A Riemann-Hilbert approach to asymptotic analysis of a bordered Toeplitz determinant and the next-to-diagonal correlations of the anisotropic square lattice Ising model.

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Abstract: It is a classical result (e.g. see [MW73],[DIK13]) that the diagonal correlations $\langle \sigma_{0,0}\sigma_{N,N}\rangle$ in the square lattice Ising model are expressed as the determinant of an $N\times N$ Toeplitz matrix corresponding to a specific symbol. Various approaches, including Riemann-Hilbert and operator-theoretic techniques have been used to study the large N asymptotics of such *structured* determinants (Toeplitz, Hankel, Fredholm, etc.). In 1987 Au-Yang and Perk expressed the spin-spin *next-to-diagonal* correlations of the anisotropic square lattice Ising model in terms of a *bordered Toeplitz* determinant [AYP87],[AYJP01]. In this work, we relate this particular bordered Toeplitz determinant to the 12-entry of the 2×2 matrix solution of the well known Riemann-Hilbert problem associated with Toeplitz determinants. We will use this connection to find the large N asymptotics of the given bordered Toeplitz determinant, and assuming the validity of the connection with the Ising model, of the next-to-diagonal correlations $\langle \sigma_{0,0}\sigma_{N,N-1}\rangle$. The results of our Riemann-Hilbert analysis, which have also recently been fully and independently confirmed both by operator-theoretic and numerical methods, has lead to an unexpected physical behavior if one accepts the results of Au-Yang and Perk. We believe that this work, indirectly implies that there could be a flaw in the derivation of the connection between the next-to-diagonal correlations and the bordered Toeplitz determinant. In the beginning of this talk before getting into the details of the problem at hand, I will briefly discuss what a Riemann-Hilbert problem is, and why its analytical solution reveals crucial asymptotic information, particularly for a given structured determinant. This is a joint work with Alexander Its.

References

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