Lecture 31

Construction of the Likelihood Function

1. Additive noise

Suppose the noise is modeled as additive and mutually
indep. of the unknown \( X \). Then

\[
Y = f(X) + E
\]

where \( X \in \mathbb{R}^n, Y, E \in \mathbb{R}^m \).

Assume the prob. distr. of the noise is known, i.e.,

\[
\mu_E(B) = \mathbb{P}(E \in B) = \int_B \mu_{\text{noise}}(e) \, de
\]

For fixed \( x \), the assump. of mut. indep. of \( X \) and \( E \)
ensures the prob. dens. of \( E \) is unchanged when
conditioned on \( X = x \). Thus,

\[
\Pr(y \mid x) = \Pr_{\text{noise}}(y - f(x))
\]

since \( E = Y - f(X) \)

Hence from Bayes’ formula for IP’s

\[
\Pr(x \mid y) = \frac{\Pr_{\text{prior}}(x) \Pr(y \mid x)}{\Pr(y)} = \frac{\Pr_{\text{prior}} \Pr_{\text{noise}}(y - f(x))}{\Pr(y)}
\]

Bayes’ formula

If \( X \) and \( E \) are not mutually indep., we need to
know the cond’l density of the noise, i.e.,

\[
\mu_E(B \mid x) = \int_B \mu_{\text{noise}}(e \mid x) \, de
\]

Then

\[
\Pr(y \mid x, e) = \int_{\mathbb{R}^m} \Pr(y \mid x, e) \Pr_{\text{noise}}(e \mid x) \, dx
\]

When both \( X = x \) and \( E = e \) are fixed, \( Y \) is determined
from \( y = f(x) + e \). So

\[
\Pr(y \mid x, e) = \delta(y - f(x) - e)
\]

So

\[
\Pr(y \mid x) = \int_{\mathbb{R}^m} \delta(y - f(x) - e) \Pr_{\text{noise}}(e \mid x) \, dx
\]

\[
= \Pr_{\text{noise}}(y - f(x) \mid x)
\]
Then Bayes' formula for IP's implies
\[
\Pi(x|y) = \Pi_{pr}(x) \Pi_{noise}(y-f(x)|x) / \Pi(y)
\]

2. Multiplicative noise

Suppose we have a simple real-val'd meas & the obs model contains multiplicative noise mult. indep. with the unknown. Then we can write
\[
y = E f(x)
\]
(application - a signal amplified by a noisy amplifier)

If \( \Pi_{noise} \) is the prob dens. of \( E_x \) and
\[
\Pi(y|1,x,e) = \delta(y-f(x,e)) \quad \text{so that}
\]
\[
\Pi(y|1,x) = \int_{\mathbb{R}^k} \delta(y-f(x,e)) \Pi_{noise}(e|x) \, de
\]
then in this case we have
\[
\Pi(y|1,x) = \int_{\mathbb{R}} \delta(y-e f(x)) \Pi_{noise}(e) \, de
\]

change var's
\[
e = \frac{v}{f(x)} = \frac{1}{f(x)} \int_{\mathbb{R}} \delta(y-v) \Pi_{noise}(\frac{y}{f(x)}) \, dv
\]
\[
= \frac{1}{f(x)} \Pi_{noise}(\frac{y}{f(x)})
\]

Bayes' formula for IP's implies
\[
\Pi(x|y) = \Pi_{pr}(x) \Pi_{noise}(y|f(x)) / f(x) \Pi(y)
\]

Now suppose the for. model is not completely known. Let \( A(v) \in \mathbb{R}^{m \times n} \) den. a m \times n dep. on a param. vec. \( v \in \mathbb{R}^k \) & assume the deterministic model who noise is \( y = A(v)x, \ y \in \mathbb{R}^m, \ x \in \mathbb{R}^n \).
Assume the meas'ment is corrupted by additive noise mut indep with \( X \) and \( V \). Then
\[
y = A(v)x + V
\]
If $\Pi_{\text{noise}}$ is the prob. density of $E$ which is mut. indep. with $X + V$, we have

$$\Pi(y | x, v) = \Pi_{\text{noise}}(y - A(v)x)$$

Further, assuming that $X + V$ are mut. indep. and $V$ has density $\Pi_{\text{param}}$, we obtain the likelihood density

$$\Pi(y | x) = \int_{\mathbb{R}^k} \Pi(y | x, v) \Pi_{\text{param}}(v) \, dv$$

$$= \int_{\mathbb{R}^k} \Pi_{\text{noise}}(y - A(v)x) \Pi_{\text{param}}(v) \, dv.$$  

**Example: Decovolution**

(1) $$g(x) = \int_{-1}^{1} k(x-x') f(x') \, dx', \quad -1 < x < 1$$

Suppose we approx. the kernel $k(x)$ as a linear combination of Gaussian kernels

$$k(x) = \sum_{j=1}^{M} v_j \phi_j(x)$$

where

$$\phi_1(x) = \frac{1}{\sqrt{2\pi \sigma_1^2}} \exp \left( -\frac{1}{2\sigma_1^2} x^2 \right)$$

$$\phi_j(x) = \frac{1}{\sqrt{2\pi \sigma_j^2}} \exp \left( -\frac{1}{2\sigma_j^2} x^2 \right) - \frac{1}{\sqrt{2\pi \sigma_{j-1}^2}} \sqrt{2\pi \sigma_{j-1}^2} \exp \left( -\frac{1}{2\sigma_{j-1}^2} x^2 \right)$$

where the variances $\sigma_j^2$ are chosen to be fixed and arranged in growing order.

Assume the coeff. $v_j$ are poorly known.

Assume the obs. is corrupted by additive Gaussian noise.

Let $x_1, \ldots, x_n$ be discretization pts of $[-1, 1]$ and $w_j, 1 \leq j \leq n, \text{ weights for a quadrature}$
formula of the integral (1). Then denoting the noisy obs. of \( g(x) \) at pts \( x \) by a vector \( y \), and the additive noise by a vector \( e \), and \( F = [f(x_1), \ldots, f(x_n)]^T \), the discrete obs. model is

\[
y = \sum_{j=1}^{M} y_j A_j F + e
\]

For a stochastic model, assume the additive noise is zero mean white noise with variance \( \sigma^2 \). Then the likelihood function is

\[
\Pi(y | x, \nu) \propto \exp \left( -\frac{1}{2\sigma^2} \| y - \sum_{j=1}^{M} y_j A_j F \| \right)
\]

which is a Gaussian density w.r.t. the par. \( \nu \).

Now by formula (n) the possibility of finding a closed-form post-density depends on the density \( \Pi_{\text{param}} \) of \( \nu \). If \( \Pi_{\text{param}} \) is Gaussian, one can find an explicit formula for \( \Pi(x | y) \).