Chapter 1

The Real Numbers

In a beginning course in calculus, the emphasis is on introducing the techniques of the subject;i.e., differentiation and integration and their applications. An advanced calculus course builds on this foundation, extending the techniques and re-examining the fundamentals in a more rigorous light. A rigorous treatment of calculus requires more than just a heuristic understanding of the real numbers. On the other hand, a systematic construction of the real numbers is perhaps more than is required at this level. Therefore, in this chapter, we shall present a set of axioms which are sufficient to imply all properties of the real numbers relevant to our purposes. These axioms describe a set of abstract objects, called numbers, whose properties include the properties of the real numbers of our experience. However, these axioms allow us to deduce a much more detailed description of the real numbers than is possible using only our computational experience. In particular, the notion of a limit is fundamental to the development of calculus and limits require that the underlying structure be complete. The axiomatic construction of the reals presented here ensures that the real numbers are complete.

For computational purposes we are used to representing each real number as a decimal expansion while for purposes of visualization, we often find it convenient to think of the real numbers as being in one to one correspondence with the points of a line. We may also have heard of some special subsets of the real numbers, including the natural numbers, the integers as well as the rational and irrational numbers.

Several properties of the real numbers will be developed from the axioms of this chapter and each of these results will play an important role in subsequent chapters in providing a rigorous treatment of the calculus of functions of one real variable. In addition we will introduce various concepts regarding sets of real numbers, so-called topological properties of sets. Finally, we will discuss some strategies that can be employed in constructing proofs of mathematical statements.

Terminology

We will speak frequently in this chapter about sets of real numbers. A ***set of real numbers*** is just a collection of real numbers and the numbers in the set are referred to as the ***elements*** of the set. A set of numbers may be defined by simply listing the numbers in the set. Alternatively, the set may be defined by specifying a membership rule against which any number can be tested in order to determine whether or not it belongs to the set. If M is a set, all of whose elements are contained in another set, $Q,$ then we say that M is a ***subset*** of Q and we use the notation $M \subset Q$ to indicate this. A set which contains no elements is said to be ***empty***. For example, if the set is defined by a self contradictory membership rule, then the set will be empty. We use the symbol, $\emptyset,$ to denote the empty set.

The Real Numbers

The real numbers contain several important subsets of real numbers:

- ***Natural Numbers***- sometimes called the ”counting numbers”, these are the numbers $1, 2, 3, \ldots$ The set of natural numbers will be denoted by $\mathbb{N}$
- ***Integers***- the integers consist of the natural numbers together with the numbers

$0, -1, -2, \ldots$ The set of integers will be denoted by Z. Note that $\mathbb{N}$ is a subset of Z

- ***Rational numbers***- any number that can be expressed as the quotient of two integers is a rational number. This set is denoted by Q.
- ***Irrational numbers***- every real number that is not rational is said to be irrational

The real numbers are composed of the set of all rational numbers together with the irrational numbers. We denote the set of all real numbers by R. An even larger set of numbers, the complex numbers, will not be discussed here.

Representing Real Numbers

For computational purposes it is often convenient to represent real numbers by their (unique) decimal expansion; e.g., $\frac{5}{4} = 1.250$, $\frac{1}{3} = .333\ldots$, $\sqrt{2} = 1.414\ldots$, $\pi = 3.141592\ldots$ We will show that if the decimal representation terminates (all entries after some point are zeroes, like $1.25000\ldots$) or if the representation repeats some block of digits (like $0.123123123\ldots$) or if it eventually repeats a block (like $23.68753123123123\ldots$) then the number that is represented is rational. Then numbers which can be proved to be irrational, like $\sqrt{2}$, must have decimal representations in which there are no repeating blocks.

For purposes of visualization, it is often helpful to think of the ***real line*** in which we consider points on the line as corresponding to real numbers. In particular, if we label one point on the line as corresponding to zero, then the points to the left may be taken to correspond to the negative reals while the positive reals then correspond to points to the right of the zero. Of course this visual interpretation of the real numbers is not sufficiently precise for our purposes. For example, there is no way to know whether there might be some very small subintervals in this line which contain no rational numbers. Our axiomatic development will show that there can be no such intervals.

The connection between the decimal representation for a real number x and the image of x as a point on the real line can be made as follows:

- Let x denote a real number. Then there exists an integer N such that $N \leq x < N+1$.
- Divide the interval $[N, N+1]$ into 10 equal subintervals,
  $I_k = \{x : N + \frac{k}{10} \leq x < N + \frac{k+1}{10}\}$ $k = 0, 1, \ldots 9$.
- Determine p, $0 \leq p \leq 9$ such that $x \in I_p$. Set $d_1 = p$
- Divide I_p into 10 equal subintervals
  $I_k = \{x : N + \frac{d_1}{10} + \frac{k}{10^2} \leq x < N + \frac{d_1}{10} + \frac{k+1}{10^2}\}$ $k = 0, 1, \ldots 9$.
- Again determine p, $0 \leq p \leq 9$ such that $x \in I_p$. Set $d_2 = p$.
- Divide I_p into 10 equal subintervals
  $I_k = \{x : N + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{k}{10^3} \leq x < N + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{k+1}{10^3}\}$ $k = 0, 1, \ldots 9$.
- Continue in this way, restricting x to a subinterval of length 10^{-n} for $n = 1, 2, \ldots$ where d_n indicates in which of the nine intervals of length 10^{-n} the number x is to be found. This leads to a decimal expansion $\delta_n = N.d_1 d_2 d_3 \ldots d_n$ where $|x - \delta_n| \leq 10^{-(n+1)}$

The proof that $\delta_\infty = x$ depends on the completeness axiom which will be given below.

A similar construction can be used to produce the binary or ternary representation for $x \in R$. It is sufficient to consider the interval $[0, 1]$.

- Let $x \in [0, 1] = I$.

- Divide I into two equal subintervals, $I_k = \left\{\frac{k}{2} \leq x < \frac{k+1}{2}\right\}, k = 0, 1.$
- Let $b_1 = 0$ if $x \in I_0$ and let $b_1 = 1$ if $x \in I_1$.
- Divide the interval $\left\{\frac{b_1}{2} \leq x \leq \frac{b_1+1}{2}\right\}$ into equal subintervals $I_k = \left\{\frac{b_1}{2} + \frac{k}{2^2} \leq x < \frac{b_1}{2} + \frac{k+1}{2^2}\right\}, k = 0, 1.$
- Let $b_2 = 0$ if $x \in I_0$ and let $b_2 = 1$ if $x \in I_1$.
- Continuing in this way leads to an expansion $\beta_n = 0.b_1b_2b_3\ldots.b_n$ where $|x - \beta_n| \leq 2^{-(n+1)}$ and $b_k = 0$ or 1 for every k. This is the binary expansion for x.

