Math. Ann. 255, 379–394 (1981)

Mathematische Annalen

The Moduli of Weierstrass Fibrations Over $\mathbb{P}^1$

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1. Introduction

Let k be an algebraically closed field of characteristic $\neq 2, 3$. Let $X \xrightarrow{p} Y$ be a flat proper map of reduced irreducible k-schemes such that every geometric fibre is either

(a) an elliptic curve,
(b) a rational curve with a node, or
(c) a rational curve with a cusp.

Moreover we assume that a section $s: Y \to X$ is given, not passing through the nodes or cusps of the fibres. Following Artin [A], let us call such a map a *Weierstrass fibration.*

In this paper we will focus our attention on the case when $Y = \mathbb{P}^1$. Using geometric invariant theory we will construct coarse moduli spaces for these objects; this boils down to the calculation of stability. The most interesting case from this viewpoint is when X is a *rational* surface; here stability depends entirely on the singular fibres of the map p. We can also describe, in this case, the semistable compactification of the moduli space in some detail.

Weierstrass fibrations over a *curve* Y arise most naturally as follows. Let $\tilde{X} \xrightarrow{\tilde{p}} Y$ be a minimal elliptic surface over Y with a section S. By contracting every component of every fibre of $\tilde{p}$ which does not meet S, one obtains a normal surface X which maps to Y and is a Weierstrass fibration with only rational double points. X is uniquely determined by $\tilde{X}$; moreover, two non-isomorphic elliptic surfaces cannot give rise to the same Weierstrass fibration X ($\tilde{X}$ is obtained from X by resolving the double points). We will denote this $1-1$ correspondence by saying that the elliptic surface $\tilde{X}$ *contracts to* the Weierstrass fibration X, and that a Weierstrass fibration X with only rational double points (at singular points of singular fibres) *resolves to* the elliptic surface $\tilde{X}$.

0025-5831/81/0255/0379/$03.20

2. The Weierstrass Form

We wish to describe briefly a canonical form for a Weierstrass fibration $X \xrightarrow{p} Y$, exhibiting X as a divisor in a $\mathbb{P}^2$-bundle over the base Y. Let S denote the given section of p, i.e. S is the image of s. S is a divisor on X, and is carried isomorphically onto Y by p.

Theorem (2.1) (Weierstrass form). *Let X, Y, p, S be as above, and let $L = p_*[O_X(S)/O_X]$. Assume that the general fibre of p is smooth. Then L is invertible and X is isomorphic to the closed subscheme of $\mathbb{P} = \mathbb{P}(L^{\otimes 2} \oplus L^{\otimes 3} \oplus O_Y)$ defined by*

$$(2.2) \quad y^2 z = x^3 + Axz^2 + Bz^3 ,$$

where $A \in \Gamma(Y, L^{\otimes -4})$, $B \in \Gamma(Y, L^{\otimes -6})$, and $[x, y, z]$ is the global coordinate system of $\mathbb{P}$ relative to $(L^{\otimes 2}, L^{\otimes 3}, O_Y)$. Moreover, the pair (A, B) is unique up to isomorphism, and the discriminant $4A^3 + 27B^2 \in \Gamma(Y, L^{\otimes -12})$ vanishes at a point $q \in Y$ precisely when the fibre X_q is singular.

Proof. The statement is essentially that of Theorem 1′ of [ITM]; the difference is the existence of possible singular fibres. The proof, however, is identical.

The above theorem gives us one way to construct Weierstrass fibrations; indeed, by careful choice of the coefficients A and B one may prescribe the points $q \in Y$ where the fibre X_q is singular (with certain restrictions). Moreover, if Y is a smooth curve and if X has only rational double points as singularities, one may compute many of its numerical characters (or, equivalently, those of the elliptic surface $\tilde{X}$ to which X resolves) by using the explicit embedding, given above, of X as a divisor in a rather simple 3-fold (a more intrinsic approach is also possible). Let us recall the basic results.

Proposition (2.3). *Let $\tilde{p}: \tilde{X} \to Y$ be a minimal elliptic surface over a smooth curve Y. Assume that a section $s: Y \to \tilde{X}$ with image S exists. Then*

(i) $\tilde{p}_* O_{\tilde{X}} = O_Y$
(ii) $R^1 \tilde{p}_* O_{\tilde{X}}$ *is invertible on* Y
(iii) $\dim H^1(\tilde{X}, O_{\tilde{X}}) = \dim H^0(Y, R^1 \tilde{p}_* O_{\tilde{X}}) +$ *(genus of Y)*
(iv) $\dim H^2(\tilde{X}, O_{\tilde{X}}) = \dim H^1(Y, R^1 \tilde{p}_* O_{\tilde{X}})$
(v) $R^1 \tilde{p}_* O_{\tilde{X}} \cong \tilde{p}_*[O_{\tilde{X}}(S)/O_{\tilde{X}}] \cong s^* O_{\tilde{X}}(S)$
(vi) $(S \cdot S) = \deg R^1 \tilde{p}_* O_{\tilde{X}} = -\chi(O_{\tilde{X}})$
(vii) *the canonical class $K_{\tilde{X}}$ of $\tilde{X}$ is an integral linear combination of fibres of $\tilde{p}$.*

Proof. Let us only sketch these well-known results. Since every fibre of $\tilde{p}$ is a connected curve of arithmetic genus one, (i) and (ii) follow immediately by base change. The Leray spectral sequence $H^p(Y, R^q_{\tilde{p}_*} O_{\tilde{X}}) \Rightarrow H^{p+q}(\tilde{X}, O_{\tilde{X}})$ yields the exact sequence

$$0 \to H^1(Y, O_Y) \to H^1(\tilde{X}, O_{\tilde{X}}) \to H^0(Y, R^1_{\tilde{p}_*} O_{\tilde{X}}) \to H^2(Y, O_Y)$$

and since $H^2(Y, O_Y) = 0$, (iii) follows. The collapsing of the $p + q = 2$ terms in the spectral sequence, except for the $p = q = 1$ term, yields (iv). Applying $\tilde{p}_*$ to the exact sequence

$$0 \to O_{\tilde{X}} \to O_{\tilde{X}}(S) \to O_{\tilde{X}}(S)|_S \to 0$$

gives

$$0 \to O_Y \xrightarrow{f} \tilde{p}_* O_{\tilde{X}}(S) \to s^* O_{\tilde{X}}(S) \to R^1 \tilde{p}_* O_{\tilde{X}} \to R^1 \tilde{p}_* O_{\tilde{X}}(S) \to 0 .$$

By base change to the fibres of $\tilde{p}$ one sees that f is an isomorphism and $R^1_{\tilde{p}_*} O_X(S) = (0)$; this gives (v). The penultimate assertion follows from (iii), (iv), and (v), by applying Riemann-Roch to $R^1_{p_*} O_X$ on Y. A proof of (vii) may be found in [Sh], Chap. VII, Sect. 3. QED.

