1 Lecture 11

Definition 1 The Fourier Transform of a function defined on \( \mathbb{R}^n \) is given by

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx
\]

Note: A sufficient condition for the existence of the Fourier transform is \( f \in L^1(\mathbb{R}) \), in which case \( \hat{f} \in C_0(\mathbb{R}) \), but may not belong to \( f \in L^1(\mathbb{R}) \).

Theorem 1 If the Fourier transform of both \( f(x) \) and \( f'(x) \) exist, then

\[
\mathcal{F}(f'(x))(\xi) = i \xi \hat{f}(\xi)
\]

More generally, if \( f(x) \) and all its derivatives up to order \( m \) have Fourier transforms, then

\[
\mathcal{F}(f^{(k)}(x))(\xi) = (i \xi)^k \hat{f}(\xi), \quad k = 1, \ldots, m
\]

If \( f \) and \( \hat{f} \in L^1(\mathbb{R}) \), then the inverse Fourier transform is defined by

\[
\mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix \cdot \xi} d\xi
\]

Definition 2 A function \( \phi(x) \) defined on \( \mathbb{R} \) is a test function if \( \phi \) is in \( C_0^\infty(\mathbb{R}) \) and has compact support. We write \( \phi \in C_0^\infty(\mathbb{R}) \).

(Note: This notation varies from book to book.)

Theorem 2 \( C_0^\infty(\mathbb{R}) \) is a dense subspace of \( L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \).

This theorem allows us to extend the Fourier transform to \( L^2 \)-functions as follows:

Theorem 3 For every \( f \in L^2(\mathbb{R}) \) there exists a unique \( \hat{f} \in L^2(\mathbb{R}) \) such that \( \hat{f} = \mathcal{F}(f) \) in the sense that if \( \{\phi_n(x)\} \in C_0^\infty(\mathbb{R}) \) is such that \( \|\phi_n - f\|_2 \to 0 \) as \( n \to \infty \), then \( \hat{\phi}_n = \mathcal{F}(\phi_n) \in L^2(\mathbb{R}) \) is such that \( \|\hat{\phi}_n - \hat{f}\|_2 \to 0 \) as \( n \to \infty \),

Examples:

1) \( g(x) = \begin{cases} 
1 - |x| & \text{if } |x| < 1 \\
0 & \text{if } |x| > 1 
\end{cases} \)

\( g(x) \in L^2(\mathbb{R}) \), is continuous and has compact support but is not differentiable. Since \( g \in L^2(\mathbb{R}) \), \( \hat{g} \) exists by the previous theorem, and one can show

\[
\hat{g}(\xi) = \frac{2}{\xi^2}(1 - \cos \xi)
\]

Note that \( i \xi \hat{g}(\xi) = \frac{2i}{\xi^3}(1 - \cos \xi) \in L^2(\mathbb{R}) \). Thus, \( \mathcal{F}(g'(x)) = i \xi \hat{g}(\xi) \) and we can define the derivative of \( g \) in the \( L^2 \)-sense by

\[
g'(x) = \mathcal{F}^{-1}(i \xi \hat{g}(\xi)) = \begin{cases} 
-\text{sgn}(x) & \text{if } |x| < 1 \\
0 & \text{if } |x| > 1 
\end{cases}
\]
2) 

\[ p(x) = \begin{cases} 
  |x| & \text{if } |x| < 1 \\
  0 & \text{if } |x| > 1 
\end{cases} \]

\[ p(x) \in L^2(\mathbb{R}), \] has compact support but is not continuous and is not differentiable. Since 
\[ p \in L^2(\mathbb{R}), \] \( \hat{p} \) exists by the previous theorem, and one can show

\[ \hat{p}(\xi) = \frac{2}{\xi^2}(\xi \sin \xi - (1 - \cos \xi)) \]

So \( i\xi \hat{p}(\xi) = 2i \sin \xi - \frac{2i}{\xi}(1 - \cos \xi) \notin L^2(\mathbb{R}) \). So \( p \) does not have a derivative in the sense of \( L^2 \).

**Theorem 4 (Sobolev Embedding Theorem in \( \mathbb{R}^1 \)).** Suppose \( f \in L^2(\mathbb{R}) \) and \( \xi^p \hat{f}(\xi) \in L^2(\mathbb{R}) \) for \( p \leq M \). Then \( f \) has derivatives in the \( L^2 \)-sense of all orders less than or equal to \( M \) and these \( L^2 \)-derivatives belong to \( C_0 \) for \( j \leq M - 1 \).