The ternary expansion proceeds in a similar fashion:

- Let $x \in [0,1] = I$.
- Divide I into three equal subintervals, $I_k = \left\{\frac{k}{2} \leq x < \frac{k+1}{2}\right\}, k = 0, 1, 2.$
- Let $t_1 = 0, 1, 2$ if $x \in I_0$,I_1 or I_2
- Divide the interval $\left\{\frac{t_1}{3} \leq x \leq \frac{t_1+1}{3}\right\}$ into equal subintervals $I_k = \left\{\frac{t_1}{3} + \frac{k}{3^2} \leq x < \frac{t_1}{3} + \frac{k+1}{3^2}\right\}, k = 0, 1, 2.$
- Let $t_2 = k$ if $x \in I_k$.
- Continuing in this way leads to an expansion $\tau_n = 0.t_1t_2t_3\ldots.b_n$ where $|x - \tau_n| \leq 3^{-(n+1)}$ and $t_k = 0, 1$ or 2 for every k. This is the ternary expansion for x.

The ternary expansion is useful in discussing the so called Cantor set.

Axiomatic Definition of the Real Numbers

While this rather primitive description of the reals is sufficient for a discussion of calculus on an elementary level, a more precise knowledge is needed for a more rigorous treatment of the subject. Therefore we will present a set of axioms from which it will be possible to deduce all the essential properties of the reals. These axioms are grouped into three classes: the field axioms, the order axioms and the completeness axiom.

Field Axioms

A field is a set of object, $x, y, z, \ldots$ called real numbers together with two binary operations, addition, $x + y$, and multiplication $x \bullet y$ which satisfy the following set of axioms:

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. $\exists\ 0 \in R$ such that $x + 0 = x\ \forall x \in R$
4. $\forall x \in R\ \exists\ w \in R$ such that $x + w = 0$
5. $x \bullet y = y \bullet x$
6. $(x \bullet y) \bullet z = x \bullet (y \bullet z)$
7. $\exists\ 1 \in R,\ 1 \neq 0,$ such that $x \bullet 1 = x\ \forall x \in R$
8. $\forall x \in R,\ x \neq 0,\ \exists\ w \in R$ such that $x \bullet w = 1$
9. $x \bullet (y + z) = x \bullet y + x \bullet z$

Although the real numbers form a field, they are not the only example of a field.

The element w in axiom 4 is usually denoted by $-x$. Then ***subtraction***, $x - y$ is defined to mean $x + (-y)$. The element w in axiom 8 is usually denoted by x^{-1} and then ***division*** by x, $\frac{y}{x}$, may be defined as $y \cdot x^{-1}$.

Note that if $z = y \cdot x^{-1}$, then $z \cdot x = y \cdot x^{-1} \cdot x = y$; i.e., z is the unique number satisfying $z \cdot x = y$. In the case of ***division by zero***, we would have $z = \frac{y}{0}$ which requires z to be the unique number such that, $z \cdot 0 = y$. If $y = 0$ then this condition is satisfied for every z and if $y \neq 0$, then there is no z for which this condition is satisfied. As a result, we are forced to consider division by zero as an undefined operation. No consistent meaning can be given to the expression $\frac{y}{0}$ and hence division by zero must be ruled out of our computational system.

Order Axioms

The subset P of R is called the positive reals and it satisfies:

10 if $x, y \in P$ then $x + y \in P$
11 if $x, y \in P$ then $x \cdot y \in P$
12 if $x \in P$ then $-x \notin P$
13 if $x \in R$ then exactly one of the following must hold: $x = 0$, $x \in P$ or $-x \in P$.

Any field which satisfies axioms 10 through 13 is said to be an ordered field. The real numbers are an ordered field but they are not the only example of an ordered field.

Mathematical Induction

The natural numbers are a special ordered subset of the reals. In fact,$\mathbb{N}$ is the smallest subset of R with the following properties:

Induction Properties: A set S is said to be an ***inductive set*** if:

$a)$ $1 \in S$ and
$b)$ $x + 1 \in S$ whenever $x \in S$.

Since $\mathbb{N}$ is the smallest subset of R which is an inductive set, it follows that any subset of R that is an inductive set must contain $\mathbb{N}$. This is the basis of the principle of mathematical induction.

Example Mathematical Induction

We will use induction to prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \qquad (1.1)$$

We note first that (1.1) holds when $n = 1$, so 1 belongs to the set S of numbers, n, for which (1.1) is true. Next, suppose $m \in S$, which is to say

$$1 + 2 + 3 + \ldots + m = \frac{m(m+1)}{2}.$$

Adding $m + 1$ to both sides of this last equation, leads to

$$1 + 2 + 3 + \ldots + m + m + 1 = \frac{m(m+1)}{2} + m + 1 = \frac{(m+1)(m+2)}{2}.$$

But this is just (1.1) with n replaced by $m+1$, and it follows that $m+1 \in S$ whenever $m \in S$. Then S contains the natural numbers which means that (1.1) holds for all natural numbers.

Inequalities

In any ordered field, and in the real numbers in particular, we can define $x < y$ to mean $y - x \in P$, and $x > y$ to mean $x - y \in P$. In addition, then $x \leq y$ means $x < y$ or $x = y$ and $x \geq y$ means $x > y$ or $x = y$. The order axioms imply the following properties for the inequality symbol, $<$.

Theorem ***1.1***- *For* $a, b, c \in R$

i) if $a \neq 0$, then $a^2 = a \cdot a > 0$
ii) if $a < b$ and $b < c$ then $a < c$
iii) if $a < b$ then for any c, $a + c < b + c$
iv) if $a < b$ and $0 < c$ then $a \cdot c < b \cdot c$

Absolute Value

Definition *For* $x \in R$, *we define* $|x|$ *to mean* $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Then we can prove

Theorem ***1.2***- *For* $a, b \in R$

i) $|a \cdot b| = |a| \cdot |b|$
ii) $|a + b| \leq |a| + |b|$ (triangle inequality)

Sets of Real Numbers

We will be concerned in what follows with various sets of real numbers, that is, sets whose elements are real numbers. A set consisting of only finitely many points is called a finite set and, similarly, a set if infinite if it contains an infinite number of points. An important example of a set in $\mathbb{R}$ is the intervals.

Intervals

Using the notation of absolute value and inequalities, we can describe certain subsets of R and visualize them as subsets of the real line. For $a, b \in R$ with $a < b$ we define:

$[a,b] = \{x \in R : a \leq x \leq b\}$
$(a,b) = \{x \in R : a < x < b\}$
$[a,b) = \{x \in R : a \leq x < b\}$, and $(a,b] = \{x \in R : a < x \leq b\}$

Bounded and Unbounded Sets

A set S in R is bounded above if there exists a number M such that $x \leq M$ for every x in S. Then M is called an **upper bound** for S and we may indicate this by writing $M = UB(S)$. Similarly, a set S in R is bounded below if there exists a number m such that

$x \geq m$ for every x in S. Then m is called a **lower bound** for S (i.e., $m = LB(S)$). A set which is bounded above and bounded below as well is a **bounded set**. More succinctly, a set S is bounded if there exists a number B such that $|x| \leq B$ for every x in $S;$ i.e., S is contained in the interval $(-B,B)$ with $B < \infty$.

