq x$$

Then (a) and (b) follow by theorem 2.12.

Problem 2.24 Show that

$$\lim_{x\to 0}\frac{Sin(x)}{x} = 1.$$

Solution: Since $f(x) = \dfrac{Sin(x)}{x}$ is even $\left(i.e.,\ f(-x) = f(x)\right)$, it will be sufficient to consider only positive values of x. Referring to the figure once again, note that PQ is perpendicular to the radius OR and PS is perpendicular to the radius OP. Then it is evident from the figure that the areas of the triangles OPQ, and OPS are related to the area of the circular sector, OPR as follows:

Area of $\Delta OPQ \leq$ area of sector OPR $\leq$Area of ΔOPS.

We can express these area explicitly in terms of x, and write this last string of inequalities as

$$\frac{1}{2}Sin(x)Cos(x) \leq \frac{1}{2}x \leq \frac{1}{2}Tan(x)$$

Now, dividing through by $\frac{1}{2}Sin(x)$, (which is valid so long as $x > 0$), we obtain

$$Cos(x) \leq \frac{x}{Sin(x)} \leq \frac{1}{Cos(x)} \qquad or \qquad Cos(x) \leq \frac{Sin(x)}{x} \leq \frac{1}{Cos(x)}.$$

Now, using the result of the previous problem, together with theorem 2.12, the result follows.

Problem 2.25 Evaluate the limits,

(a) $\lim_{x\to 0}\dfrac{1 - Cos(x)}{x}$ (b) $\lim_{x\to 0}\dfrac{1 - Cos(x)}{x^2}$

Solution: Write

$$\frac{1 - Cos(x)}{x} = \frac{1 - Cos(x)}{x}\frac{1 + Cos(x)}{1 + Cos(x)} = \frac{Sin(x)^2}{x(1 + Cos(x))}$$

$$= \left(\frac{Sin(x)}{x}\right)^2\frac{x}{1 + Cos(x)}$$

and

$$\frac{1 - Cos(x)}{x^2} = \left(\frac{Sin(x)}{x}\right)^2\frac{1}{1 + Cos(x)}.$$

Now we can use the results of the previous two problems together with theorem 2.11 in order to obtain

$$\lim_{x\to 0}\frac{1 - Cos(x)}{x} = 0 \qquad and \qquad \lim_{x\to 0}\frac{1 - Cos(x)}{x^2} = \frac{1}{2}.$$

Exercises

1. Use both the definition of limit and a sequence approach to establish

$\lim_{x \to 2} \frac{1}{1-x} = -1$

2. Use both the definition of limit and a sequence approach to establish $\lim_{x \to 0} \frac{x^2}{|x|} = 0$

3. Use both the definition of limit and a sequence approach to establish $\lim_{x \to 1} \frac{x}{1+x} = 1/2$

4. Show that the limit: $\lim_{x \to 0} \frac{x}{|x|}$ does not exist

5. Show that the limit: $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist

6. Show that the limit: $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist

7. Show that the limit: $\lim_{x \to 0} \frac{1}{\sqrt{x}}$ does not exist

8. Prove that if $a_n \geq 0 \ \forall n$ and $a_n \to A,$ then $\sqrt{a_n} \to \sqrt{A}$

9. Find $\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$ or show the limit does not exist

10. Find $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$ or show the limit does not exist

11. Find $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x + 2x^2}$ or show the limit does not exist

12. Find $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ or show the limit does not exist

13. Find $\lim_{x \to 0} \sqrt{x} \ \sin\left(\frac{1}{x}\right)$ or show the limit does not exist

14. Find $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$ or show the limit does not exist

15. Find $\lim_{x \to 0} x \cos\left(\frac{1}{x^2}\right)$ or show the limit does not exist

16. Given that $x - \frac{1}{6}x^3 \leq \sin x \leq x \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)$

17. Given that $x - \frac{1}{6}x^3 \leq \sin x \leq x \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\sin x}{\sqrt{x}}\right)$

18. Given that $1 - \frac{1}{2}x^2 \leq \cos x \leq 1 \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\cos x - 1}{x}\right)$

19. Given that $1 - \frac{1}{2}x^2 \leq \cos x \leq 1 \ \ for \ \ x \geq 0,$ find $\lim_{x \to 0} \left(\frac{\cos x - 1}{\sqrt{x}}\right)$