Corollary (2.4). *Let $\tilde{X}$ be a minimal elliptic surface over $\mathbb{P}^1$, with section S. Then $R^1_{\tilde{p}_*} O_{\tilde{X}} \cong O_{\mathbb{P}^1}(-N)$ for some $N \geqq 0$, and $\tilde{X}$ is a product $\Leftrightarrow N = 0$. Moreover, if $N > 0$, then*

(i) $\dim H^1(\tilde{X}, O_{\tilde{X}}) = 0$

(ii) $\dim H^2(\tilde{X}, O_{\tilde{X}}) = N - 1$

(iii) $(S \cdot S) = -N$

(iv) $K_{\tilde{X}} \sim (N-2)F$ *where F is the class of a fibre of $\tilde{p}$.*

Proof. Let $\tilde{X}$ contract to the Weierstrass fibration X, and put X in the Weierstrass form (2.2). If degree of $R^1 \tilde{p}_* O_{\tilde{X}}$ is positive, then the coefficient A and B of the form must vanish, and hence $\tilde{X}$ will have no smooth fibres, a contradiction. Using this representation the second statement is clear. The preceding proposition now implies (i), (ii), and (iii) immediately. Since the base curve is $\mathbb{P}^1$, all fibres are linearly equivalent so that $K_{\tilde{X}} \sim MF$ for some $M \in \mathbb{Z}$. By applying adjunction to the section S we find

$$-2 = \operatorname{degree} K_{\mathbb{P}^1} = (S \cdot S) + (S \cdot K_{\tilde{X}}) = (-N) + M ,$$

proving (iv). QED.

From now on fix N to be strictly positive.

This Corollary allows us to be more specific in describing Weierstrass fibrations over $\mathbb{P}^1$ by their Weierstrass form. Let us denote by V_N the vector space $\Gamma(\mathbb{P}^1, O_{\mathbb{P}^1}(N))$.

Corollary (2.5). *Let $\tilde{X} \xrightarrow{\tilde{p}} \mathbb{P}^1$ be a minimal elliptic surface over $\mathbb{P}^1$ with section S, contracting to the Weierstrass fibration $X \xrightarrow{p} \mathbb{P}^1$. Let $N = -\deg R^1_{\tilde{p}_*} O_X$. Then X is isomorphic to the closed subscheme of $\mathbb{P} = \mathbb{P}(O_{\mathbb{P}^1}(2N) \oplus O_{\mathbb{P}^1}(3N) \oplus O_{\mathbb{P}^1})$ defined by*

$$(2.6) \quad y^2 z = x^3 + Axz^2 + Bz^3$$

as in (2.2), *where $A \in V_{4N}$ and $B \in V_{6N}$. Moreover,*

(i) *$4A^3 + 27B^2 \in V_{12N}$ is not identically zero, and vanishes at $q \in \mathbb{P}^1 \Leftrightarrow$ the fibre $\tilde{X}_q$ is singular*

(ii) *For every $q \in \mathbb{P}^1$, either $v_q(A) \leqq 3$ or $v_q(B) \leqq 5$.*

(iii) *Every pair of forms $(A, B) \in V_{4N} \oplus V_{6N}$ satisfying* (i) *and* (ii) *define, via* (2.6), *a Weierstrass fibration X with only rational double points, which resolves to a minimal elliptic surface $\tilde{X} \xrightarrow{\tilde{p}} \mathbb{P}^1$ with $R^1_{\tilde{p}_*} O_{\tilde{X}} \cong O_{\mathbb{P}^1}(-N)$.*

Proof. The only serious point to notice is that condition (ii) is equivalent to the Weierstrass fibration X defined by (2.6) having only rational double points; this is well-known (see for example [K]). QED.

Let $T_N \subset V_{4N} \oplus V_{6N}$ be the open set of those pairs of forms (A, B) satisfying (i) and (ii) above. Given the Weierstrass fibration X as above, the pair (A, B) is unique up to isomorphism, in the following sense. The multiplicative group k^* acts on T_N by $(\lambda, (A, B)) \rightsquigarrow (\lambda^{4N}A, \lambda^{6N}B)$. The group $\mathrm{SL}(V_1)$ acts on T_N in the obvious manner; $V_K = \mathrm{Symm}^K V_1$. These actions commute and hence define an action of $k^* \times \mathrm{SL}(V_1)$ on T_N.

Proposition (2.7). *Two pairs of forms in T_N give rise to isomorphic Weierstrass fibrations $\Leftrightarrow$ they are in the same orbit of the action of $k^* \times \mathrm{SL}(V_1)$.*

Proof. Again, this is well-known; the action of $\mathrm{SL}(V_1)$ corresponds to changes of coordinates in $\mathbb{P}^1$, and that of k^* corresponds to admissible changes of coordinates in the $\mathbb{P}^2$-bundle. QED.

Corollary (2.8). *The set of isomorphism classes of minimal elliptic surfaces $X \xrightarrow{\tilde{p}} \mathbb{P}^1$ (or, equivalently, of Weierstrass fibrations $X \xrightarrow{p} \mathbb{P}^1$ with only rational double points) with $\deg R^1_{\tilde{p}_*} O_{\tilde{X}} = -N$ and with fixed section is in $1-1$ correspondence with the set of orbits $T_N/k^* \times \mathrm{SL}(V_1)$.*

The central problem with which we are concerned is the following: to what extent can a geometric structure be put on the above set of orbits? In particular, can one construct a coarse moduli space (in the sense of [GIT]) for Weierstrass fibrations? For this purpose we will use the techniques of geometric invariant theory. To discuss the problem geometrically, we will make the standard identification of elements of $V_{4N} \oplus V_{6N}$ with the geometric points of the scheme $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*) = \mathrm{Spec}(S(V_{4N}^* \oplus V_{6N}^*))$; needless to say, this scheme (and the open subscheme defined by the conditions (i) and (ii) of Corollary (2.5), which we shall also call T_N] inherits the action of the group $k^* \times \mathrm{SL}(V_1)$.