The intervals $[a,b],\ (a,b),\ [a,b)$ and $(a,b]$ are all bounded sets. Each the following intervals is an example of an unbounded set:

$$[a,\infty) = \{x \in R : x \geq a\} \qquad (a,\infty) = \{x \in R : x > a\}$$
$$(-\infty,a] = \{x \in R : x \leq a\} \qquad (-\infty,a) = \{x \in R : x < a\}$$
and $(-\infty,\infty) = R$.

For each of these intervals it is clear that there exists no B such that $|x| \leq B$ for all x in the interval. Here the symbols $-\infty$ and ∞ are not intended to represent elements of R. They are simply symbols with the following order properties: $-\infty < x$ and $x < \infty$ for all real numbers x.

Least Upper Bound or Sup of S

A number β is said to be the ***least upper bound*** ($Lub(S)$) or ***supremum***, ($\sup(S)$) for S if and only if:

$$\beta \text{ is an upper bound for } S$$
$$\text{and} \quad \beta \leq b \quad \forall\ b = UB(S)$$

An equivalent way of defining the $Lub(S)$ is to say $\beta = \sup(S)$ if and only if:

$$\beta \text{ is an upper bound for } S$$
$$\forall \varepsilon > 0 \quad \beta - \varepsilon \neq UB(S)$$

Greatest Lower Bound or Inf of S

The ***greatest lower bound*** for S ($Glb(S)$) or the ***infimum***,($\inf(S)$) is defined in an analogous way:. $\lambda = \inf(S)$ if and only if

$$\lambda \text{ is a lower bound for } S$$
$$\text{and} \quad \lambda \geq p \quad \forall\ p = LB(S)$$

Equivalently $\lambda = \inf(S)$ if and only if

$$\lambda \text{ is a lower bound for } S$$
$$\forall \varepsilon > 0 \quad \lambda + \varepsilon \neq LB(S)$$

Example *Sup and Inf*

The set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is bounded below by -3 but this is not the greatest lower bound. Clearly 0 is the greatest lower bound for $S;.i.e.\ 0 = \inf S.$ Similarly, $A = \{1 - e^{-n} : n \in \mathbb{N}\}$is bounded above by 10 but 10 is not the least upper bound for this set; The least upper bound is. $\sup A = 1$.

Note that it is not necessarily the case that the sup or inf for a set belongs to the set.

The Completeness Axiom

14 Every nonempty set of real numbers having an upper bound, has a least upper bound.

An ordered field which satisfies the completeness axiom is called a ***complete ordered field***. The real numbers are essentially the only example of a complete ordered field. All the properties of the real numbers may now be deduced from axioms 1 through 14. These include:

Theorem ***1.3***: *Every nonempty set of real numbers having a lower bound, has a greatest lower bound.*

Theorem ***1.4*** *(Archimedean Property) The following statements are equivalent:*

i) If $x \in R,$ then $\exists n \in N$ such that $x < n$

ii) For real numbers a, b with $0 < a < b,$ there exists a positive integer n such that $na > b$

iii) For all positive real numbers, y there exists $n \in N$ such that $0 < \frac{1}{n} < y$

Statement i) of the Archimedean property is the assertion that there is no "largest" real number. The other two statements assert that there are no "infinitesimally small" real numbers. The following corollary follows immediately from the Archimedean property.

Corollary ***1.5***: *Between any two real numbers there is a rational number and an irrational number*

The corollary precludes the possibility that there is a "gap" in the real number line; i.e., there is no interval in $\mathbb{R}$ that is devoid of either rational numbers or irrational numbers.

Properties of Subsets of $\mathbb{R}$

In our development of calculus on $\mathbb{R}$ it will be convenient to have some definitions from set theory and basic topology. All this means is that we are going to define some terms that will make our discussion of limits more efficient later on.

Our discussion of the topological properties of sets of real numbers, will begin with the notion of a *neighborhood*.

- For $a \in \mathbb{R}$ and $\varepsilon > 0,$ the set $N_\varepsilon(a) = \{x \in \mathbb{R} : a - \varepsilon < x < a + \varepsilon\} = (a - \varepsilon, a + \varepsilon)$ is called an $\varepsilon -$ ***neighborhood of*** a
- a ***deleted neighborhood of*** a, denoted by $\mathring{N}_\varepsilon(a)$ consists of $N_\varepsilon(a)$ with the point a removed.

The notion of a neighborhood can be used now to define several additional concepts. Here G denotes an arbitrary non-empty set in $\mathbb{R}$.

- A point p is an ***accumulation point*** for the set G if every $\varepsilon -$ *neighborhood* of p meets G in a point other than $p;$ i.e.,$\forall \varepsilon > 0, \mathring{N}_\varepsilon(p) \cap G \neq \emptyset$
- A point p is an ***isolated point*** for the set G if $\exists \varepsilon > 0$ such that $\mathring{N}_\varepsilon(p) \cap G = \emptyset$

- A point p is an ***interior point*** for the set G if $\exists\varepsilon > 0$ such that $N_\varepsilon(p) \subset G$
- A set $G \subset \mathbb{R}$ is an ***open set*** if every point of G is an interior point
- The ***complement*** of a set G is the set of all x that are not in G. It is denoted by G^C.
- For any set G, $(G^C)^C = G$.
- A set $F \subset \mathbb{R}$ is ***closed*** if F^C is open
- A set $K \subset \mathbb{R}$ is ***compact*** if it is both closed and bounded

Unions and Intersections

If A and B are two sets, then the ***union*** of A and B is denoted by $A \cup B$ and consists of all the points of A together with all the points of B. The ***intersection*** of A and B is denoted by $A \cap B$ and consists of all the points that belong to both A and B. If A and B are both open then $A \cup B$ is open. If A and B are both closed, then $A \cap B$ is closed.

- $S_1 = (a,b)$ All points in S_1 are interior points so S_1 is open. Also, all points in S_1 are accumulation points for S_1. The points a and b are accumulation points for S_1 but do not belong to S_1 and none of the points in S_1 are isolated.
- $\mathbb{R} = (-\infty,\infty)$ All points are interior points so $(-\infty,\infty)$ is open and its complement, the empty set $\emptyset$ is closed. On the other hand, all the points of $\emptyset$ are interior points so $\emptyset$ is open and $(-\infty,\infty)$ is closed; i.e., both these sets are both open and closed. Note that since $\emptyset$ has no points, it is not false to say all its points are interior points.
- $S_2 = [a,b]$ All points in S_1 are accumulation points and its complement is $(-\infty,a) \cup (b,\infty)$ is open, being the union of two open sets. Then $[a,b]$ is closed.
- $\mathbb{N}$ has no interior points, no accumulation points and all its points are isolated.
- $\mathbb{Q}$ all its points are accumulation points but none of its points are interior points and none are isolated. All irrationals are accumulation points but do not belong to $\mathbb{Q}$
- $S_2 = \{\frac{1}{n} : n \in \mathbb{N}\}$ all its points are isolated so it has no interior points. Zero is the only accumulation point but 0 is not in S_2.

