Lecture 25

Notes on HW 2 (comments)

1. An "inverse crime" can only be committed when working with simulated data. It occurs when
   a) the num. sim. data is produced by the same mtd used to invent the data
      Ex: Using the same psf to blur an image and create a data set as is used in the deconvolution m/x
   b) using the same discretization / mesh in producing the simulated data as in the inversion
      Ex: Suppose we define a conductivity distribution a certain mesh M, solve the F.P. to generate voltage data, then formulate a least-squares problem with the cond. distr. discretized on the same mesh M.

These render the IP better posed than it otherwise would be, and you cannot accurately assess the accuracy or value of the reconstr. alg.
2. Different regularization methods result in reconstructions with different features. We can control what features we want to penalize or reward through our choice of:
   a) filter factors
   b) penalty function
   c) discretization

3. Different parameter choice methods use different information about the problem to strike a balance between minimizing the residual, $\|Ax-m\|_2$, and the penalty term $\|Lx\|_2$.

   We saw the Morozov discrepancy principle requires an estimate of the measurement error, while L-curve and GCV do not.

   The condition number plot of the singular values are valuable tools in analyzing the IP and choosing a regularization technique.
Def: Let $A$ be an injective bounded linear op. $X \to Y$, $X, Y$ normed spaces. Then a family of cont. ops $R_\alpha : Y \to X$ with $\alpha > 0$ with the prop. of pointwise convergence

\[ \lim_{\alpha \to 0} R_\alpha Ax = x \quad \forall x \in X \]

is called a regularization scheme for the operator $A$, and $\alpha$ is the regularization parameter.

If $R_\alpha$ is linear $\forall \alpha$, then the reg. scheme is a linear reg. scheme.

Note: If $Ax = m$, then (x) is equiv. to

\[ R_\alpha m \to A^{-1}m \quad \text{as} \quad \alpha \to 0 \quad \text{for all} \quad m \in R(A). \]

Thm: Let $A : X \to Y$ be an inj. comp. lin. op. with sing. sys. $(\mu_n, \phi_n, \gamma_n)$ nc IN and let

$g : (0, \infty) \times (0, \|A\|] \to \mathbb{R}$

be a bld. func. s.t. for each $\alpha > 0$ $\exists c(\alpha) > 0$ s.t.

1. $\lim_{\alpha \to 0} g(\alpha, \mu) = 1$, $0 < \mu \leq \|A\|$
2. $|g(\alpha, \mu)| \leq c(\alpha) \mu$, $0 < \mu \leq \|A\|$

Then the bld. lin. op.'s $R_\alpha : Y \to X$, $\alpha > 0$, def'd by

\[ R_\alpha m = \sum_{n=1}^{\infty} \mu_n g(\alpha, \mu_n) (m, \gamma_n, \phi_n), \quad m \in Y \]

describe a reg. scheme with $\|R_\alpha\| \leq c(\alpha)$. 

Pf. of Thm 4.9: (i)

By (4.11) + (4.16),

$$\|R_a \Phi_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left| q(a, \mu_n)(\Theta, \phi_n) \right|^2$$

$$\leq \left| c(a) \right|^2 \sum_{n=1}^{\infty} \left| q(a, \mu_n) \right|^2 \|\Theta, \phi_n\|^2$$

which proves (4.19)

$$(R_a \Phi, \phi_n) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(a, \mu_n)(\Theta, \phi_n)$$

$$= q(a, \mu_n)(\phi, \phi_n)$$

So

$$\|R_a \Phi - \Phi\|^2 = \sum_{n=1}^{\infty} \left| (R_a \Phi - \Phi, \phi_n) \right|^2$$

$$= \sum_{n=1}^{\infty} \left| q(a, \mu_n) - 1 \right|^2 \|\phi, \phi_n\|^2$$

(since $A$ is inj). Let $\phi \in X$, $\Phi \not \equiv 0$ be given.

Let $1 \geq M$. Then $\exists N \in \mathbb{N}$ s.t.

$$\sum_{n=0}^{N-1} \|\phi, \phi_n\|^2 \leq \frac{\epsilon}{2(M+1)^2}$$

By (4.17), $\exists \alpha_0 > 0$ s.t.

$$\left| q(a, \mu_n) - 1 \right|^2 \leq \frac{\epsilon}{211\|\phi\|^2} \forall n = 1, 2, \ldots, N$$

Then by (4.11),

$$\|R_a \Phi - \Phi\|^2 \leq \epsilon \sum_{n=1}^{\infty} \|\phi, \phi_n\|^2 + \frac{\epsilon}{2} \leq \epsilon$$

$0 \leq \epsilon\phi_n$. So $R_a \Phi \equiv \Phi \forall \phi \in X$
Thm 4.9: encompasses both $T$-reg + TSVD

Thm 4.10 (TSVD):
Let $A : X \rightarrow Y$ be an inj. empty Im. op. with sing. sys. $(\mu_n, \phi_n, \gamma_n), n \in \mathbb{N}$. Then

$$R_m f = \sum_{\mu_n > \mu_m} \frac{1}{\mu_n} (f, \phi_n) \phi_n$$

describes a reg. scheme with reg. par. $m \rightarrow 0$ and $\|R_m\| = \frac{1}{\mu_m}$.