3. The Construction of T_N/k^*

Since the actions of k^* and $\mathrm{SL}(V_1)$ on T_N commute, $\mathrm{SL}(V_1)$ acts on the set of orbits T_N/k^* and $T_N/k^* \times \mathrm{SL}(V_1) = (T_N/k^*)/\mathrm{SL}(V_1)$. We will therefore try to construct the desired quotient space in two stages; the first is the construction of T_N/k^*. This can be done by mimicking the constriction of $\mathbb{P}^N$ as a quotient of $\mathbb{A}^{N+1} \backslash \{0\}$ under the usual equivalence. Since k^* acts with different weights on the subspaces $\mathbb{V}(V_{4N}^*)$ and $\mathbb{V}(V_{6N}^*)$ of $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*)$, there are some differences. These can be avoided by the following construction.

Let $S^n(W)$ denote the n^{th} symmetric power of a vector space W, so that $S(W) = \bigoplus_{n \geq 0} S^n(W)$ is the symmetric algebra of W. The natural inclusions $S^3 V_{4N}^* \hookrightarrow S(V_{4N}^*)$ and $S^2(V_{6N}^*) \hookrightarrow S(V_{6N}^*)$ induce a k-linear map $S^3(V_{4N}^*) \oplus S^2(V_{6N}^*) \to S(V_{4N}^*) \otimes_k S(V_{6N}^*) \cong S(V_{4N}^* \oplus V_{6N}^*)$ which extends to the map $S(S^3(V_{4N}^*) \oplus S^2(V_{6N}^*)) \to S(V_{4N}^* \oplus V_{6N}^*)$. This map of k-algebras induces $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*) \xrightarrow{f} \mathbb{V}(S^3(V_{4N}^*) \oplus S^2(V_{6N}^*))$. The action of an element λ of k^* on $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*)$ is given by multiplication by λ^{4N} on the subspace $\mathbb{V}(V_{4N}^*)$ and by multiplication by λ^{6N} on the subspace $\mathbb{V}(V_{6N}^*)$; the action of λ on $\mathbb{V}(S^3(V_{4N}^*) \oplus S^2(V_{6N}^*))$ is given by multiplication by λ^{12N}.

Proposition (3.1). *The morphism f is finite onto its image, and commutes with the actions of k^* defined above. Moreover, each fibre of f is a subset of an orbit of k^*.*

Proof. It follows directly from the fact that the map

$$S(S^i(U)) \to S(U), \qquad \text{for } U \text{ a vector space over } k,$$

is a finite map of k-algebras, that the morphism f is finite. Moreover, if the action of $\lambda \in k^*$ on U is given by multiplication by λ^K, then the action of λ on $S^i(U)$ is multiplicative by λ^{iK}; this implies that f commutes with the given actions. Finally, if 2 points of $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*)$ are represented by the pairs of forms (A_1, B_1) and (A_2, B_2), then these points are in the same fibre of f is and only if $A_1^3 = A_2^3$ and $B_1^2 = B_2^2$. Letting λ be a primitive $12N^{\text{th}}$ root of unity in k, one sees that if this condition holds then the two pairs of forms differ by some power of λ. QED.

Define Z_N to be the image of f. Z_N is a closed subvariety of $\mathbb{V}(S^3(V_{4N}^*) \oplus S^2(V_{6N}^*))$ and, by Proposition (3.1), two points of $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*)$ are in the same orbit of k^* if and only if their images Z_N are. We have thus reduced the problem to the construction of Z_N/k^*. Since 0 is in the closure of every orbit we can only hope to construct the space $Z_N \backslash \{0\}/k^*$; this variety is obtained easily by projecting $Z_N \backslash \{0\}$ into $\mathbb{P}(S^3 V_{4N}^* \otimes S^2 V_{6N}^*)$, which is the quotient of $\mathbb{V}(S^3 V_{4N}^* \oplus S^2 V_{6N}^*) \backslash \{0\}$ by the given action of k^*. Define M_N to be the quotient variety of $Z_N \backslash \{0\}$ constructed above. $\mathrm{SL}(V_1)$ acts on M_N in the natural manner [it is an $\mathrm{SL}(V_1)$-invariant subvariety of $\mathbb{P}(S^3 V_{4N}^* \oplus S^2 V_{6N}^*)$]. We have proved the

Proposition (3.2). *Two points of $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*) \backslash \{0\}$ are in the same orbit of the action of $\mathrm{SL}(V_1) \times k^*$ if and only if their images in M_N are in the same orbit of the action of $\mathrm{SL}(V_1)$ on M_N.*

Define $E_N \subset M_N$ to be the image of T_N under the quotient map $\mathbb{V}(V_{4N}^* \oplus V_{6N}^*) \backslash \{0\} \to M_N$. $E_N \cong T_N/k^*$ and is a parameter space for those Weierstrass fibrations with a smooth generic fibre and $p_g = N - 1$.

4. The Construction of $M_N/\mathrm{SL}(V_1)$

The construction of the quotient of M_N by the action of $\mathrm{SL}(V_1)$ is performed using the techniques of geometric invariant theory (we refer the reader to [GIT] for the definitions of stability and semi-stability). Let $M_{N,s}$ (respectively $M_{N,ss}$) be the invariant open subvariety of stable (resp. semi-stable) points of M_N. Let R_N be the homogeneous coordinate ring of M_N, and let $\bar{W}_N = \operatorname{Proj} R_N^{\mathrm{SL}(V_1)}$. By geometric invariant theory, we have a diagram

$$\begin{tikzcd}
M_{N,s} \arrow[r, hook] \arrow[d, "\pi_N"'] & M_{N,ss} \arrow[r, hook] \arrow[d, "\bar{\pi}_N"] & M_N \\
W_N \arrow[r, hook] & \bar{W}_N ,
\end{tikzcd}$$

where W_N is open in $\bar{W}_N$ and the square is cartesian. Moreover, the maps π_N and $\bar{\pi}_N$ have the following properties:

(4.1) For $x, y \in M_{N,s}$, $\pi_N(x) = \pi_N(y)$ if and only if x and y are in the same orbit of $\mathrm{SL}(V_1)$.

(4.2) For $x, y \in M_{N,ss}$, $\bar{\pi}_N(x) = \bar{\pi}_N(y)$ if and only if the closure of the orbits of x and y intersect in $M_{N,ss}$.
The variety W_N is a geometric quotient of $M_{N,s}$ by the action of $\mathrm{SL}(V_1)$.

