An Overview of Algebraic Surfaces

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Preface

These notes are a faithful record of five lectures given at the Summer School on Algebraic Geometry, held at the Bilkent International Center for Advanced Studies of Bilkent University, in August of 1995. The intention of the lectures was to give a quick overview of the classification of algebraic surfaces. In the first lecture a brief digest of the general theory is given, concentrating on the major invariants for algebraic surfaces and on the standard theorems relating them. In the next three lectures the standard constructions of algebraic surfaces of special type are presented; rational and ruled surfaces in the second lecture, abelian, $K3$, Enriques, and hyperelliptic surfaces in the third lecture, and elliptic surfaces in the fourth lecture. In the last lecture the geography of surfaces of general type is discussed, along with some examples of particular constructions which are the most well-known to illustrate the situation. In general the approach is not at all systematic, with hardly any proofs given; my intention is only to display some of the highlights of the classification of algebraic surfaces, hoping to whet the reader's appetite for a more detailed study.

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Contents

1 Algebraic Surfaces: Basic Invariants

In this first section we present a quick digest of the general theory of algebraic surfaces, simply to fix notation and to set the stage. The basic references for algebraic surfaces would include [Z], [E], [S2], [Be], and [BPV]. There are also sections of [S1], [GH], and [Ha] which deal specifically with surfaces.

1.1 The Definition of an Algebraic Curve

We will work over the field of complex numbers $\mathbf{C}$. With this point of view an algebraic curve is, first of all, a Riemann surface, that is, a complex manifold of dimension one. Being a complex manifold means that at every point there is a local complex coordinate z, and the change-of-coordinate functions are holomorphic. Hence if both z and w are local complex coordinates in the neighborhood of a point p, then $z = T(w)$ and $w = S(z)$ near p, where T and S are (inverse) holomorphic functions.

For a compact Riemann surface X to be an algebraic curve, its field of global meromorphic functions $\mathcal{M}(X)$ must be sufficiently rich. To be precise:

Definition 1.1 *A compact Riemann surface X is an* algebraic curve *if and only if*

(a) for every two points p, q on X there is a global meromorphic function f on X having different values at p and q, and

(b) for every point p on X there is a global meromorphic function f on X such that f is a local coordinate at p.

The first condition is usually referred to as "separating points"; the second as "separating tangents".

It is a basic theorem that *every* compact Riemann surface is an algebraic curve. The field of functions $\mathcal{M}(X)$ is an extension field of $\mathbf{C}$ of transcendence degree one.

1.2 The Definition of an Algebraic Surface

We take our definition of an algebraic surface directly from that of an algebraic curve. Let X be a compact connected complex manifold of dimension two; this means that at each point of X there are two local complex coordinates (z, w) and change-of-coordinate maps are holomorphic. Let $\mathcal{M}(X)$ be the field of global meromorphic functions on X.

Definition 1.2 *X is an* algebraic surface *if and only if the field $\mathcal{M}(X)$ separates points and tangents, i.e.,*

(a) for every two points p, q on X there is a global meromorphic function f on X having different values at p and q, and

(b) for every point p on X there are global meromorphic functions f and g on X such that (f, g) are local coordinates at p.

This implies rather readily that the field of functions $\mathcal{M}(X)$ has transcendence degree two; the converse is true, but only for surfaces: there are counterexamples in dimension three and higher.

The compactness of X implies that in fact there are finitely many meromorphic functions $f_1, \ldots, f_n$ which separate points and tangents. We may use these functions to map X to projective space $\mathbf{P}^n$, via the mapping $x \to [1 : f_1(x) : \cdots : f_n(x)]$; then X becomes embedded in $\mathbf{P}^n$. By Chow's theorem X is an algebraic subset of $\mathbf{P}^n$, i.e., defined by the vanishing of a set of homogeneous polynomials. Therefore an algebraic surface in the above sense is indeed a projective algebraic surface.

It is trivial that every smooth projective algebraic surface is algebraic in the above sense: the ratios of the homogeneous coordinates on $\mathbf{P}^n$ restrict to meromorphic functions on X which separate points and tangents.

One of the basic examples of projective algebraic surfaces are the *hypersurfaces* in $\mathbf{P}^3$, defined by the vanishing of a single homogenous polynomial in the four projective coordinates. More generally, one has *complete intersections* in $\mathbf{P}^n$, by taking the common zeroes of a set of $n-2$ homogeneous polynomials, whose zero locus is a surface. If the polynomials have degrees $(d_1, \ldots, d_{n-2})$, then such a surface is said to be a *complete intersection of type* $(d_1, \ldots, d_{n-2})$.

