1 Lecture 14

1.1 The SVE, SVD and Regularization

Reference: *Rank-Deficient and Discrete ill-posed problems* Christian Hansen, SIAM

A Fredholm integral equation of the first kind
Find \( f(t) \) such that
\[
\int_0^1 K(s, t)f(t)dt = g(s)
\]
with \( 0 \leq s \leq 1 \) and \( \|K\|^2 \equiv \int_0^1 \int_0^1 K(s, t)^2dsdt \leq C \). Moreover \( g \) is the data and \( K(s, t) \) is the kernel arising from the mathematical model. Often, \( g \) is only known at discrete points \( s_1, \ldots, s_m \). This lead to
\[
\int_0^1 k_i(t)f(t)dt = b_i
\]
where \( K_i(t) = K(s_i, t) \) and \( b_i = g(s_i) \). The problem of determining \( f(t) \) in equation (1) and (2) is ill-posed.

Example:
Suppose \( f(t) = \sin(2\pi pt), \ p = 1, 2, \ldots \). Then \( g(s) = \int_0^1 K(s, t)\sin(2\pi pt)dt \). By Riemann-Lebesque lemma, \( \lim_{p \to \infty} g(s) = 0 \). But as \( p \to \infty \), \( \sin(2\pi pt) \) becomes highly oscillatory. So small changes in \( g \) can correspond to large changes in \( f \).

1.2 The Singular Value Expansion (SVE)

The SVE theorem says that any kernel \( K \) such that \( \|K\| \leq C \) can be written as
\[
K(s, t) = \sum_{i=1}^{\infty} s_i u_i(s) v_i(t)
\]
\( u_i \) and \( v_i \) are the singular functions of \( K \). They are orthonormal i.e \( \langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 1 \) if \( i = j \) and 0 if \( i \neq j \). The \( s_i \) are the singular values of \( K \) with \( s_1 \geq s_2 \geq \ldots \geq 0 \) and \( \sum_{n=1}^{\infty} s_n^2 = \|K\|^2 \) (so \( s_n \) decays faster than \( 1/\sqrt{n} \)).
Also
\[
\int_0^1 K(s, t)v_i(t)dt = s_i u_i(s), \quad i = 1, 2, \ldots
\]
Multiply equation (1) with $s_i u_i(s)$ to get

$$
\int_0^1 K(s,t)f(t)dt s_i u_i(s) = s_i u_i(s)g(s)
$$

$$
\int_0^1 \left( \sum_{j=1}^{\infty} s_j u_j(s)v_j(t)f(t)dt \right) s_i u_i(s) = s_i u_i(s)g(s)
$$

$$
\int_0^1 \sum_{j=1}^{\infty} s_j u_j(s)\langle v_j, f \rangle s_i u_i(s)ds = \int_0^1 s_i u_i(s)g(s)ds
$$

$$
s_i^2 \langle v_i, f \rangle = s_i \langle u_i, g \rangle
$$

$$
\langle v_i, f \rangle v_i(t) = \frac{\langle u_i, g \rangle}{s_i} v_i(t)
$$

$$
\sum_{i=1}^{\infty} \langle v_i, f \rangle v_i(t) = \sum_{i=1}^{\infty} \frac{\langle u_i, g \rangle}{s_i} v_i(t)
$$

Since the left side from the last equation is the projection of $f(t)$ onto the span $\{v_i\}_{i=1}^{\infty}$ we get

$$
f(t) = \sum_{i=1}^{\infty} \frac{\langle u_i, g \rangle}{s_i} v_i(t)
$$

Observations:

1. The smoother the kernel, the faster the $s_i$ decay.
2. The smaller the $s_i$, the more oscillatory $u_i$ and $v_i$ will be.
3. The factor $s_i^{-1}$ in (3) amplifies high frequency oscillations in $g$.

### 1.3 A Characterization of Ill-posedness

If there exist a positive real number $\alpha$ such that the singular values satisfy $s_i = O(n^{-\alpha})$, then $\alpha$ is called the **degree of ill-posedness**. If $\alpha \leq 1$, the problem is mildly ill-posed. If $\alpha \geq 1$, the problem is moderately ill-posed. If $s_i = O(e^{-\alpha n})$ the problem is **severely** ill-posed.

**What is regularization?** A method of incorporating further information about the desired solution to stabilize the problem. These can have lots(!) of different forms. The dominating approach is: Consider the **constrained** minimization problem, minimize $p(t)$ where $p(t) = \| \int_0^1 K(s,t)f(t)dt - g(s) \|_2$ subject to one of the following:

1. $f$ belongs to a specified subset.
2. some measure of $f$ (e.g. its norm or derivative) is less than some upper bound. i.e. $w(f) \leq \delta$
3. $p(t) \leq \alpha$
4. minimize a linear combination of $p(t)$ and a measure of $f$, $w(f)$, $\min\{p(t)^2 + \lambda^2 w(f)^2\}$.

Here $\delta, \alpha, \lambda$ are called regularization parameters.