The non-complete variety W_N is the optimal solution to the problem of constructing a quotient; it is a true orbit space for those orbits contained in $M_{N,s}$. The projective variety $\bar{W}_N$ is not an orbit space, and should be viewed as a compactification of W_N. Three questions remain:

(4.3) What are explicit conditions [in terms of the pair of forms (A,B)] for stability?

(4.4) Are points of M_N which represent elliptic surfaces (i.e., those points of E_N) stable, or semi-stable?

(4.5) What are explicit conditions for two distinct orbits in $M_{N,ss} \backslash M_{N,s}$ to be in the same fibre of the map $\bar{\pi}_N$?

Before addressing these questions, let us remark that the dimension of $\bar{W}_N$ can be computed as $\dim \mathbb{V}(V_{4N}^* \oplus V_{6N}^*) - \dim(\mathrm{SL}(V_1) \times k^*)$, anticipating the existence of stable points. This gives $\dim \bar{W}_N = 10N - 2$.

5. Conditions for Stability and Semi-Stability

Let (A,B) be a pair of forms in $V_{4N} \oplus V_{6N}$ which represent the point r of M_N. For any point q in $\mathbb{P}^1$, let $v_q(A)$ and $v_q(B)$ denote the order of vanishing of A and B, respectively, at q. With this notation we can state the

Proposition (5.1). (a) *The point r of M_N represented by the pair (A,B) is not semi-stable if and only if there is a point $q \in \mathbb{P}^1$ such that*

$$v_q(A) > 2N \quad \text{and} \quad v_q(B) > 3N.$$

(b) *The point r is not stable if and only if there is a point $q \in \mathbb{P}^1$ such that*

$$v_q(A) \geqq 2N \quad \text{and} \quad v_q(B) \geqq 3N.$$

Proof. (a) Assume r is not semi-stable. The numerical criterion for semi-stability (see [GIT]) then states that there exists a non-trivial one-parameter subgroup f: $k^* \to \mathrm{SL}(V_1)$ of $\mathrm{SL}(V_1)$ such that the weights of r with respect to f are all positive. Let us explicitly compute these weights by choosing a basis T_0, T_1 of V_1 so that the action of $f(\lambda)$ on V_1 is given diagonally by $f(\lambda) \cdot T_0 = \lambda^e T_0$, $f(\lambda) \cdot T_1 = \lambda^{-e} T_1$ for all $\lambda \in k^*$, and some $e > 0$. Then $f(\lambda)$ acts on forms in V_{4N}^* and V_{6N}^* by sending

$$\sum_{i=0}^{4N} a_i T_0^i T_1^{4N-i} \quad \text{to} \quad \sum_{i=0}^{4N} a_i \lambda^{2ei-4eN} T_0^i T_1^{4N-i}$$

and

$$\sum_{j=0}^{6N} b_j T_0^j T_1^{6N-j} \quad \text{to} \quad \sum_{j=0}^{6N} b_j \lambda^{2ej-6eN} T_0^j T_1^{6N-j},$$

thereby sending the coordinates

$$a_i \quad \text{to} \quad \lambda^{2ei-4eN} a_i$$

and

$$b_j \quad \text{to} \quad \lambda^{2ej-6eN} b_j.$$

From the construction of the space M_N, coordinates in M_N are $a_i a_j a_k$ for i, j, and $k \leqq 4N$ and $b_l b_m$ for l and $m \leqq 6N$; these are the coordinates of $S^3(V_{4N})$ and $S^2(V_{6N})$. Assume that in the chosen basis the point r has coordinates $[a_i a_j a_k; b_l b_m]$. Then $f(\lambda)$ acts on M by sending such a point r to

$$[\lambda^{2e(i+j+k)-12eN} a_i a_j a_k; \lambda^{2e(l+m)-12eN} b_l b_m].$$

The weights of r with respect to f are those exponents

$$2e(i+j+k)-12eN \quad \text{for which} \quad a_i a_j a_k \neq 0,$$

and

$$2e(l+m)-12eN \qquad \text{for which} \quad b_l b_m \neq 0.$$

Since r is not semi-stable, all these weights are positive; i.e.,

$$(5.2) \quad \begin{array}{l} \text{whenever } i+j+k-6N \leqq 0,\ a_i a_j a_k = 0, \quad \text{and} \\ \text{whenever } l+m-6N \leqq 0,\ b_l b_m = 0. \end{array}$$

In particular, $a_n^3 = 0$ for $n \leqq 2N$ and $b_n^2 = 0$ for $n \leqq 3N$, or

$$(5.3) \quad a_n = 0 \quad \text{for} \quad n \leqq 2N \quad \text{and} \quad b_n = 0 \quad \text{for} \quad n \leqq 3N.$$

Thus if q is the point of $\mathbb{P}^1$ with coordinates $[T_0, T_1] = [0, 1]$, $v_q(A) > 2N$ and $v_q(B) > 3N$.

Conversely, if q is a point of $\mathbb{P}^1$ satisfying the vanishing conditions of (5.1)(a), choose a basis $[T_0, T_1]$ of V_1 so that q is the point $[0, 1]$ in these coordinates. To show that the point r represented by (A, B) is then not semi-stable, it suffices to find a one-parameter subgroup f of $\mathrm{SL}(V_1)$ so that the weights of r with respect to f are all positive. The reader may check that such an f is given by $f(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where the matrix acts on V_1 in the usual manner with the chosen basis $\{T_0, T_1\}$.

(b) Assume r is not stable. The numerical criterion for stability (see [GIT]) then states that there is a non-trivial one-parameter subgroup f of $\mathrm{SL}(V_1)$ such that the weights of r with respect to f are all non-negative. Again choose a basis $\{T_0, T_1\}$ of V_1 so that f is diagonalized as above; let this basis induce the coordinates $[a_i a_j a_k; b_l b_m]$ for r. The same computation of the weights in (a) implies, under the assumption that r is not stable, that

$$(5.4) \quad \begin{array}{l} \text{whenever } i+j+k-6N < 0,\ a_i a_j a_k = 0, \quad \text{and} \\ \text{whenever } l+m-6N < 0,\ b_l b_m = 0. \end{array}$$

In particular, $a_n = 0$ for $n < 2N$ and $b_n = 0$ for $n < 3N$. Thus if q is the point $[0, 1]$ of $\mathbb{P}^1$ in these coordinates, $v_q(A) \geqq 2N$ and $v_q(B) \geqq 3N$. The argument to prove the converse is identical to that used in part (a) and will be omitted. QED.