1.3 Functions and Forms

The concept of a differentiable function or a meromorphic function on an algebraic surface is straightforward. That of a form may be less familiar; let us briefly review it.

Suppose that (z, w) are local coordinates on X. If we write $z = x + iy$ and $w = u + iv$ then (x, y, u, v) become real coordinates on X. A differentiable 1-form on X is, locally, an expression of the form

$$f_1(x,y,u,v)dx + f_2(x,y,u,v)dy + f_3(x,y,u,v)du + f_4(x,y,u,v)dv$$

where the f_i are differentiable functions. Upon changes of coordinates such an expression transforms to another such, and a global 1-form can be thought of as a collection of such expressions, one for every choice of local coordinates, all of which transform to one another. It is sufficient to give a set of such expressions, at least one of which are valid at each point of X.

It is more useful to substitute the variables z, $\overline{z}$, w, and $\overline{w}$ for the 4 real variables; the obvious relationships between these four and x, y, u, v make the concepts completely equivalent. In this notation then a 1-form is a collection of compatible expressions of the form

$$g_1(z, \overline{z}, w, \overline{w})dz + g_2 d\overline{z} + g_3 dw + g_4 d\overline{w}$$

with the g_i's differentiable.

Note that under holomorphic changes of coordinates, say to new complex coordinates (s, t), the dz and dw terms transport to the ds and the dt terms, while the $d\overline{z}$ and $d\overline{w}$ terms transport to the $d\overline{s}$ and the $d\overline{t}$ terms. Therefore the 1-form written above naturally decomposes *globally* into the dz, dw part (called the $(1,0)$ part) and the $d\overline{z}, d\overline{w}$ part (called the $(0,1)$ part).

Higher forms are defined as sets of compatible expressions which are linear combinations of terms like

$$f(z, \overline{z}, w, \overline{w}) da \wedge \ldots db$$

where f is differentiable and the expressions $da, \ldots, db$ stand for any one of the differentials dz, $d\overline{z}$, dw, and $d\overline{w}$. The basic exterior algebra laws that $da \wedge da = 0$ and $da \wedge db = -db \wedge da$ hold. A term like this has a *type* (p, q), where there are p dz and dw's, and q $d\overline{z}$ and $d\overline{w}$'s. The possible types are restricted by $0 \leq p, q, \leq 2$ of course; the form above is then called a $(p+q)$-form, and for the same reasons as above any n-form decomposes naturally into its (p, q) parts, for all p, q with $p + q = n$.

For example, a 2-form is locally an expression of the form

$$\begin{gathered} f(dz \wedge dw) \\ +g_1(dz \wedge d\overline{z}) + g_2(dz \wedge d\overline{w}) + g_3(dw \wedge d\overline{z}) + g_4(dw \wedge d\overline{w}) \\ +h(d\overline{z} \wedge d\overline{w}) \end{gathered}$$

and the first line is the $(2, 0)$ part, the second line is the $(1, 1)$ part, and the third line is the $(0, 2)$ part.

Let $\mathcal{E}^n$ denote the sheaf of differentiable n-forms on X $(0 \leq n \leq 4)$; denote by $\mathcal{E}^{p,q}$ the sheaf of $(p+q)$-forms of type (p, q). The global sections H^0 of these sheaves are the global forms.

In addition to these differentiable forms, there are the holomorphic and meromorphic forms. These forms, always of type $(p, 0)$, have holomorphic (respectively meromorphic) functions as coefficients. The sheaf of holomorphic p-forms is denoted by Ω^p; Ω^0 is more commonly denoted by $\mathcal{O}$, the sheaf of holomorphic functions. The sheaf of meromorphic two-forms is denoted $\mathcal{M}^2$.

To be more explicit, a section of $\mathcal{O} = \Omega^0$ is a holomorphic function; a section of Ω^1 is a 1-form of the form $f(z, w)dz + g(z, w)dw$ with f and g holomorphic; and a section of Ω^2 is a 2-form of the form $f(z, w)dz \wedge dw$ with f holomorphic. A section of clM^2 is a 2-form of the form $f(z, w)dz \wedge dw$ with f meromorphic.

1.4 Divisors

A divisor on X is locally defined by a single nonzero meromorphic function f; one thinks of the divisor as the locus $f = 0$, but when f has multiple roots this is taken into account also. Two such meromorphic functions locally define the same divisor when their ratio has no zero or poles.

This then is how divisors are globalized: one takes a meromorphic function $f(z, w)$ for every choice of local coordinates (z, w), and requires that in an open set where two choices of local coordinates exist, the ratio of the given functions has no zeros or poles.

Of course it is sufficient to give a meromorphic function on each open set of an open covering of the surface, which satisfy the ratio condition.