We have now several properties of the real numbers that all involve the notion of accumulation points.

Theorem ***1.6*** *A set $F \subset \mathbb{R}$ is closed if and only if F contains all its accumulation points.*

Theorem ***1.7*** *For $S \subset \mathbb{R}$, let $\alpha = \sup S$. Then either α belongs to S or α is an accumulation point for S. Similarly, if $\beta = \inf S$, then either β belongs to S or β is an accumulation point of S.*

Theorem ***1.8*** *(Bolzano-Weierstrass theorem) Every bounded, infinite subset of $\mathbb{R}$ must have at least one accumulation point.*

Theorem ***1.9*** *(Nested interval theorem) Let $I_n, I_{n+1}, \ldots$ denote a family of closed and bounded intervals such that I_{n+1} is contained (together with its endpoints) in I_n for every $n \in N$.*

$$Then \bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Example *Nested Intervals*

It is important to realize that the intervals in theorem 1.9 must be both closed and bounded in order for the conclusions of the theorem to hold.

(a) Consider the nested intervals $I_n = \{x \in R : 0 < x < 1/n\}$. If $x \in I_n$ for every $n,$ then it follows from the definition of I_n that $x > 0.$ On the other hand, if $x < 1/n$ for every $n,$ then x must also satisfy $x \leq 0.$ Since these two traits are mutually exclusive, it follows that there is no x that belongs to every $I_n.$ This does not violate the theorem, however, since the intervals I_n are bounded but not closed.

(b) The nested intervals $I_n = \{x \in R : n \leq x < \infty\}$ have no point that is common to all of them since this number, $x,$ would then have to satisfy $x > n$ for all $n.$ This would contradict theorem 1.4.The nested interval theorem is not violated by this example since the intervals are closed but not bounded.

(c) The intervals I_k having length 10^{-k} that were described in the process of illustrating the connection between the decimal expansion of a real number and its location on the real line were an example of a set of nested intervals. In addition, the lengths of these intervals shrinks to zero with increasing $k.$ In the next chapter we are going to see that the intersection of these intervals is not only not empty, it contains just a single point. In particular, the intersection contains the unique x in R associated with the decimal expansion that generated the sequence of nested intervals.

Proofs

In what follows we will frequently be attempting to prove that a given statement is true or that it is false. There are certain strategies that can be followed in constructing proofs. These include:

Proof by induction

Problem 1.5 illustrates the use of the induction property of the real numbers in order to prove a statement. Essentially such a proof consists of showing the statement in question is true for some integer, $n,$ usually $n = 1,$ and then showing that if the statement is assumed to hold when $n = N,$ then it must hold for $n = N + 1.$

Direct Proof

A direct proof is one that demonstrates the validity of the statement to be proved by a straightforward application of known principles. For example, to show that any number that has a decimal representation that is repeating must be a rational number, we follow the steps taken in problem 1.2 to show that a number with a repeating decimal representation can be converted into the quotient of two integers.

Proof by contradiction

In this type of proof, we assume that the statement we are trying to prove is false and proceed to show that this leads to a contradiction; i.e., a statement that is clearly false. In problem 1.4, we prove $\sqrt{2}$ is irrational by supposing it is a rational number and then showing this leads to a result that is clearly false.

Use of counter-examples

The most effective way to prove a statement is false is to find a counter-example, that is an example for which the statement clearly does not hold. For example, the statement " $2^n - 1$ is prime for every n" is true when $n = 1, 2, 3$ but the case $n = 4$ is a counter-example that shows the statement is false. Similarly the statement that "if $f(x)$ has an absolute minimum at a point, then the derivative of f must be zero there" is proved to be false by the counter-example $f(x) = |x|$.

Proof of the contrapositive

The statement "if A then B" is logically equivalent to "if not B then not A". The second statement is the ***contrapositive*** of the first statement and to say they are logically equivalent means they are both true or they are both false. Therefore, in order to show that a statement is true it is sufficient to show that its contrapositive is true. For example, to prove "if n^2 is even, then n must be even" it is sufficient to show the contrapositive of the statement, i.e., "if n is odd, then n^2 must be odd". The second statement is easier to prove than the first. See problem 1.3.

The Converse

The converse of the statement "if A then B" is the statement, "if B then A". A statement and its converse are not logically equivalent but if both are true, then we say "A if and only if B". For example, we will prove in problem 1.20 that a set $S \subset \mathbb{R}$ is closed if and only if S contains all its accumulation points. To do this we have to show "if S is closed then S contains all its accumulation points" and then show the converse, "if S contains all its accumulation points, then S is closed".

Solved Problems

Representation of the reals

Problem 1.1 Convert each of the following rational numbers to a decimal representation;

1. $\frac{1}{8}, \quad \frac{11}{80}, \quad \frac{111}{800}$
2. $\frac{1}{7}, \quad \frac{1}{27}, \quad \frac{1}{271}$
3. $\frac{1}{6}, \quad \frac{17}{66}, \quad \frac{2493}{9900}$

Solution: By carrying out the division (e.g., using a calculator) we find

1. $\frac{1}{8} =.12500\ldots, \quad \frac{11}{80}.137500\ldots, \quad \frac{111}{800} =.1387500\ldots$
2. $\frac{1}{7} = 0.142857142857\ldots, \quad \frac{1}{27} =.037037\ldots, \quad \frac{1}{271} =.0036900369\ldots$
3. $\frac{1}{6} =.166\ldots, \quad \frac{17}{66} =: .2575757\ldots, \quad \frac{2493}{9900} =: .251818\ldots$

The decimal representations in group 1 are all examples of what are called ”terminating” decimal representations; they consist of all zeroes from some point onwards. The second group are all examples of ”periodic” decimal representations; they consist of the same block of digits, repeated over and over. The length of the block of digits that is repeated is called the ”period” of the representation. The rational numbers in the third group are examples of ”eventually periodic” representations. They become periodic after some point in the expansion. The first two numbers in this group begin to repeat after one digit while the third number begins to repeat after two digits.