From Proposition (5.1) we can draw one immediate conclusion, addressing the question (4.4).

Corollary (5.5). *For $N \geqq 2$, $E_N \subset M_{N,s}$ and thus $\pi_N(E_N)$ is a coarse moduli space for Weierstrass fibrations with fixed section, smooth generic fibre, and $p_g = N-1$.*

Proof. If $(A,B) \in T_N$, then $v_q(A) \leqq 3$ or $v_q(B) \leqq 5$, for every $q \in \mathbb{P}^1$. Hence if $N \geqq 2$, the condition of Proposition (5.1)(b) for the point r of E_N represented by (A,B) to be not stable is not satisfied. Hence r is stable. QED.

We will not address the problem of describing W_N or the compactification $\bar{W}_N$ for $N \geqq 2$ more fully in this paper. Let us simply remark that when $N=2$, the condition for stability of the point r given by Proposition 5.1(b) is equivalent to the condition that $v_q(A) \leqq 3$ or $v_q(B) \leqq 5$ for all $q \in \mathbb{P}^1$. However, it is not true that $E_2 = M_{2,s}$. Consider the example

$$A(T_0, T_1) = -3(T_0)^2(T_0 - T_1)^2(T_0 + T_1)^2(T_1)^2$$

$$B(T_0, T_1) = 2(T_0)^3(T_0 - T_1)^3(T_0 + T_1)^3(T_1)^3$$

of a pair of forms in $V_8 \oplus V_{12}$. The conditions for stability are met, but the discriminant $4A^3 + 27B^2$ is identically 0, and so the point of $M_{2,s}$ represented by (A,B) is not in E_2. This case $(N=2)$ of elliptic $K3$ surfaces has been studied by Jayant Shah and a detailed discussion appears in [S].

By Corollary (2.4), the case $N=1$ occurs precisely when the surface X is a rational surface. We will call such a fibration $X \xrightarrow{p} \mathbb{P}^1$ a *rational Weierstrass fibration*, and the associated elliptic surface $\tilde{X} \xrightarrow{\tilde{p}} \mathbb{P}^1$ a *rational elliptic surface with section*. We note that the pathology exhibited above for the $N=2$ case does not occur for rational Weierstrass fibrations:

Proposition (5.6). *Assume that the point $r \in M_1$, represented by the pair (A,B), is stable. Then the rational Weierstrass fibration X defined by $y^2z = x^3 + Axz^2 + Bz^3$ has a smooth generic fibre.*

Proof. Assume X has a singular generic fibre. Then the discriminant $4A^3 + 27B^2 \in V_{12}$ is identically 0. If we choose an affine coordinate t in $\mathbb{P}^1$, we may write A as a quartic polynomial and B as a sextic polynomial in t. Factor A and B into linear factors:

$$A(t) = a \prod_{i=1}^{4} (t - c_i), \qquad B(t) = b \prod_{i=1}^{6} (t - d_i).$$

Since $4A^3 + 27B^2$ is identically 0, we must have

$$4a^3 \prod_{i=1}^{4} (t - c_i)^3 = -27b^2 \prod_{i=1}^{6} (t - d_i)^2.$$

By comparing the order of vanishing of the two sides of this equation at the c_i's and d_i's one readily deduces that

$$A(t) = a(t-g)^2(t-h)^2 \quad \text{and} \quad B(t) = b(t-g)^3(t-h)^3$$

for some $g, h \in k$, and with $4a^3 + 27b^2 = 0$. Hence at the point $t = g$, for example, A vanishes to order 2 and B to order 3; thus r is not stable by Proposition (5.1)(b). QED.

The above proposition assures us that a stable point of M_1 actually represents an *elliptic* surface; this is not the case for semi-stable points. For example, the Weierstrass fibration $y^2z=x^3-3t^2xz^2+2t^3z^3$ is represented by a semi-stable point of M_1, but the discriminant vanishes identically. However, not all elliptic rational Weierstrass fibrations are stable; the reader can check that the surface $y^2z=x^3+t^3xz^2+t^4z^3$ has a smooth generic fibre, but Proposition (5.1) implies that it is not even semi-stable. Let us turn to the problem of characterizing geometrically those rational Weierstrass fibrations which are stable and semi-stable.

6. The Geometric Characterization of Stability

To characterize the stability of a rational Weierstrass fibration in terms of the singular fibres of the associated elliptic surface $\tilde{X}$, we must determine, given a pair of forms (A,B) defining $\tilde{X}$, and a point $q\in\mathbb{P}^1$, what the fibre X_q is. If the discriminant

$$D=4A^3+27B^2$$

is non-zero at q, then X_q is a smooth elliptic curve whose J-invariant is given by

$$J=4A^3/D.$$

However, if $D=0$ at q, the fibre of $\tilde{X}$ over q is a singular curve of arithmetic genus one. Kodaira [Ko] has classified all such fibres and we shall adopt his notation for them. Neron [N] has taken Kodaira's classification one step further and has shown that the singular fibre $\tilde{X}_q$ depends only on the order of vanishing of A, B, and D at q; moreover, he calculates explicitly, given $v_q(A)$, $v_q(B)$, and $v_q(D)$, the fibre $\tilde{X}_q$. We reproduce these results in Table (6.1). In column 1 appears Kodaira's name for the singular fibre, and in column 2 appears a self-explanatory graph of the components of the singular fibre, with multiplicities indicated if unequal to 1. Columns 3, 4, and 5 contain the orders of vanishing of A, B, and D at q. In column 6 the value of the J-invariant appears, and the multiplicity $m(J)$ to which J takes on this value is in column 7.

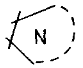
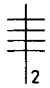
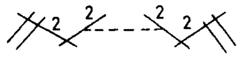
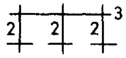
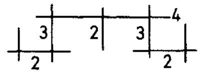
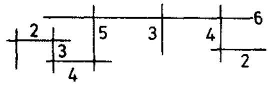
A quick inspection of Table (6.1) now gives desired characterization of stability.