With this definition, two collections of local meromorphic functions form the same divisor if and only if their union is a valid divisor, that is, each function of one set agrees (up to multiplication by a nowhere zero holomorphic function) with each function of the other set, where they are both defined.

Divisors form an abelian group Div(X), by multiplying the corresponding functions; the group operation on divisors is usually denoted by addition however.

Divisors are used to organize meromorphic functions and forms, by serving as bounds on the possible poles. If D is a divisor on X, locally defined by f, then a term $g(z,w)$ or $g(z,w)dz$ or $g(z,w)dz \wedge dw$ of a p-form is said to *have poles bounded by* D if the product fg is holomorphic.

We may then introduce the sheaves $\Omega^p[D]$ of meromorphic p-forms with all terms having poles bounded by D; as above, it is more common to write $\mathcal{O}[D]$ for $\Omega^0[D]$.

The zeroes and poles of a meromorphic function always have codimension one; therefore on a surface they form curves. A divisor D then gives rise to a finite formal sum $\sum_i n_i C_i$ where the C_i are curves on X and the n_i are integers, representing the order of the local function along the curve C_i. This representation is called a *Weil divisor*; on a smooth surface, there is a 1-1 correspondence between Weil divisors and divisors. When there is any possibility of confusion, ordinary divisors are called *Cartier divisors*.

If f is a *global* meromorphic function, then in every local coordinate one can take the function f; this gives *divisor of* f, denoted by div(f). We note that $\mathrm{div(fg)} = \mathrm{div(f)} + \mathrm{div(g)}$ and $\mathrm{div(1/f)} = -\mathrm{div(f)}$. Such divisors are called *principal divisors*, and form a subgroup of the group Div(X) of all divisors.

Two divisors are said to be *linearly equivalent* if their difference is principal; this equivalence relation is simply congruence modulo the subgroup of principal divisors.

Cohomology does not "see" linear equivalence, in the sense that if $D_1 = D_2 + \mathrm{div(f)}$, then multiplication by f induces natural isomorphisms between $H^i(\Omega^p[D_1])$ and $H^i(\Omega^p[D_2])$.

The *Picard group* Pic(X) is the group of divisors modulo linear equivalence. It may be identified with the first cohomology group $H^1(\mathcal{O}^*)$ of the sheaf of nowhere zero holomorphic functions; given a divisor, the ratios of the local functions give a 1-cocycle for this sheaf. Changing local coordinates on an open set changes the cocycle by a coboundary.

If X is a projective surface, lying in $\mathbf{P}^n$, then if one takes a hyperplane H defined by a linear polynomial, this polynomial may be taken as the numerator of a local defining equation for a divisor on X, where any nonzero denominator is chosen. If one changes the hyperplane one obtains a linearly equivalent divisor, so that the linear equivalence class is unique; this class is called the *hyperplane class*.

Divisors are rather well-behaved for maps $\pi : X \to Y$ of algebraic surfaces; if D is a divisor on Y, then one defines $\pi^*(D)$ as a divisor on X by taking as the local function at a point $p \in X$ to be the composition $f \circ \pi$, where f is the local function for D at the image $\pi(p)$. The divisor $\pi^*(D)$ is called the *pullback* of D.

1.5 The Canonical Class and the Plurigenera

Let ω be a meromorphic 2-form, which is therefore locally of the form $f(z,w)dz \wedge dw$. Suppose that one changes coordinates, to (s,t), where $z = z(s,t)$ and $w = w(s,t)$. Then

$$dz = (\partial z/\partial s)ds + (\partial z/\partial t)dt$$

and

$$dw = (\partial w/\partial s)ds + (\partial w/\partial t)dt$$

so that

$$dz \wedge dw = (\frac{\partial z}{\partial s}\frac{\partial w}{\partial t} - \frac{\partial z}{\partial t}\frac{\partial w}{\partial s})ds \wedge dt$$

and this coefficient, being the Jacobian J of the change of coordinates map, is nowhere zero and holomorphic. Therefore the coefficient function $f(z, w)$ is transformed to fJ, and these two meromorphic functions define the same divisor locally. Hence we may define the *divisor* K of ω by taking this coefficient function in any local expression for ω, and since this is well-defined up to multiplying by a nowhere zero holomorphic function, the divisor is well-defined; it is denoted by $\mathrm{div}(\omega)$. We have

$$\mathrm{div}(\mathrm{f}\omega) = \mathrm{div}(\mathrm{f}) + \mathrm{div}(\omega)$$

for a meromorphic function f and a meromorphic 2-form ω. Such a divisor is called a *canonical divisor* on X.

