1 Lecture 10

Recall, a problem is said to be well-posed if

1. there exists a solution
2. the solution is unique
3. the solution depends continuously on the data.

In terms of an operator equation \( A\phi = f \) where \( A : U \rightarrow V \), \( U,V \) subsets of normed space \( X \) and \( Y \), respectively, this means \( A : U \rightarrow V \) is bijective and \( A^{-1} : V \rightarrow U \) is continuous. So there are 3 types of ill-posedness

1. If \( A \) is not surjective (onto), then \( A\phi = f \) is not solvable for all \( f \in V \) (nonexistence of a solution).
2. If \( A \) is not injective (one to one), then \( A\phi = f \) may have more than one solution (nonuniqueness).
3. If \( A^{-1} \) exists, but is not continuous, then the solution does not depend continuously on the data (instability).

These properties are not completely independent.

**Theorem 1** If \( A : X \rightarrow Y \) is a bounded linear operator mapping a Banach space \( X \) bijectively into a Banach space \( Y \), then \( A^{-1} : Y \rightarrow X \) is bounded and therefore continuous.

**Theorem 2** Let \( A : U \subset X \rightarrow Y \) be a compact operator. Then \( A\phi = f \) is ill-posed if \( \dim U \) is not finite.

**Proof**: (by contradiction) Suppose \( \dim U = \infty \) and \( A^{-1} : Y \rightarrow U \) exists and is continuous. Since \( I = A^{-1}A \), \( I \) is the product of a compact operator and a continuous operator and is therefore compact. But the identity operator on an \( \infty \)-dimensional Banach space is not compact. Hence \( \dim U < \infty \).

**Theorem 3** If \( K \in \mathcal{L}(H) \) is compact and \( \{e_n\} \) is a orthonormal basis for \( H \), then \( Ke_n \rightarrow 0 \) as \( n \rightarrow \infty \).

**Proof**: Suppose not. Then there exists \( \epsilon_0 > 0 \) and a subsequence \( n_j \) of \( \mathbb{N} \) such that \( \|Ke_{n_j}\| \geq \epsilon_0 \) for all \( j \in \mathbb{N} \). Since \( \|e_n\| = 1 \) for all \( j \) and \( K \) is compact there exists a subsequence \( \{f_i = e_{n_{i_j}}\}_{i=1}^{\infty} \) of \( \{e_{n_i}\} \) such that \( Kf_i \rightarrow u \) as \( i \rightarrow \infty \) for some \( u \in H \). Since \( \|Kf_i\| \geq \epsilon_0 \) for all \( i \) and \( u \neq 0 \). Also note \( (Kf_i, u) = (f_i, K^*u) = (f_i, z) \rightarrow 0 \) since \( \{f_i\} \) is an orthonormal set. But this contradicts \( (Kf_i, u) \rightarrow 0 \).

**Corollary 1** If \( K : H \rightarrow H \) is compact, linear and \( K^{-1} \) exists and \( H \) is \( \infty \)-dimensional, then \( K^{-1} \) is unbounded.

**Proof**

Let \( \{e_n\} \) be an orthonormal basis for \( H \). Then \( Ke_n \rightarrow 0 \) as \( n \rightarrow \infty \). Thus, \( K \) is not bounded away from 0. Hence \( K^{-1} \) is unbounded.
**Theorem 4** Let $A : H \to H$ be a linear operator. If $\{K_n\}$ is a sequence of compact operators such that $K_n \to A$ in operator norm, then $A$ is compact.

**Theorem 5** Let $Au(x) = \int_{\Omega} K(x, y)u(y)dy$, with kernel $K : \Omega \to \mathbb{R}$. If $K$ is Hilbert-Schmidt (i.e. square integrable), then $A$ is compact.

**Proof**: Let $\{\Phi_i(x)\}$ be an orthonormal basis for $L^2(\Omega)$. Then $\{\Phi_i(x)\Phi_j(x)\}$ is an orthonormal basis for $L^2(\Omega \times \Omega)$. Write

$$K(x, y) = \sum_{i,j=1}^{\infty} k_{ij} \Phi_i(x)\Phi_j(x)$$

(where convergence is in $L^2(\Omega)$ - norm) and

$$k_{ij} = \int_{\Omega} \int_{\Omega} K(x, y)\Phi_i(x)\Phi_j(y)dx dy$$

Then

$$\|K\|_{L^2(\Omega \times \Omega)}^2 = \int_{\Omega} \int_{\Omega} |K(x, y)|^2 dx dy = \sum_{i,j=1}^{\infty} |k_{ij}|^2$$

Define the operator $A_n \in (L^2(\Omega))$ by

$$(A_n u)(x) = \int_{\Omega} K_n(x, y)u(y)dy$$

where

$$K_n(x, y) = \sum_{i,j=1}^{n} k_{ij} \Phi_i(x)\Phi_j(y).$$

since $D(A_n)$ is finite, $A_n$ is compact. Now

$$\|(A - A_n)u\|^2 = \left\| \int_{\Omega} (K(x, y) - K_n(x, y))u(y)dy \right\|^2$$

$$= \left( \int_{\Omega} \left( \int_{\Omega} |(K(x, y) - K_n(x, y))u(y)dy| dx \right)^2 \right.$$

use Hölder’s inequality

$$\leq \int_{\Omega} \left( \int_{\Omega} |(K(x, y) - K_n(x, y))|^2 dy \right) \left( \int_{\Omega} |u(x)|^2 dy \right) dx$$

$$= \int_{\Omega} \int_{\Omega} |(K(x, y) - K_n(x, y))|^2 dy dx \int_{\Omega} |u(x)|^2 dy$$

So $\|(A - A_n)\|^2 \leq \int_{\Omega} \int_{\Omega} |(K(x, y) - K_n(x, y))|^2 dy dx$. Now

$$\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} |(K(x, y) - K_n(x, y))|^2 dy dx = \lim_{n \to \infty} \sum_{i,j=n+1}^{\infty} |k_{ij}|^2 = 0$$

So $A_n \to A$ in the operator norm, so $A$ is compact.
Examples:
1) Weakly singular integral operators under the conditions of Vainikko. (See Lecture 9)
2) Recall the 1D heat conduction problem from Lecture 1: Find $f(x)$, the initial temperature distribution of a finite insulated rod, given the temperature $u(x, 1) = g(x)$.

We derived the Fredholm integral equation of the 1st kind

$$g(x) = \int_0^\pi K(x, \xi) f(\xi) d\xi$$

where $K(x, \xi) = \sum_{n=1}^{\infty} e^{-n^2} \sin(nx) \sin(n\xi)$.

Exercise: Show the integral operator is compact.