Theorem (6.2). *Let r be a point of M_1 represented by the pair of forms (A,B). Let X be the rational Weierstrass fibration defined by (A,B). Then r is stable if and only if X has a smooth generic fibre and the associated elliptic surface $\tilde{X}$ has only reduced fibres.*

Proof. Assume r is stable. Proposition (5.6) then states that X has a smooth generic fibre. Using Proposition (5.1) and Table (6.1), we see that $\tilde{X}$ cannot have a fibre of type I_0^*, I_N^*, II^*, III^*, or IV^*; all the other fibres in Kodaira's list are reduced.

Conversely, if all fibres of $\tilde{X}$ are reduced, then Table (6.1) gives bounds on the simultaneous orders of vanishing of A and B; these bounds are precisely those insuring the stability of the point r, by Proposition 5.1. QED.

Table (6.1). Fibres of rational elliptic surfaces with section $y^2z=x^3+Axz^2+Bz^3$; $D=4A^3+27B^2$; $J=4A^3/D$

Name	Graph	$v_q(A)$	$v_q(B)$	$v_q(D)$	J	$m(J)$
I_0		0	0	0	$\neq 0, 1, \infty$	–
I_0		0	K	0	1	$2K$
I_0		L	0	0	0	$3L$
I_1		0	0	1	∞	1
I_N		0	0	N	∞	N
I_0^*		2	3	6	$\neq 0, 1, \infty$	–
		$L\geqq 3$	3	6	0	$3L-6$
		2	$K\geqq 4$	6	1	$2K-6$
I_N^*		2	3	$N+6$	∞	N
II		$L\geqq 1$	1	2	0	$3L-2$
III		1	$K\geqq 2$	3	1	$2K-3$
IV		$L\geqq 2$	2	4	0	$3L-4$
IV^*		$L\geqq 3$	4	8	0	$3L-8$
III^*		3	$K\geqq 5$	9	1	$2K-9$
II^*		4	5	10	0	2

Remark. A rational elliptic surface with section can also be constructed by taking a pencil of (generically smooth) cubics in $\mathbb{P}^2$ and blowing up the base points. If one analyses the stability of such pencils [under the action of $\mathrm{Aut}(\mathbb{P}^2)$] one obtains the same answer as the above for Weierstrass fibrations: stability depends only on the singular fibres of the associated elliptic surface. For details, see [M].

Inspired by the above theorem, let us adopt the following terminology.

Definition (6.3). Let $X \xrightarrow{p} \mathbb{P}^1$ be a rational Weierstrass fibration with a smooth generic fibre. We say that X is *stable* if the associated elliptic surface $\tilde{X}$ has only reduced fibres.

7. The Moduli Space for Stable Rational Weierstrass Fibrations

As noted in Sect. 4, the variety W_1 constructed there is an orbit space for the action of $\mathrm{SL}(V_1)$ on the stable points $M_{1,s}$ of M_1. Therefore, by Corollary (2.8) and Proposition (3.2), there is a $1-1$ correspondence between geometric points of W and isomorphism classes of stable rational Weierstrass fibrations. In this section we wish to investigate the role of the scheme structure of W_1; it should reflect properties of *families* of stable fibrations. To discuss this more precisely, let us make the following

Definition (7.1). A stable rational Weierstrass fibration over a k-scheme T is a T-map

$$X \xrightarrow{p} Q$$

together with a section $s: Q \to X$ of p such that

(i) Q is a $\mathbb{P}^1$-bundle over T,

(ii) the fibre $X_t \underset{s_t}{\overset{p_t}{\rightleftarrows}} Q_t$ is a stable rational Weierstrass fibration for every geometric point t in T.

The central property of W_1 which is implied by its structure as a scheme is given by the

Theorem (7.2). *Given a stable rational Weierstrass fibration over a k-scheme T, there is a unique morphism $T \to W_1$, extending the set-theoretic map sending a geometric point t of T to the point of W representing the isomorphism class of the fibre at t.*

Proof. As usual, it suffices to prove the theorem for T affine, $=\mathrm{Spec}R$. Let $X \xrightarrow{p} T \times \mathbb{P}^1$ be a stable rational Weierstrass fibration over T, with section $s: T \times \mathbb{P}^1 \to S \subset X$. Let $L = p_*[\mathcal{O}_X(S)/\mathcal{O}_X]$, and let the element (A,B) in $\Gamma(T \times \mathbb{P}^1, L^{\otimes -4}) \oplus \Gamma(T \times \mathbb{P}^1, L^{\otimes -6})$ define the fibration X, as in Theorem (2.1). Since the fibration is rational, we have $L_t \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ for every t in T by Corollary (2.4). Since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = (0)$, we have by base change $L \cong p_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$ where $p_2: T \times \mathbb{P}^1 \to \mathbb{P}^1$ is the natural projection. Hence the pair (A,B) can be considered as an element of the R-module $[\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \oplus \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))] \otimes_k R$ by the Kunneth formula. Thus the pair (A,B) induces a map $T \to \mathbb{V}(V_4^* \oplus V_6^*)$, which clearly has image in $\mathbb{V}(V_4^* \oplus V_6^*) \setminus \{0\}$. The assumption of stability implies that the map in fact has image in the inverse image of $M_{1,s}$ under the quotient map $\mathbb{V}(V_4^* \oplus V_6^*) \setminus \{0\} \to M_1$. Hence by composing with the quotient map $M_{1,s} \to W_1$ we define the desired map $T \to W_1$. If we choose a different pair (A', B') to define the fibration X, there will exist a T-valued point of $\mathrm{SL}(2) \times k^*$ sending (A,B) to (A',B'); and since W_1 is a geometric quotient of the inverse image of $M_{1,s}$ in $\mathbb{V}(V_4^* \oplus V_6^*)$, the pair (A',B') will induce the same map $T \to W_1$. QED.

Theorem (7.2) implies that the variety W_1 is a *coarse moduli space* for stable rational Weierstrass fibrations (see [GIT], Chap. 5, Sect. 1). Its compactification $\bar{W}_1$ constructed in Sect. (4) does not have the nice functorial properties which W has; nevertheless, it is interesting to see explicitly which fibrations are represented by points of $\bar{W}_1 \setminus W_1$. Let us then turn our attention to this subvariety, which we shall refer to as the *strictly semi-stable* points of M_1, (or, by abuse of notation, of $\mathbb{V}(V_4^* \oplus V_6^*)$).