Note that if ω_1 and ω_2 are nonzero meromorphic 2-forms on X, then there is always a global meromorphic function f on X such that $\omega_1 = f\omega_2$. Therefore

$$\mathrm{div}(\omega_1) = \mathrm{div}(\mathrm{f}\omega_2) = \mathrm{div}(\mathrm{f}) + \mathrm{div}(\omega_2)$$

so that the two canonical divisors $\mathrm{div}(\omega_1)$ and $\mathrm{div}(\omega_2)$ are linearly equivalent. Therefore the linear equivalence class of canonical divisors is well-defined; this is called the *canonical class* of divisors, and is usually denoted by K (as is any particular canonical divisor).

The *geometric genus* of the surface X is the dimension of the space $H^0(\Omega^2)$ of global holomorphic 2-forms on X. If $K = \mathrm{div}(\omega)$ is a canonical divisor, and f is any global meromorphic function with poles bounded by K, then $f\omega$ is holomorphic. This gives an isomorphism between $H^0(\mathcal{O}[K])$ and $H^0(\Omega^2)$. The geometric genus is commonly denoted by p_g.

Generalizing this, we may define the *plurigenera* of X, to be the dimensions of the spaces $H^0(\mathcal{O}[nK])$ as n varies; precisely,

$$P_n = \dim H^0(\mathcal{O}[nK])$$

for $n \geq 0$. Then $P_0 = 1$, $P_1 = p_g$, etc.

1.6 Kodaira Dimension

The growth rate κ of the plurigenera is an important invariant of X, called the *Kodaira dimension*. Precisely, if the plurigenera are all 0, then we set $\kappa = -\infty$ (some authors set $\kappa = -1$ for this case). Otherwise κ is the integer such that the plurigenera sequence P_n grows like n^κ, in the sense that κ is minimal such that P_n/n^κ is bounded. For algebraic surfaces, using the Riemann-Roch Theorem or otherwise, we have that $\kappa \leq 2$, so that the four possible values of κ are $-\infty$, 0, 1, or 2.

The situation is illustrated by briefly studying the corresponding invariant for curves. There the canonical class has degree $2g - 2$, where g is the genus of the curve. Therefore if $g = 0$, the canonical divisor is negative, and all plurigenera are 0: we have Kodaira dimension equal to $-\infty$. If $g = 1$, then 0 is a canonical divisor, so all plurigenera equal 1 and $\kappa = 0$.

Finally if $g \geq 2$ then the plurigenera grow: by Riemann-Roch one has $P_n = n(2g-2)+1-g$ for $n \geq 2$, which is linear in n, so $\kappa = 1$.

The Kodaira dimension is obviously a rather coarse invariant, but it serves well as a starting tool to classify surfaces. As the case of curves indicates, the "general" situation is that κ is maximal, equal to 2. The surfaces with $\kappa = 2$ are therefore said to be *of general type*, while those with $\kappa \leq 1$ are *of special type*. The classification of algebraic surfaces begins by first classifying those of special type; then attacking those of general type.

1.7 Numerical Invariants: q, $h^{p,q}$, b_i, e, χ

There are several other important numerical invariants which it is well to be aware of. Most involve some form of cohomology.

We begin with the *Betti numbers* b_i, which are the ranks of the simplicial (or singular) homology groups:

$$b_i = \mathrm{rank}\mathrm{H}^{\mathrm{i}}(\mathrm{X}, \mathbf{Z}) = \dim_{\mathbf{R}} \mathrm{H}^1(\mathrm{X}, \mathbf{R}) = \dim_{\mathbf{C}} \mathrm{H}^{\mathrm{i}}(\mathrm{X}, \mathbf{C}).$$

The Betti numbers b_i are defined for $0 \leq i \leq 4$.

The *Euler number* e is the alternating sum of the Betti numbers:

$$e = \sum_{i=0}^{4} (-1)^i b_i.$$

It is also the alternating sum of the numbers of i-simplices in any triangulation of X.

Next we have the *irregularity* q, defined as the dimension of the space of global holomorphic 1-forms:

$$q = \dim H^0(\Omega^1).$$

More generally we have the *Hodge numbers*, which we may define as the dimensions of the cohomology groups of the sheaves of p-forms:

$$h^{p,q} = \dim H^q(\Omega^p).$$

Note that the irregularity $q = h^{1,0}$ and the geometric genus $p_g = h^{2,0}$.