**Proof:**
\( \xi^p \hat{f}(\xi) \in L^2(\mathbb{R}) \) for \( p \leq M \) implies \( f^{(p)} = \mathcal{F}^{-1}((i\xi)^p \hat{f}(\xi)) \) in the \( L^2 \)-sense and \( (1 + \xi^2)^{M/2} \hat{f}(\xi) \in L^2(\mathbb{R}) \). We must show \( \xi^p \hat{f}(\xi) \in L^1(\mathbb{R}) \) for \( q \leq M - 1 \) since this implies

\[ \mathcal{F}^{-1}((i\xi)^q \hat{f}(\xi)) \in C_0(\mathbb{R}) \] and 

\[ \mathcal{F}^{-1}((i\xi)^q \hat{f}(\xi)) = f^{(q)}(x) \text{ a.e.} \]

By Hölder’s inequality

\[ \int_{\mathbb{R}} |\xi|^q |\hat{f}(\xi)|d\xi = \int_{\mathbb{R}} \frac{|\xi|^q}{(1 + \xi^2)^{M/2}}(1 + \xi^2)^{M/2} |\hat{f}(\xi)|d\xi \]
\[ \leq \left( \int_{\mathbb{R}} \frac{|\xi|^{2q}}{(1 + \xi^2)^M}d\xi \right)^{1/2} \left( \int_{\mathbb{R}} (1 + \xi^2)^M |\hat{f}(\xi)|^2d\xi \right)^{1/2} \]
\[ \leq C \left( \int_{\mathbb{R}} \frac{|\xi|^{2q}}{(1 + \xi^2)^M}d\xi \right)^{1/2} \]
\[ < \infty \text{ provided } q \leq M - 1. \]

End of proof.

Back to \( \mathbb{R}^n \): Let us define a multi-index \( \alpha, \alpha = (\alpha_1, \ldots, \alpha_m) \) and the partial derivative operator \( D^\alpha \) by

\[ D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_m} \right)^{\alpha_m} \]

Define \( |\alpha| = \alpha_1 + \ldots + \alpha_m \) and \( \alpha! = (\alpha_1)! \cdots (\alpha_m)! \).

(i) If \( u, D^\alpha u \in L^2 \), then \( \mathcal{F}(D^\alpha u) = (i\xi)^\alpha \hat{u}(\xi) \in L^2 \).

(ii) If \( u, x^\alpha D^\alpha u \in L^2 \), then \( \mathcal{F}(x^\alpha D^\alpha u) = (iD_\xi)^\alpha \hat{u}(\xi) \in L^2 \).
(i) implies if $u$ is smooth, then $\hat{u}$ decays rapidly at $\infty$.
(ii) implies if $u$ decays rapidly at $\infty$, then $\hat{u}$ is smooth.

Let us define, for $s \geq 0$, the Sobolev spaces

$$H^s(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \}$$

with

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi)^2 d\xi$$

and

$$(u, v)_s = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi)^2 \hat{v}(\xi) d\xi$$

Note: 1) $H^0 = L^2$
2) $s \geq t \geq 0$ implies $H^s \subset H^t \subset H^0$

**Theorem 5** Any differential operator of order $M$ is a bounded linear mapping from $H^s$ into $H^{s-m}$ for $s > m$.

**Proof:**
Define a function $P_m(\xi)$ by

$$P_m(\xi) = \sum_{|\alpha| \leq m} C^\alpha \xi^\alpha$$

and define an operator $P(D_x) : H^m(\mathbb{R}^n) \rightarrow \mathbb{R}$ by $(D_x)u(x) = \mathcal{F}^{-1}(P_m(i\xi)\hat{u}(\xi))$. Then by definition of the Inverse Fourier transform

$$P(D_x)u(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{|\alpha| \leq m} C^\alpha(i\xi)^\alpha \hat{u}(\xi) d\xi$$

$$= \sum_{|\alpha| \leq m} C^\alpha \int_{\mathbb{R}^n} e^{ix \cdot \xi} (i\xi)^\alpha \hat{u}(\xi) d\xi$$

$$= \sum_{|\alpha| \leq m} C^\alpha D_x^\alpha u(x) \quad \text{by (i)}$$

For $s > m$,

$$\|P(D_x)u(x)\|_{s-m} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-m} P_m(i\xi)^2 \hat{u}(\xi)^2 d\xi$$

$$\leq \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left( \frac{P_m(i\xi)^2}{(1 + |\xi|^2)^{m}} \right) \hat{u}(\xi)^2 d\xi$$

$$\leq C_m \|u\|_s^2$$

where $C_m = \max_{\xi} \frac{|P_m(i\xi)|^2}{(1 + |\xi|^2)^m} < \infty$.

End of proof.