8. The Strictly Semi-Stable Fibrations

Let us adopt the following notation: let W_{sss} denote the closed subvariety $\bar{W}_1 \setminus W_1$ of strictly semi-stable points of $\bar{W}_1$; similarly let M_{sss} and V_{sss} denote the inverse images of W_{sss} in M_1 and $\mathbb{V}(V_4^* \oplus V_6^*)$ respectively. As a corollary of Proposition 5.1, we have

Proposition (8.1). *The point r of M_1 represented by the pair (A, B) is in M_{sss} if and only if there is a point q of $\mathbb{P}^1$ such that either*

(i) $v_q(A) = 2$ and $v_q(B) \geqq 3$,

or

(ii) $v_q(A) \geqq 2$ *and* $v_q(B) = 3$.

It is easy to see from this description that V_{sss}, and hence both M_{sss} and W_{sss}, are irreducible schemes.

Using this characterization and Table (6.1) we have

Proposition (8.2). *Let r be a point of M_1 represented by the pair (A, B), and assume that the fibration X defined by (A, B) has a smooth generic fibre. Then r is in M_{sss} if and only if the associated elliptic surface $\tilde{X}$ has a fibre of type I_N^* for some $N \geqq 0$.*

To complete the description of the strictly semi-stable fibrations, we would like to analyse not only which fibrations are in V_{sss} or M_{sss}, which is described above, but also how the quotient map

$$V_{sss} \to M_{sss} \to W_{sss}$$

distinguishes between these fibrations. The answer is given by the following

Theorem (8.3). *Let (A_1, B_1) and (A_2, B_2) be two points of V_{sss} defining fibrations X_1 and X_2 which have smooth generic fibres. Let J_1 and J_2 be the values of the J-invariant of the singular fibres on X_1 and X_2 of type I_N^*, whose existence is insured by Proposition (8.2). Then (A_1, B_1) and (A_2, B_2) have the same image in W_{sss} if and only if $J_1 = J_2$.*

Remark. This theorem implies that surfaces with a fibre of type I_0^* are classified in W_1 by the finite J-value of that fibre; however, surfaces with a fibre of type I_N^* for $N \geqq 1$ are all mapped to one point of $\bar{W}_1$ (that corresponding to $J = \infty$). It is not hard to see that if the Weierstrass fibration has no smooth fibre, then it is also represented in $\bar{W}_1$ by the point of W_{sss} corresponding to $J = \infty$. Since the map $M_{1,ss} \to \bar{W}_1$ is surjective, the variety $\bar{W}_1$ can be stratified as follows:

$$\bar{W}_1 = W_1 \cup Y \cup \{\omega_\infty\},$$

where

(i) points of W_1 classify completely the stable rational Weierstrass fibrations up to isomorphism,

(ii) $Y \cong \mathbb{A}^1$, and classifies the fibrations with a smooth generic fibre and a singular fibre of type I_0^* by the J-value of that singular fibre.

(iii) ω_∞ represents all the fibrations with a smooth generic fibre and a singular fibre of type I_N^* with $N \geqq 1$, and also all the fibrations with no smooth fibres.

To return to the proof of the theorem, let us check that the numbers J_1 and J_2 are well-defined:

Proposition (8.4). *Let $\tilde{X}$ be a rational elliptic surface with section, containing a singular fibre of type I_N^* and one of type I_M^*. Then*

(i) $N=M=0$,

(ii) *there are no other singular fibres on $\tilde{X}$.*

(iii) *Every fibre on $\tilde{X}$ has the same J-invariant. We say that $\tilde{X}$ is a "$J=$constant" surface. Moreover,*

(iv) *Every rational elliptic surface with section with $J=$constant, not equal to* 0 *or* 1, *has two fibres of type I_0^* and no other singular fibres, and*

(v) *Two $J=$constant, not equal to* 0 *or* 1, *rational elliptic surfaces with section are isomorphic if and only if the J-values are equal.*

Proof. Since $\tilde{X}$ is rational, the discriminant $D=4A^3+27B^2$ is a form of degree 12. At the points q_1 and q_2 of $\mathbb{P}^1$ where the fibre is of type I_N^* and I_M^*, D vanishes to order $N+6$ and $M+6$ respectively, by Table (6.1). Thus $N=M=0$, and D vanishes only at q_1 and q_2; hence there are no other singular fibres. To prove (iii), choose an affine coordinate t of $\mathbb{P}^1$ such that q_1 and q_2 are $t=0$ and $t=\infty$. Represent the Weierstrass fibration X obtained from $\tilde{X}$ by the pair of forms $((A,B)$. By Table (6.1), $v_0(A)\geqq 2$, $v_\infty(A)\geqq 2$, $v_0(B)\geqq 3$, and $v_\infty(B)\geqq 3$. Since A is a quartic and B a sextic, we must have $A(t)=at^2$ and $B(t)=bt^3$ for some $a,b\in k$, not both 0. The J-value of the fibre over t is then $J(t)=4a^3/4a^3+27b^2$, independent of t.

To prove (iv) and (v), assume that on X, J is constant, unequal to 0 or 1. Since J is not 0 or 1, any pair of forms A and B representing X are not identically zero; hence we may choose an affine coordinate t of $\mathbb{P}^1$ such that at $t=\infty$ neither form is 0. Let $A(t)$ and $B(t)$ be the polynomials of degrees 4 and 6 respectively of the Weierstrass equation as above. We have

$$J=4A(t)^3/4A(t)^3+27B(t)^2$$

or

$$27JB(t)^2=(4-4J)A(t)^3$$

for this constant value of J. For the square of the sextic B to equal the cube of the quartic A up to a unit factor, one readily checks that there must be a quadratic polynomial $q(t)=u(t-r)(t-s)$ such that $A(t)=aq(t)^2$ and $B(t)=bq(t)^3$ for some non-zero a, b in k. From Table (6.1) it is clear now that X has a fibre of type I_0^* at $t=r$ and $t=s$, and no other singular fibres. Finally, assume that X_1 and X_2 are both $J=$constant, unequal to 0 or 1, rational elliptic surfaces with section. By (iv), we may represent X_1 and X_2 by the pair of forms (A_1,B_1) and (A_2,B_2), and if we choose an affine coordinate t such that the two singular fibres are at $t=0$ and $t=\infty$, we must have

$$A_1=a_1t^2, \quad B_1=b_1t^3$$

and

$$A_2=a_2t^2, \quad B_2=b_2t^3.$$

Assuming that the J-values of the two surfaces are equal, we have

$$4a_1^3/(4a_1^3+27b_1^2)=4a_2^3/(4a_2^3+27b_2^2)$$

or equivalently

$$(a_1/a_2)^3=(b_1/b_2)^2.$$

By choosing one of the 4th roots of a_1/a_2 appropriately, we obtain a λ such that

$$\lambda^4=a_1/a_2 \quad \text{and} \quad \lambda^6=b_1/b_2,$$

and hence by Proposition (2.7), X_1 and X_2 are isomorphic. QED.