Another definition avoiding cohomology is to consider the $\overline{\partial}$ operator on (p,q)-forms, which (locally) sends a term $f(z,w)da \wedge \ldots db$ to

$$(\partial f/\partial \overline{z}) da \wedge \ldots db \wedge d\overline{z} + (\partial f/\partial \overline{w}) da \wedge \ldots db \wedge d\overline{w}.$$

This maps (p,q)-forms to $(p,q+1)$-forms, and $\overline{\partial} \circ \overline{\partial} = 0$. Then one defines

$$H^{p,q} = \ker(\overline{\partial} : \mathrm{H}^0(\mathcal{E}^{\mathrm{p,q}}) \to \mathrm{H}^0(\mathcal{E}^{\mathrm{p,q+1}}))/\mathrm{image}(\overline{\partial} : \mathrm{H}^0(\mathcal{E}^{\mathrm{p,q-1}}) \to \mathrm{H}^0(\mathcal{E}^{\mathrm{p,q}}))$$

and $h^{p,q} = \dim H^{p,q}$. These numbers are defined for $0 \leq p,q \leq 2$.

These invariants have various relationships, which form the heart of Hodge theory for algebraic surfaces; these are summed up as follows:

(a) $h^{p,q} = h^{q,p} = h^{2-p,2-q} = h^{2-q,2-p}$

(b) $b_i = \sum_{p+q=i} h^{p,q}$

(c) $b_0 = b_4 = 1$

The second property comes from the Hodge decomposition

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q=n} H^{p,q}$$

where $H^n(X, \mathbf{C})$ has its DeRham interpretation. The symmetry properties (a) are induced by conjugation and the $*$ operator.

One often arranges the Hodge numbers in a "Hodge diamond"

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

with 5 rows; because of the above identifications we see that the Hodge diamond looks like

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

whose rows sum to the Betti numbers. Therefore we have that

$$b_1 = b_3 = 2q \quad \text{and} \quad \mathrm{b_2 = 2p_g + h^{1,1}}$$

while the alternating sum of the rows gives the Euler number

$$e = 2 + 2p_g + h^{1,1} - 4q.$$

Finally we have the holomorphic Euler-Poincare characteristic $\chi = \chi(\mathcal{O})$, the alternating sum of the dimensions of the cohomology groups of the sheaf $\mathcal{O}$ of holomorphic functions:

$$\chi = \dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O}) + \dim H^2(\mathcal{O}) = 1 - q + p_g.$$

1.8 The Neron-Severi Group and the Lefschetz $(1,1)$ Theorem

On any algebraic surface we have the exponential short exact sequence of sheaves

$$0 \to \underline{\mathbf{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

where the map on the right is $\exp(2\pi i -)$. This is also exact on global sections, so we have an exact sequence starting at the H^1 level which begins

$$0 \to H^1(\mathbf{Z}) \to H^1(\mathcal{O}) \to H^1(\mathcal{O}^*) \to H^2(\mathbf{Z}) \to H^2(\mathcal{O}).$$

Using the standard names for the invariants we see that this is

$$0 \to \mathbf{Z}^{2q} \to \mathbf{C}^q \to \mathrm{Pic(X)} \to \mathrm{H}^2(\mathbf{Z}) \to \mathrm{H}^2(\mathcal{O}).$$

We see therefore that the Picard group Pic(X) of divisors modulo linear equivalence has a continuous part $\mathbf{C}^q/\mathrm{image}(\mathbf{Z}^{2q})$ (which is a complex torus of dimension q) and its image in $H^2(\mathbf{Z})$ is its discrete part. This map to $H^2(\mathbf{Z})$ is called the *Chern class map* on divisors. The image, modulo torsion, is called the *Neron-Severi group* of X, denoted by NS(X). It is a finitely generated free abelian group; its rank ρ is called the *Picard number* of X, and tells us how many discrete parameters divisors on X depend on.

Note that any class in $H^2(\mathbf{Z})$ which comes from a divisor must then go to zero in $H^2(\mathcal{O})$. This map factors as

$$H^2(\mathbf{Z}) \subset H^2(\mathbf{C}) \to H^2(\mathcal{O})$$

which shows that such a class, when considered in $H^2(\mathbf{C})$, is in the kernel of the map to $H^2(\mathcal{O}) \cong H^{0,2}$. A conjugation argument shows that it is also zero in $H^{2,0}$, and so must lie in $H^{1,1}$.

The converse of this is the Lefschetz $(1,1)$ Theorem: a class in $H^2(\mathbf{Z})$ is the class of a divisor if and only if, when considered in $H^2(\mathbf{C})$, it lies in the $H^{1,1}$ part.

Thus the divisors on X are controlled by how the complex subspace $H^{1,1}$ of $H^2(\mathbf{C})$ intersects the discrete subgroup $H^2(\mathbf{Z})$. For surfaces with $p_g = 0$, for which therefore $H^2(\mathbf{C}) = H^{1,1}$, this is no condition, but for surfaces of positive geometric genus this can be a subtle arithmetic problem.

