Lecture 18

[Discussion of inverse crimes]

Thm: The Tikhonov-reg. min-norm sol for \( m = Ax + e \)

is given by \( \chi_\delta = V \Sigma_\delta^+ U^T m \)

where \( A = U \Sigma V^T \) is the SVD of \( A \), \( \Sigma = \text{diag} (s_1, ..., s_n) \)

\[
\Sigma_\delta^+ = \text{diag} \left( \frac{s_1}{s_1^2 + \delta}, ..., \frac{s_n}{s_n^2 + \delta} \right)
\]

Pf: Write \( x \in \mathbb{R}^n \) as a linear comb. of the column vectors of \( V \).

\[
x = \sum_{j=1}^{r} \alpha_j V(j) = V \alpha
\]

Then

\[
\| Ax - m \|^2 + \| x \|^2 = \sum_{j=1}^{r} (s_j \alpha_j - m_j)^2
\]

\[
+ \sum_{j=r+1}^{n} m_j^2 + \delta \sum_{j=1}^{n} \alpha_j^2
\]

\[
= \sum_{j=1}^{r} (s_j^2 + \delta) (\alpha_j^2 - 2 \frac{s_j \alpha_j}{s_j^2 + \delta} \alpha_j) + \delta \sum_{j=r+1}^{n} \alpha_j^2
\]

\[
+ \sum_{j=1}^{n} m_j^2 \quad \text{where} \quad m' = U^T m.
\]

Completing the square in first term yields

\[
\| Ax - m \|^2 + \| x \|^2 = \sum_{j=1}^{r} \left( s_j^2 + \delta \right) \left( \alpha_j^2 - 2 \frac{s_j \alpha_j}{s_j^2 + \delta} \alpha_j \right) + \delta \sum_{j=r+1}^{n} \alpha_j^2
\]

\[
\delta \sum_{j=r+1}^{n} \alpha_j^2 - \sum_{j=1}^{r} \left( s_j \alpha_j \right)^2 + \frac{s_j^2}{s_j^2 + \delta} m_j^2
\]

This attains its min when \( \alpha_j = \left\{ \begin{array}{ll}
\frac{s_j \alpha_j}{s_j^2 + \delta} m' & 1 \leq j \leq r \\
0 & r+1 \leq j \leq n
\end{array} \right. \)

\[
\text{ie, } \alpha = \sum_{\delta}^+ m'
\]
Consider the quadratic functional $F_s(x)$.

$$F_s(x) = \|Ax - m\|^2 + \delta \|x\|^2$$

It can be proven that $F_s(x)$ has a unique minimizer $x_0$. The minimizer $x_0$

such that

\[ \frac{d}{dt} \left( \langle Ax, Aw \rangle - 2 \langle m, Aw \rangle \right) + \frac{d}{dt} \left( \|Ax + Aw\|^2 - \|Ax\|^2 \right) = 0 \]
In case of the generalized Influence method, we arrive at the following equations:

(1) \[ x_8 = (A^T A + \delta I) A \]

(2) \[ x_8 = (A^T A + \delta I)^2 A \]

Next we will derive a computationally attractive version of (1). It's called an 'attached form'.

So we get

\[ \langle A x_8, A w \rangle + \delta \langle x_8, w \rangle = 0 \]

and by taking transpose and by taking transpose we get the traditional form

\[ \langle A^T A x_8, A^T w \rangle + \delta \langle x_8, w \rangle = 0 \]

So finally we get the traditional form

\[ \langle (A^T A + \delta I) x_8, A^T w \rangle = 0 \]

For any we \( A^T w \) and we get

\[ x_8 = (A^T A + \delta I)^{-1} A^T w \]

So the minimizing \( x_8 \) satisfies

\[ x_8 = (A^T A + \delta I)^{-1} A^T w \]