Remark. The conspicuous omission of the J=constant, equal to 0 or 1, surfaces will be remedied in the next section.

Proof of (8.3). Assume first that $J_1=J_2$. Recall that two points of V_{sss} map to the same point of W_{sss} if and only if their orbits under the action of $\mathrm{SL}(V_1)\times k^*$ have common closure; we will exhibit this common closure as follows. Proposition (8.1) implies that we may choose a basis $[T_0, T_1]$ of V_1 inducing bases of V_4 and V_6 and coordinates $[a_ia_ja_k; b_lb_m]$ of M_1 such that

$$A_1=a_0T_1^4+a_1T_0T_1^3+a_2T_0^2T_1^2$$

and

$$B_1=b_0T_1^6+b_1T_0T_1^3+b_2T_0^2T_1^4+b_3T_0^3T_1^3$$

with a_2 and b_3 not both zero. The element

$$g_\lambda=\begin{bmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{bmatrix}$$

of SL(2) sends the pair (A_1,B_1) to the pair $(g_\lambda A_1, g_\lambda B_1)$ where

$$g_\lambda A_1=\lambda^4a_0T_1^4+\lambda^2a_1T_0T_1^3+a_2T_0^2T_1^2$$

and

$$g_\lambda B_1=\lambda^6b_0T_1^6+\lambda^4b_1T_0T_1^3+\lambda^2b_2T_0^2T_1^4+b_3T_0^3T_1^3.$$

Letting λ approach 0, we see that the point of V_{sss} represented by the pair $(a_2T_0^2T_1^2, b_3T_0^3T_1^3)$ is in the closure of the orbit of (A_1,B_1). Notice that this pair defines a J=constant surface Z_1, and $J_1=4a_2^3/4a_2^3+27b_3^2$. By making the same construction with the pair (A_2,B_2), we find that there is a point of V_{sss}, represented by the pair $(c_2T_0^2T_1^2, d_3T_0^3T_1^3)$, in the closure of the orbit of (A_2,B_2). Moreover, this pair represents a J=constant surface Z_2 and $J_2=4c_2^3/4c_2^3+27d_3^2$. If neither a_2 nor b_3 are 0, then J_1 is unequal to 0 or 1, and since $J_1=J_2$ neither is J_2. Hence by Proposition (8.4), (v), Z_1 is isomorphic to Z_2 and are in the same orbit; thus (A_1,B_1) and (A_2,B_2) have the same image in W_{sss}.

If $a_2=0$ and $b_3\neq0$, then $J_1=0$ and $J_1=0$ and therefore so does J_2. We infer that $c_2=0$ and now Z_1 and Z_2 are defined by the pairs $(0, b_3T_0^3T_1^3)$ and $(0, d_3T_0^3T_1^3)$, and are clearly isomorphic. A similar argument is made for the case $a_2\neq0$, $b_3=0$. This proves the sufficiency of the condition of the theorem.

Conversely, assume $J_1 \neq J_2$. As above, the point (A_1, B_1) of V_{sss} has a point of the form $(aT_0^2T_1^2, bT_0^3T_1^3)$ in the closure of its orbit; similarly the point (A_2, B_2) has a point $(cT_0^2T_1^2, dT_0^3T_1^3)$ in the closure of its orbit. Moreover,

$$J_1 = 4a^3/4a^3 + 27b^2 \quad \text{and} \quad J_2 = 4c^3/4c^3 + 27d^2.$$

It suffices to show that these two derived pairs do not map to the same point of W_{sss}; to do this, it suffices to show that the closures of their orbits are disjoint. As seen above, these two pairs represent surface with a constant J-invariant. Let N be the subvariety of V_{sss} consisting of all points representing surfaces with constant J-invariant. The set N is invariant under the action of $\mathrm{SL}(V_1) \times k^*$ and the J-value is an invariant function on N. Since $J_1 \neq J_2$, this function separates the two orbits in N. Hence their closures are disjoint. QED.

This completes our analysis of the strictly semi-stable fibrations.

9. The $J=0$, $J=1$ Surfaces

Let $\tilde{X}$ be a rational elliptic surface with section, and $J=$constant. If $J \neq 0$ or 1, Proposition (8.4)(iv), (v) states that $\tilde{X}$ has two fibres of type I_0^* and no other singular fibres; moreover, there is only one such surface up to isomorphism. It is represented in $\bar{W}_1$ by a point of W_{sss}, and two $J=$constant, $\neq 0$ or 1 surfaces with distinct J-values are represented by distinct points. Hence the points of W_{sss} representing such surfaces classify them completely.

Let us now turn to the case $J=1$. The surface $\tilde{X}$ is given here by a pair of forms (A, B) and B must be identically 0. The surface $\tilde{X}$ is then completely determined by the quartic form A and one can easily determine from Table (6.1) the singular fibres of $\tilde{X}$ given the order of vanishing of A [*see Table* (9.1)].

Table (9.1). $J \equiv 1$ surfaces

$v_q(A)$	Fibre $\tilde{X}_q$
0	Smooth elliptic
1	Type *III*
2	Type I_0^*
3	Type *III**

Table (9.2). $J \equiv 0$ surfaces

$v_q(B)$	Fibre $\tilde{X}_q$
0	Smooth elliptic
1	Type *II*
2	Type *IV*
3	Type I_0^*
4	Type *IV**
5	Type II*

We merely wish to remark that, unlike the other $J=$constant surfaces, there are $J \equiv 1$ surfaces which are stable; in face, by Proposition (5.1) we have that a $J \equiv 1$ surface $\tilde{X}$ is stable if and only if $\tilde{X}$ has exactly four fibres of type *III*.

Finally, let us perform the above analysis for $J=0$. Here $\tilde{X}$ will be given by a pair of forms (A, B), and A must be identically 0; thus $\tilde{X}$ depends only on the sextic form B and the singular fibres are determined by the order of vanishing of B [see Table (9.2)].

Again there are stable $J=0$ surfaces $\tilde{X}$; $\tilde{X}$ is stable if and only if $\tilde{X}$ has only singular fibres of types *II* and *IV*.

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Received November 4, 1980