We can, in fact, give a direct proof that any number that is the quotient of two integers must have a repeating decimal representation. Suppose that $x = \frac{m}{n}$ where m and n have no factors in common. In dividing n into $m,$ at the first step in the long division there will be a remainder r_1 which must lie between 1 and $n-1;$ i.e., $1 \leq r_1 \leq n-1.$ At each step of the division there will be such a remainder, and since they all lie between 1 and $n-1,$ there can be at most $n-1$ steps in the division process before the remainders begin to repeat. For example, in the example 2 above, when $1/7$ is converted to a decimal, there are 6 remainders before they begin to repeat. In any case, since there can be at most $n-1$ remainders in converting $\frac{m}{n}$ to a decimal, the period in the decimal representation of this rational number is at most equal to $n-1;$ i.e., the representation is necessarily repeating.

Problem 1.2 Convert the following decimal representations to a rational number:

$a)$ $x = .0027100271\ldots$ $\quad y = .12343434\ldots$

$b)$ $x = .5000$ $\quad y = .4999\ldots$

Solution:

a) $x = .0027100271\ldots$ (5 digit repeating pattern)

$$10^5 x = 271.0027100271\ldots$$

$$(10^5 - 1)x = 271 \qquad i.e., \qquad x = \frac{271}{10^5 - 1} = \frac{271}{99999}$$

$y = .12343434\ldots$

$$100y = 12.3434\ldots$$ (2 digit eventually repeating pattern)

$$10^4 y = 1234.3434\ldots$$

$$(10^4 - 10^2)y = 1222 \qquad y = \frac{1222}{10^4 - 10^2} = \frac{1222}{9900}$$

b) $x = .5000$ $\qquad y = .4999\ldots$

$$10x = 5.000 \qquad 10y = 4.999\ldots$$

$$x = \tfrac{1}{2} \qquad 100y = 49.999$$

$$90y = 45 \quad \text{so} \quad y = \tfrac{1}{2}$$

This last example illustrates why we have to disqualify representations terminating in a string of 9’s if we want the association between real numbers and decimal representations to be one to one. Note that we have defined an algorithm whereby any number that has a repeating decimal representation can be converted to a quotient of two integers; i.e., any number that has a repeating decimal representation is rational. The contrapositive of this statement is that any number that is irrational has a decimal representation that is nonrepeating This result, combined with the result from the previous problem establishes that a number is rational if and only if it has a repeating decimal representation. Conversely, a number is irrational if and only if its decimal representation is not repeating .

Problem 1.3 Prove that if n^2 is even, then n must be even.

Solution: We will prove this fact by proving the contrapositive assertion, that if n is odd, then n^2 is odd.

An odd integer has the form $n = 2m+1$ for some integer, m. Then

$$\begin{aligned} n^2 &= (2m+1)^2 = 4m^2+4m+1 \\ &= 2(2m^2+m)+1 = 2N+1 \end{aligned}$$

where $N = 2m^2+2m$ is an integer. Thus n^2 is odd whenever n is odd. The contrapositive of this statement asserts that every integer with an even square must itself be even.

Problem 1.4 Prove that $\sqrt{2}$ is irrational.
Solution: Suppose $\sqrt{2} = \dfrac{p}{q}$ where the fraction has been reduced to lowest terms; i.e., p and q have no factors in common except ± 1. Then this assumption implies that $p^2 = 2q^2$, which means that p^2 is an even integer. Then p must be an even integer by the result in the previous problem, and we can write $p = 2m$ for some integer, m. But this leads to $4m^2 = 2q^2$ or $q^2 = 2m^2$ which implies that q must also be even. Since our original assumption implies that p and q have no factors in common, this is a contradiction to our assumption that $\sqrt{2} = \dfrac{p}{q}$. Then there are no integers p, q such that $\sqrt{2} = \dfrac{p}{q}$.

Mathematical Induction
Problem 1.5 Use mathematical induction to prove that for every positive integer M,

$$\sum_{k=1}^{M} k^2 = \frac{M(M+1)(2M+1)}{6} \qquad (1)$$

Solution: Let the set of integers M for which (1) is valid be denoted by A. Then $1 \in A$ since $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$. Now suppose that A contains all the integers from 1 up to the positive integer m. Then it follows from (1) that

$$\begin{aligned} \sum_{k=1}^{m} k^2 + (m+1)^2 &= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 \\ &= \frac{m(m+1)(2m+1)}{6} + \frac{6(m^2+2m+1)}{6} \\ &= \frac{2m^3+9m^2+13m+6}{6} \end{aligned}$$

Now

$$(m+1)(m+2)(2m+3) = 2m^3+9m^2+13m+6$$

hence

$$\sum_{k=1}^{m+1} k^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

and since this is just (1) with M replaced by $m+1$, we have proved that $m+1 \in A$ whenever $m \in A$. Then A is an inductive set and by the principle of mathematical induction, $A = N$.

Problem 1.6 Use mathematical induction to prove that for $x \geq -1$,

$$(1+x)^n \geq 1+nx \qquad \forall n \in N \qquad (1)$$

Solution: Let the set of integers n for which (1) is valid be denoted by A. Clearly $1 \in A$. Now suppose that A contains all the integers from 1 up to the positive integer m. Then it follows from multiplying both sides of (1) (with $n = m$) by the positive number $(1+x)$, that

$$(1+x)^{m+1} \geq (1+mx)(1+x) = 1+(m+1)x+mx^2 \geq 1+(m+1)x$$

i.e.,

$$(1+x)^{m+1} \geq 1+(m+1)x.$$

Since this is just (1) with m replaced by $m+1,$ we have proved that $m+1 \in A$ whenever $m \in A.$ Then A is an inductive set and by the principle of mathematical induction, $A = N.$

Properties of $\mathbb{N}$ and $\mathbb{R}$

Problem 1.7 Prove that the set $\mathbb{N}$ of natural numbers has no upper bound.

Solution:We make use of the fact that $\mathbb{N}$ is an inductive set. In fact, we sometimes **define** $\mathbb{N}$ by saying that 1 belongs to $\mathbb{N}$ and if n is any element of $\mathbb{N}$, then $n+1$ is also an element of $\mathbb{N}$. Suppose there is an upper bound for $\mathbb{N}$. Since $\mathbb{N}$ is a nonempty subset of R, then $\mathbb{N}$ has a least upper bound, say $\alpha.$ Then $\alpha - 1$ is not an upper bound for $\mathbb{N}$, which is to say, there exists $n \in \mathbb{N}$ such that $\alpha - 1 < n.$ But in this case $\alpha < n+1$ and since $\mathbb{N}$ is an inductive set, $n+1 \in \mathbb{N}.$ This contradiction to the definition of α means there can be no upper bound for $\mathbb{N}$.