Proof: Let $q(m, \mu) = \begin{cases} 1, & \mu > \mu_m \\ 0, & \mu < \mu_m \end{cases}$

Then $q$ satisfies conditions (4.16) + (4.17).

$$\|R_m f\|^2 = \sum_{\mu_n > \mu_m} \frac{1}{\mu_n^2} |(f, \phi_n)|^2 \leq \frac{1}{\mu_m^2} \sum_{\mu_n > \mu_m} |(f, \phi_n)|^2 \leq \frac{1}{\mu_m^2} \|f\|^2$$

So $\|R_m\| \leq \frac{1}{\mu_m}$.

Note $R_m g_m = \frac{\phi_m}{\mu_m}$.

So $\|R_m\| = \frac{1}{\mu_m}$.
Tikhonov regularization

Thm 4.13: Let $A : X \to Y$ be a compact linear operator. Then for each $\alpha > 0$, the operator $\alpha I + A^*A : X \to X$ is bij. and has a bounded inverse. Furthermore, if $A$ is injective, then

$$(4.27) \quad R_\alpha = (\alpha I + A^*A)^{-1}A^*$$

describes a regular scheme with $\|R_\alpha\| \leq \frac{1}{2\alpha}$.

**Pf:**

Note that for all $\phi \in X$ and $\alpha > 0$

$$\alpha \|\phi\|^2 = \alpha \langle \phi, \phi \rangle \leq \alpha \langle \phi, \phi \rangle + \langle A\phi, A\phi \rangle = \langle \phi, \phi \rangle + \langle A^*A\phi, \phi \rangle = \langle \alpha \phi + A^*A\phi, \phi \rangle$$

Thus, $(\alpha I + A^*A)$ is injective.

Let $(\mu_n, \phi_n, q_n), \ n \in \mathbb{N}$, be a singular system for $A$.

Let $Q : X \to N(A)$ denote the orthogonal projection.

Then $T : X \to X$ defined by

$$T\phi = \sum_{n=1}^{\infty} \frac{1}{\alpha + \mu_n^2} (\phi, \phi_n) \phi_n + \frac{1}{\alpha} Q(\phi)$$

is bounded (check) + $(\alpha I + A^*A)T = T(\alpha I + A^*A) = I$, i.e., $T = (\alpha I + A^*A)^{-1}$.

If $A$ is inj, then for the unique solution $\phi_\alpha$ of

$$(4.29) \quad A\phi_\alpha + A^*A\phi_\alpha = A^*f$$

we have
(4.30) \[ \phi_\alpha = \sum_{n=1}^{\infty} \frac{M_n}{\alpha + \mu_n^2} (f, g_n) \phi_n \]

(since \( \phi_\alpha = (\alpha I + A^*A)^{-1} A^*f \)

\[ = \left( \sum_{n=1}^{\infty} \frac{1}{\alpha + \mu_n^2} \right) (f, \phi_n) \phi_n + \frac{1}{\alpha} \text{Q}(A^*f) \]

\[ (A^*f, \phi_n) = \mu_n (f, g_n) \]

\[ = \sum_{n=1}^{\infty} \frac{M_n}{\alpha + \mu_n^2} (f, g_n) \phi_n + \frac{1}{\alpha} \text{Q}(A^*f) \]

Let \( \phi : x \to N(A) \)

so \( \text{Q} = \int \text{N}(A) \)

\[ \text{Q} = \sum \langle \varphi, A^*f \rangle \]

Note 0 < q(\alpha, \mu) < 1 \quad q Satisfaction

cond's

(4.16) \( 1/2 < \mu \) \quad 0 < \mu < 1 \quad (4.17) \( \lim_{\alpha \to 0} q(\alpha, \mu) = 1 \quad 0 < \mu < 1 \quad (A^*f, \phi_n) = \mu_n (f, g_n) \]

with

\[ c(\alpha) = \frac{1}{2\sqrt{\alpha}} \quad (\text{since} \sqrt{\alpha} \mu < \frac{\alpha + \mu^2}{2} \quad \text{so} \quad \frac{\mu^2}{\alpha + \mu^2} < \frac{\mu^2}{2\sqrt{\alpha} \mu} = \frac{1}{2\sqrt{\alpha}} \quad \text{by Thm 4.9, } R_\alpha \text{ describes a reg. scheme.} \)