1.9 Intersection Theory, Riemann-Roch, Index Theorems

On the middle-dimensional cohomology groups $H^2(\mathbf{R})$ and $H^2(\mathbf{C})$ the cup-product induces a nondegenerate bilinear form, called the *intersection form* on X. The group of divisors, through the Chern class map, inherits this bilinear form, as does the Picard group and the Neron-Severi group NS(X).

The intersection of divisors can be defined algebraically. It suffices to define $(C_1 \cdot C_2)$ for two curves on X, by linearity. If the curves are different, then they intersect at only finitely many points $\{p_i\}$. At each $p = p_i$, let C_j be defined by the local function f_j in the local ring $\mathcal{O}_p$. Then the local intersection number

$$(C_1 \cdot C_2)_p := \mathrm{length}(\mathcal{O}_\mathrm{p}/(\mathrm{f}_1, \mathrm{f}_2))$$

is well-defined and finite, and

$$(C_1 \cdot C_2) = \sum_i (C_1 \cdot C_2)_{p_i}.$$

The case of $(C \cdot C)$ can be handled in two ways: either replace one of the C's by a linearly equivalent divisor not containing C as a component, or in an ad hoc way set $(C \cdot C)$ to be the degree of the normal bundle of C on X. In particular note that $(C \cdot C)$ may be negative.

An important invariant is $K^2 = (K \cdot K)$ where K is a canonical divisor.

We are finally in a position to state some of the most important results in surface theory.

Theorem 1.3 *Let X be an algebraic surface.*

(a) (Thom-Hirzebruch Index (or Signature) Theorem:) *The signature τ of the intersection form on $H^2(\mathbf{R})$ satisfies*
$$\tau = (K^2 - 2e)/3.$$

(b) (Hodge Index Theorem:) *The intersection form on $H^{1,1} \cap H^2(\mathbf{R})$ and hence on the Neron-Severi group* NS(X) *has one positive eigenvalue and $h^{1,1}-1$ negative eigenvalues.*

(c) (Noether's Formula:)
$$12\chi = K^2 + e.$$

(d) (Riemann-Roch Theorem:)
$$\chi(\mathcal{O}[D]) = \frac{(D \cdot (D-K))}{2} + \chi.$$

(e) (The Genus Formula:) *If C is an irreducible curve on X, then its arithmetic genus $p_a(C)$ is*
$$p_a(C) = \frac{(C \cdot (C+K))}{2} + 1.$$

Here $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F})$ is the Euler-Poincare characteristic of the sheaf in question; without specifying the sheaf one takes $\chi = \chi(\mathcal{O})$. (Here we write h^i for the dimension of a cohomology group H^i.)

1.10 Projective Invariants: Degree, Class

Suppose that $X \subset \mathbf{P}^n$ is a projective surface. Then as noted above we have the hyperplane divisor class H on X. A typical hyperplane divisor should be thought of, as a Weil divisor, as the intersection of X with a hyperplane. If we intersect X with two hyperplanes we obtain the *degree* of X. On the other hand intersecting first with X and then intersecting again, we see that this degree is equal to the self-intersection number $H^2 = (H \cdot H)$. This is a fundamental invariant of a projective surface, but note that it depends very much on how X is embedded in projective space; the same surface can be re-embedded in different ways having different degrees.

A hypersurface in $\mathbf{P}^3$ defined by taking the zeroes of a homogeneous polynomial of degree d has degree d. More generally, if X is a smooth complete intersection surface of type $(d_1, \ldots, d_n)$ in $\mathbf{P}^{n+2}$, then X has degree $d_1 d_2 \cdots d_n$.

Other projective invariants abound; one of the most common is the *class*, defined to be the degree of the dual variety of tangent hyperplanes.

1.11 Birational Maps, Blowups and Minimal Models

A holomorphic map f between algebraic surfaces is a map which, locally, is defined by two holomorphic functions. In the case of surfaces (and any varieties of higher dimension) there are always interesting maps which are not defined everywhere however. The prototype is the map from $\mathbf{C}^2$ to $\mathbf{P}^1$ sending a point (z,w) to the line $[z:w]$ through 0 and (z,w). This is not well-defined at the origin, and is not definable there as a continuous function.

So the concept of a *rational* map exists, given locally by meromorphic functions, but which may not be defined at all points. Rational maps $X \dashrightarrow Y$ are in correspondence with function field inclusions $\mathcal{M}(Y) \to \mathcal{M}(X)$. A *birational* map is a rational map with a rational inverse. A birational map is therefore determined by an isomorphism between function fields; birational equivalence is essentially this isomorphism property.