Problem 1.8 Prove that the every nonempty subset of, $\mathbb{N}$, the natural numbers, has a first element

Solution:Let T denote a nonempty subset of $\mathbb{N}$, and suppose $k \in T.$ Let S denote the intersection of T with the following finite subset of $\mathbb{N}$, $\{1, 2, \ldots, k\};$ i.e., $S = T \cap \{1, 2, \ldots, k\}.$ Since S is finite, it contains a first (smallest) element, $p.$ Now for any $t \in T,\ t \neq p,$ we must have $p < t.$ For if $t \notin S,$ then $t > k \geq p,$ so $t > p,$ and if t is an element of S, then $t > p,$ since p is the smallest element of S and $t \neq p.$ We have proved $t > p$ for all $t \in T,$ so p is the first element in $T.$ This property of the natural numbers is referred to as the ***well ordering property***.

Problem 1.9 Prove that every nonempty set of reals having a lower bound has a GLB.

Solution:Let S denote a nonempty set of real numbers having a lower bound $\beta.$ Let $T = \left\{t = -s \text{ for } s \in S\right\};$ i.e., T is the reflection of S through the origin. Then every lower bound for S is mapped onto an upper bound for $T.$

Since $\beta \leq s$ for all s in $S,$ and since $\beta \leq s$ is equivalent to $-s \leq -\beta,$ it follows that $t \leq -\beta$ for all $t \in T.$ That is, $-\beta$ is an upper bound for $T.$ Now the completeness axiom asserts the existence of a least upper bound for $T,$ call it $-\hat{\beta}.$. Then $\hat{\beta}$ is the greatest lower bound for $S.$ To see this, note first that $\hat{\beta}$ is a lower bound for S since $-\hat{\beta}$ is an upper bound for $T.$ Moreover, if there were a lower bound, $\beta',$ for S that were greater than $\hat{\beta},$ this would imply that $-\beta'$ was an upper bound for T that was less than $-\hat{\beta}.$ But this is impossible since $-\hat{\beta}$ is the least upper bound for $T.$

Problem 1.10 Prove the Archimedean property; i.e., for every $z \in R,$ there exists $n \in \mathbb{N}$ such that $z \leq n.$ This asserts that there are no infinitely large real numbers.

Solution:Let $z \in R$ and define a set $A = \{n \in \mathbb{N} : n \leq z\}$ as the set of all natural numbers less than or equal to $z.$ If A is empty, then the result follows immediately. If A is not empty, then A is bounded above by z and hence by the completeness axiom, A has a least upper bound, say $\alpha.$ Since α is the least upper bound, it follows that $\alpha - 1$ is not an upper bound for A so there exists some $m \in A$ with $\alpha - 1 < m.$ But in that case, $\alpha < m+1,$ which is to say

$m+1$ is a natural number not belonging to A (a natural number larger than z). This proves the result. Notice that the proof of the Archimedean property is quite similar to the proof in problem 1.7. This is because if there were some real number z for which the Archimedean property failed, then this z would be an upper bound for the natural numbers. Then the nonexistence of infinite reals is equivalent to the nonexistence of an upper bound for the natural numbers.

Problem 1.11 Prove the following alternative statement of the Archimedean property :

- For all positive real numbers, z there exists a unique $n \in N$ such that $n-1 \leq z < n$.

This assertion is equivalent to the Archimedean property

Solution:In the solution of problem 1.10, we showed that for any positive real number, z, the set of natural numbers, $n,$ such that $n > z$ is not empty. Then by the well ordering of $\mathbb{N}$, the set has a first element, n_0. Since n_0 is the first element in the set, it follows that n_0-1 is not in the set; i.e.,$n_0-1 \leq z < n_0$

Problem 1.12 Prove the following statements :

1. For all positive real numbers, y,z there exists $n \in \mathbb{N}$ such that $n \cdot y > z$
2. For all positive real numbers, ε there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$

Statements 1 and 2 could be interpreted as asserting that there are no infinitesimally small reals. These assertions are each equivalent to the Archimedean property

Solution: Let y,z denote positive real numbers. Then $0 < \frac{z}{y} \in R,$ and by the Archimedean property, there exists some $n \in \mathbb{N}$ such that $\frac{z}{y} < n.$ But this is just the result, 1. If we choose $z=1$ in $1,$ then we get 2. This result asserts that there is no real number that is ”infinitesimal” in the sense that it is closer to zero than $\frac{1}{n}$ for every $n \in \mathbb{N}$.

Problem 1.13 Prove that the rationals and irrationals are everywhere dense in the reals.

Solution:We have to show that between any two reals there is a rational number, and an irrational number.
Let x and y be real numbers with $0 \leq x < y.$ Then $y-x > 0$ and statement 2 of the previous problem (the Archimedean property) asserts

$$0 < \frac{1}{m} < \varepsilon = y-x \text{ for some } m \in N.$$

Using the version of the Archimedean property stated in problem 1.11, the set of natural numbers k such that $m \cdot y \leq k$ is not empty, and by the well ordering of $N,$ this set contains a first element, $n.$ Then

$$\frac{n-1}{m} < y \leq \frac{n}{m},$$

and

$$x = y-(y-x) < \frac{n}{m} - \frac{1}{m} = \frac{n-1}{m};$$

i.e.,

$$x < \frac{n-1}{m} < y$$

This proves that between any two positive reals there is a rational number. To say this more simply, we could just observe that by stepping along the positive x axis in steps of length $\frac{1}{m}$, starting from x, we would eventually, after n steps, either land on y or go slightly past y. That is, we would have $y \leq \frac{n}{m}$. Then $x < \frac{n-1}{m} < y$.

If $x < y < 0$, there is a natural number k such that $-x < k$ and then $0 < k+x < k+y$. Using the result just proved, there exists a rational number r between $k+x$ and $k+y$, and $r-k$ is a rational number between x and y.

To show there is an irrational number between x and y, use the previous result to find a rational number, r, between the real numbers, $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$. Then $r\sqrt{2}$ is an irrational number between x and y.

Properties of Subsets of $\mathbb{R}$

Problem 1.14 Let $X = \{x_n\}$ denote a set of real numbers with the property that $x_n < x_{n+1}$ for every n and suppose the set X is bounded above. Then there exists a number L, called the limit of the sequence of x-values, such that L is an accumulation point for the set X.

Solution: Under the assumptions of this problem we say that $\{x_n\}$ is an increasing sequence that is bounded above. Then the set X is a set with an upper bound, hence it has a least upper bound, say L, by the completeness axiom. Then $x_n \leq L$, for every n, and since L is the least upper bound, it follows that for every $\varepsilon > 0$, $L-\varepsilon$ is not an upper bound for the values x_n. This means that for every $\varepsilon > 0$ there is some M such that $L-\varepsilon < x_M \leq L$. Since the sequence $\{x_n\}$ is increasing, it is clear that $L-\varepsilon < x_n \leq L$, for every $n > M$. It is clear then that L is an accumulation point for the set X and in the next chapter we shall learn that L is called the limit of the increasing sequence $\{x_n\}$. A similar statement holds for a decreasing sequence that is bounded below.

Problem 1.15 Prove the nested interval theorem.

Solution: Let $I_n = [a_n, b_n]$ denote a sequence of nested closed and bounded intervals. If $I_{n+1} \subset I_n$ for every n, then $a_n \leq a_{n+1}$, and $b_{n+1} \leq b_n$ for every n. Moreover, since $a_n \leq b_1$ and $a_1 \leq b_n$ holds for all n, the set of numbers $\{a_n\}$ is bounded above and must, therefore, have a least upper bound α. Similarly, the set $\{b_n\}$ is bounded below and hence it must have a greatest lower bound β. Finally, it must be that $\alpha \leq \beta$ for if $\alpha > \beta$ then there exists an a_k such that $\beta < a_k < \alpha$. But in that case there would exist a b_j such that $\beta < b_j < a_k < \alpha$. If $j \neq k$ suppose $j > k$. Then $b_j < a_k < a_j < \alpha$, and this contradicts the assumption that $a_n \leq a_{n+1}$, and $b_{n+1} \leq b_n$ for every n. It follows that $\alpha \leq \beta$. In the case that $\alpha < \beta$, we have $\bigcap_n I_n = [\alpha, \beta]$, and if $\alpha = \beta$ then $\bigcap_n I_n = \{\alpha\}$. In either case $\bigcap_n I_n \neq \emptyset$. Note that the case $\alpha = \beta$ implies that the length of the intervals $[a_n, b_n]$ shrinks to zero and in this case $\bigcap_n I_n$ consists of a single point.

Problem 1.16 For $m \in \mathbb{N}$, let $\{S_m\}$ denote the set of numbers given by,

$$S_m = \sum_{n=1}^{m} r^{n-1} \qquad \text{where } 0 < r < 1;$$

i.e. $\quad S_1 = 1, \qquad S_2 = 1+r, \qquad S_3 = 1+r+r^2, \ldots$

Prove that $S_1 < S_2 < \cdots < S_n \to S = \frac{1}{1-r}$. This is the well known formula for the sum of a geometric series.

Solution: Since $r > 0,$ it is clear that the sequence of numbers, $\{S_m\},$ is increasing; i.e., $S_m < S_{m+1}$. In fact, for any fixed $m,$ we have

$$S_m = 1 + r + r^2 + \ldots + r^{m-1}$$

$$\textit{and} \qquad rS_m = r + r^2 + \ldots + r^{m-1} + r^m.$$

Then $S_m - rS_m = 1 - r^m,$ hence

$$S_m = \frac{1 - r^m}{1 - r}.$$

From this last expression, it follows that $\{S_m\}$ is bounded above by $S = \dfrac{1}{1-r}$. In fact, S is the least upper bound for the sequence. To see this, note that for any $\varepsilon > 0,$ we have $S - \varepsilon < S_m < S,$ provided m is sufficiently large that $m > \dfrac{\log(\varepsilon(1-r))}{\log(1-r)}$. Then for any $\varepsilon > 0$ we can find an integer m such that $S - \varepsilon < S_m < S.$ In the next chapter we will learn that this means that the sequence $\{S_m\}$ is convergent to the limit, $S.$ This sequence is the so called sequence of partial sums for the geometric series.

Problem 1.17 Let $\alpha = \sup A$ and suppose α does not belong to $A.$ Then show that α is an accumulation point for $A.$

Solution: To say $\alpha = \sup A$ means that $x \le \alpha$ for all $x \in A$ and for every $\varepsilon > 0,$ $\alpha - \varepsilon$ is not an upper bound for $A.$ That is, for every $\varepsilon > 0,$ there exists and x in A such that $\alpha - \varepsilon < x < \alpha.$ Note that since α does not belong to $A,$ we have $x < \alpha$ and not $x \le \alpha.$ But this is equivalent to saying that every ε –neighborhood, $(\alpha - \varepsilon, \alpha + \varepsilon)$ of α contains points of A; i.e., α is an accumulation point of $A.$

Problem 1.18 Prove the Bolzano-Weierstrass theorem, "Every bounded, infinite subset of $\mathbb{R}$ must have at least one accumulation point."

Solution: Let A denote a bounded, infinite subset of R; i.e., A is contained in a bounded interval, $I_1 = [a,b]$ and A contains infinitely many points. Now write $I_1 = [a,b]$ as the union of two equal parts, $I_1 = \left[a, \frac{1}{2}(a+b)\right] \cup \left[\frac{1}{2}(a+b), b\right].$ At least one of the two parts must contain infinitely many points since if this were not true, their union which equals I_1 would contain finitely many points (which contradicts what we known about I_1). Let I_2 denote one of these two intervals containing infinitely many points and write I_2 as the union of two half-intervals. Again, at least one of these two half-intervals must contain an infinite number of points and we denote by I_3 one of the half-intervals with infinitely many points. Continuing in this way, we generate a sequence of nested intervals $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \cdots$ Now it is evident that for each $n,$ I_n contains I_{n+1} together with its endpoints and the length of I_n (which equals $(b-a) \cdot 2^{-n}$) shrinks to zero as n tends to infinity. Then the nested interval theorem asserts the existence of a point, $p,$ common to all intervals.

To see that p is an accumulation point for $A,$ consider $N_\varepsilon(p) = (p - \varepsilon, p + \varepsilon)$ for $\varepsilon > 0.$ Choose an $n \in N$ sufficiently large that $(b-a) \cdot 2^{-n} < \varepsilon$. Since p belongs to every $I_n,$ and since ε is greater than the length of $I_{n+1},$ it follows that I_{n+1} is contained in $N_\varepsilon(p).$ By construction, I_{n+1} contains infinitely many points of A which means that $N_\varepsilon(p)$ contains points of A that are different from $p.$ Since p is an accumulation point for $A,$ there is a subsequence of points of A that converges to $p.$ This proves the theorem.

Problem 1.19 Show that the algorithm for generating the decimal representation for real number $x,$ does in fact, generate the decimal representation for $x.$

Solution: Let $X = \{N + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \ldots + \frac{d_n}{10^n};\ n \in N\}$. Then x is an upper bound for the set X and it follows that X has a least upper bound, say β. Then it follows from the way in which the algorithm was constructed that $\beta \leq x$. Suppose that $\beta < x$. Then there exists an integer m such that $0 < \frac{1}{m} < x - \beta$. and in that case we would have

$$N + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \ldots + \frac{d_m}{10^m} + \frac{1}{10^m} \leq \beta + \frac{1}{10^m} < \beta + \frac{1}{m} < x$$

But this violates the definition of d_m and it follows that assuming $\beta < x$ leads to a contradiction. Then $\beta = x$ which is to say, the decimal representation equals x. Another way to look at it is to note that the sequence $x_n = N + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \ldots + \frac{d_n}{10^n}$, is convergent to β and $\beta = x$.