The fundamental construction in the birational theory of surfaces is the *blowup* at a point p. Suppose that one chooses local coordinates (z,w) on X at p, so that p is $z=w=0$. Choose an open set around p, giving an open set $U \subset \mathbf{C}^2$. Replace that open set by two open sets V_1 and V_2, where V_1 has coordinates (z,u) and V_2 has coordinates (w,v). Glue these into the surface X by the change of coordinates

$$w = zu, \quad uv = 1, \quad z = wv.$$

Note that in particular these two open sets are glued together, and map to the open set U; the inverse image of p (which is $z=w=0$) is the line $z=0$ (with coordinate u) in V_1 and the line $w=0$ (having coordinate v) in V_2. Since $u = 1/v$, these lines are glued to form a $\mathbf{P}^1$ which we will denote by E. The inverse image of any other point is simply one point. Therefore we have constructed another complex manifold B, which maps to X. If we call the map π, we have the diagram

$$\begin{array}{ccc} E & \subset & B \\ \downarrow & & \downarrow \pi \\ p & \in & X \end{array}$$

It is easy to see that B is an algebraic surface if X is, and has an isomorphic field of meromorphic functions: B is birational to X.

The following basic facts are easy consequences of local computations:

Lemma 1.4 *Consider $\pi : B \to X$ as above, with exceptional curve E.*

(a) $(E \cdot E) = -1$.

(b) $(E \cdot \pi^*(D)) = 0$ *for any divisor D on X.*

(c) *π is an isomorphism between $B - E$ and $X - p$.*

(d) *If C is a curve on X having multiplicity m at p, then the proper transform C' of C on B (which is defined to be the closure of $\pi^{-1}(C' - p)$ in B) satisfies*

$$(C' \cdot E) = m \quad \text{and} \quad (\mathrm{C}')^2 = \mathrm{C}^2 - \mathrm{m}^2.$$

(e) $K_B = \pi^*(K_X) + E$, *hence* $K_B^2 = K_X^2 - 1$.

(f) *q, p_g, χ, and the plurigenera P_n are the same for B as for X.*

(g) $e(B) = e(X) + 1,\ h^{1,1}(B) = h^{1,1}(X) + 1,\ b_2(B) = b_2(X) + 1,\ \rho(B) = \rho(X) + 1.$

(h) $\mathrm{Pic(B)} = \pi^*(\mathrm{Pic(X)}) \oplus \mathbf{Z} \cdot \mathrm{E}.$

Reversing this construction is *Castelnuovo's Contraction Criterion*: if on an algebraic surface B one finds a curve E with $E \cong \mathbf{P}^1$ and $E^2 = -1$, then B is the blowup of an algebraic surface X and E is the exceptional curve. This contraction of E to a point p is unique. Such a curve E on a surface is called a (-1)-*curve*.

A surface is therefore said to be *minimal* if it has no (-1)-curves; it does not arise as the blowup of any other surface. Most of the detailed work in algebraic surfaces concentrates on minimal ones.

Blowups are so important because *any* birational map between algebraic surfaces factors through blowups in the following sense:

Proposition 1.5 *Let $f : X \dashrightarrow Y$ be a birational map between algebraic surfaces. Then there is an algebraic surface Z and maps $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$, which are compositions of blowup maps, such that the diagram*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

commutes.

As a corollary, we obtain that q, p_g, χ, and the plurigenera P_n are birational invariants, since they are invariant under blowups.

1.12 Homeomorphism and Diffeomorphism

To close this first lecture I want to briefly mention the great strides that have been made in the last decade on understanding the topology and differential topology of algebraic surfaces. Firstly, when are two algebraic surfaces homeomorphic? At least in the simply connected case, we have an answer, given by Freedman (see [F]):

Theorem 1.6 *Two simply connected surfaces are homeomorphic if and only if their cohomology groups $H^2(\mathbf{Z})$ are isomorphic, as groups with bilinear form.*

This group, modulo torsion, is free abelian of rank b_2, and the form is unimodular with $2p_g + 1$ positive eigenvalues. There are at most two such forms, depending on whether the form is even or odd, and this depends on whether the canonical class is even or not. Thus a complete answer is at hand.

The diffeomorphism question is more subtle, and there are examples of simply connected surfaces which are homeomorphic but not diffeomorphic.

There has been a flurry of activity in just this past year on the diffeomorphism question; it is now known that, for minimal surfaces of general type, the canonical class is a diffeomorphism invariant up to sign, and that the plurigenera are diffeomorphism invariants, for example. The reader may consult the recent article [FM] for more complete information.