Problem 1.20 Show that a set $S \subset \mathbb{R}$ is closed if and only if S contains all its accumulation points.

Solution: Suppose S is closed $\left(\text{i.e. } S^C \text{ is open}\right)$ but S does not contain all its accumulation points. We will show this leads to a contradiction. Our assumption means there exists a point $x_0 \in S^C$ that is an accumulation point for S. But S^C is open so x_0 is an interior point of S^C which means there is a neighborhood $N_\varepsilon(x_0)$ that is contained in S^C, which is to say there are no points of S in this neighborhood. But in that case, x_0 is not an accumulation point of S. This proves "if S is closed, then S contains all its accumulation points". Now we must show the converse.

Suppose S contains all its accumulation points, and let $x_0 \in S^C$. Since x_0 is not in S, and S contains all its accumulation points, it follows that x_0 is not an accumulation point of S. In that case there is a neighborhood of x_0 that contains no points of S; i.e., $\mathring{N}_\varepsilon(x_0) \cap S = \emptyset$ which is to say $N_\varepsilon(x_0) \subset S^C$. This means S^C is open and S is closed.

Exercises

1. Convert $.1234512345$... and $.3451234512345\ldots$ into the form $\frac{p}{q}$ in reduced terms.
2. Convert $\frac{3}{7}$ to a decimal representation. What is the 200th digit in this expansion.
3. Try to find a number in the form $\frac{p}{q}$ whose decimal representation consists of a very long repeating part. Is there any limit to how long this repeating part can be?
4. Prove that the sum of two rational numbers is rational
5. Prove that the product of two rational numbers is rational
6. Prove that the sum of a rational number and an irrational number is irrational
7. Prove that the product of a nonzero rational number and an irrational number is irrational
8. Give an example of two irrational numbers whose product is rational. Is it true that for every irrational number x there exists another irrational number y such that xy is rational?
9. Prove that if p denotes a prime real number, then $\sqrt{p}$ must be irrational.
10. Prove that the sup of a set is unique.
11. Prove that a nonempty finite set contains its sup and its inf.
12. Find the sup of the set $\{x : 3x^2 + 3 < 10x\}$

13. Let A and B denote sets of real numbers and let $C = \left\{x + y : x \in A,\ y \in B\right\}$. How are the numbers $\inf A,\ \inf B$ and $\inf C$ related?

14. For A ,B and C as in the previous problem, how are the numbers $\sup A,\ \sup B$ and $\sup C$ related?

15. Let A and B denote sets of real numbers and let $C = A \cap B$. How are the numbers $\inf A,\ \inf B$ and $\inf C$ related?

16. Let A and B denote sets of real numbers and let $C = A \cup B$. How are the numbers $\sup A,\ \sup B$ and $\sup C$ related?

17. Let A denote a set of real numbers and let $A^2 = \{x^2 \in A\}$. Is there any relation between the numbers $\sup A$ and $\sup A^2$?

18. Let a, b be real numbers with $b - a > 1$. Prove or disprove: there exists an integer n such that $a < n < b$.

19. Show that the set of rational numbers q such that $q^2 < 2$ has no sup. (Note: this does not say the set has no real number supremum).

20. Show that there exists a real number $x > 0$ such that $x^2 = 2$. Hint: Let $S = \{x > 0 : x^2 < 2\}$ and let $r = \sup S$. Then show $r^2 = 2$.

21. Show that there exists a real number $x > 0$ such that $x^3 = 5$.

22. Let A and B denote sets of real numbers and define $\delta(A,B) = \inf\left\{|x - y| : x \in A,\ y \in B\right\}$;i.e., $\delta(A,B)$ is the "distance" between A and B.

- **a**. Find $\delta(A,B)$ if $A = N$ and $B = R \backslash N$
- **b**. Find $\delta(A,B)$ if $A = N$ and $B = \mathbb{Q} \backslash N$
- **c**. What does $\delta(A,B)$ represent if A and B are finite sets?

23. Let $A = \left\{\frac{m}{10^{10}} : m \in N\right\}$. Find the largest interval (a,b) which contains no elements of A.

24. Under what conditions is $\sup A$ not an accumulation point for A?

25. Let p be an accumulation point for $A \subset R$. Show there exists a sequence of points in A that converge to p.

26. Suppose $x \in [0,1]$ has decimal expansion $0.d_1d_2\ldots d_n\ldots$ with $d_i = 0$ for $i \geq 12$. Show that x must be the left endpoint of an interval of length 10^{-12}. In the case that the decimal expansion for x has $d_i = 9$ for $i \geq 12$, show that x must be a right endpoint of an interval of length 10^{-12}.

Recall the following definitions:

- Let A denote a nonempty set of reals. The complement of A, denoted by $\complement A$, or A^C is the set of all points x not in A.
- We say that x belongs to the *interior of A*, $x \in Int(A)$, if there exists a positive ε such that $N_\varepsilon(x) \subset A$.
- We say that x belongs to the *boundary of A*, $x \in \partial A$, if for every positive ε the neighborhood $N_\varepsilon(x)$ contains points of A and points of A^c.
- We say that x is an *isolated point of* A if there exists a positive ε such that $N_\varepsilon(x)$ contains no points of A other than x.
- A set A is said to be open if all the points of A are interior points. A set A is said to be closed if A^C is open.

Using these definitions, answer the following questions:

1. Find all the interior points, isolated points, accumulation points and boundary points for
   a. $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$
   b. (a,b) and $[a,b]$
   c. $\mathbb{R}$ with $\mathbb{N}$ removed
   d. $\mathbb{R}$ with $\mathbb{Q}$ removed

2. Give an example of:
   a. A set with no accumulation points.
   b. A set with infinitely many accumulation points, none of which belong to the set.
   c. A set that contains some, but not all, of its accumulation points

3. Give an example of a set with the following properties or explain why no such set can exist:
   a. a set with no accumulation points and no isolated points
   b. a set with no interior points and no isolated points
   c. a set with no boundary points and no isolated points

4. Is every interior point of A an accumulation point? Is every accumulation point of A an interior point?

5. Let x be an interior point of A and suppose $\{x_n\}$ is a sequence of points, not necessarily in $A,$ but converging to $x.$ Show that there exists an integer N such that $x_n \in A \;\forall n > N$

6. Prove the following statements
   a. if G_n is open for every $n \in \mathbb{N},$ then $\bigcup_{n\in\mathbb{N}} G_n$ is open
   b. F is closed if and only if F contains all its boundary points

7. Find the interior and boundary for each of the following sets.
   a. $A = \left\{ \frac{1}{\sqrt{n}} \;:\; n \in N \right\}$
   b. $A = \{x \in Q \;:\; 0 < x^2 < 2\}$

8. Show that:
   a. If A and B are both open then $A \cup B$ is open.
   b. If A and B are both closed, then $A \cap B$ is closed.