1.13 The Mori Point of View

Mori theory is now seen as the best way to understand algebraic threefolds, and its point of view also lends some insight into the classification of algebraic surfaces. The Kodaira Dimension κ recedes somewhat in importance, and the canonical class K itself comes into the foreground. Specifically, Mori's approach says to consider the intersection properties of the canonical class K, and in particular we say that K is *nef* (meaning **n**umerically **e**ventually **f**ree), if $(K \cdot C) \geq 0$ for all curves C on X. Mori's first main theorem is that if K is *not* nef, then either there is a (-1)-curve on X to blow down (which is the curve meeting K negatively) or K is either rational or ruled. The latter case is the $\kappa = -\infty$ situation. In the former case one then blows down until K does become nef. Then there are three possibilities: either K is numerically trivial (which corresponds to the $\kappa = 0$ case) or K is not numerically trivial but $K^2 = 0$ (which corresponds to the $\kappa = 1$ case) or $K^2 > 0$ (which corresponds to the $\kappa = 2$ case). This all then leads to the same basic finer classification, but with a slight alteration of attitude. The reader would do well to consult [CKM] for a gentle introduction to Mori's ideas.

2 Surfaces with Negative Kodaira Dimension

2.1 The Projective Plane

Let $\mathbf{P}^2$ be the projective plane, with homogeneous coordinates $[x : y : z]$. We have that $\mathbf{P}^2 = \mathbf{C}^2 \cup \mathbf{C} \cup \{p\}$, which is a cell decomposition with one cell of dimension 0, 2, and 4. Therefore in the cell chain complex all boundary maps must be zero and

$$\begin{aligned} H^0(\mathbf{P}^2, \mathbf{Z}) &\cong \mathbf{Z} \\ H^1(\mathbf{P}^2, \mathbf{Z}) &= 0 \\ H^2(\mathbf{P}^2, \mathbf{Z}) &\cong \mathbf{Z} \\ H^3(\mathbf{P}^2, \mathbf{Z}) &= 0 \\ H^4(\mathbf{P}^2, \mathbf{Z}) &\cong \mathbf{Z}. \end{aligned}$$

Hence

$$b_1 = b_3 = 0, \quad b_2 = 1, \quad e = 3.$$

Therefore also we must have

$$q = p_g = 0, \quad \chi = h^{1,1} = 1.$$

The intersection form on $H^2(\mathbf{Z})$ must be simply (1). If we denote by H the class of a line, then H is a divisor which generates the Picard group of $\mathbf{P}^2$. The intersection formula $H^2 = 1$ means that two lines intersect in one point!

If $u = x/z$ and $v = y/z$, then (u, v) are local coordinates on the $\mathbf{C}^2$ chart of $\mathbf{P}^2$ where $z \neq 0$. The 2-form $du \wedge dv$ is holomorphic on this chart, but has a triple pole at infinity (that is, on the line $z = 0$). To check this explicitly, use coordinates $s = z/x$ and $t = y/x$ at infinity ($s = 0$ defines the line at infinity in this chart). Then $u = 1/s$ and $v = t/s$, so that $du = (-1/s^2)ds$ and $dv = (-t/s^2)ds + (1/s)dt$. Therefore

$$du \wedge dv = [(-1/s^2)ds] \wedge [(-t/s^2)ds + (1/s)dt] = (-1/s^3)ds \wedge dt$$

exhibiting a triple pole at the line H at infinity. Therefore $K = -3H$ is the canonical class, and $K^2 = 9$.

Since K is a negative divisor, so is nK for every $n \geq 1$, and hence all plurigenera P_n for the plane are zero. Therefore $\kappa = -\infty$ for $\mathbf{P}^2$.

The Riemann-Roch Theorem for $\mathbf{P}^2$ says that

$$h^0(\mathcal{O}[dH]) = (dH \cdot (d+3)H)/2 + \chi = d(d+3)/2 + 1$$

which represents the number of coefficients of a form of degree d in 3 homogeneous variables; the χ equals the h^0 since the higher cohomology is all zero. The Genus Formula for a curve C of degree d (and hence linearly equivalent to dH) is

$$p_a(C) = (dH \cdot (d-3)H)/2 + 1 = (d-1)(d-2)/2$$

which is the classical Plücker formula.

An algebraic surface is said to be *rational* if it is birational to the plane.

2.2 Quadric Surfaces

Just as for $\mathbf{P}^2$, the canonical class of any projective space is a multiple of the hyperplane class; indeed, for $\mathbf{P}^n$ we have $K = -(n+1)H$. The adjunction formula then states that if X is a complete intersection surface in $\mathbf{P}^n$ of type $(d_1, \ldots, d_{n-2})$, then $K_X = (\sum_i d_i - n - 1)H$. The first case to apply this is for a smooth quadric surface X of degree 2 in $\mathbf{P}^3$. Then $K = -2H$, and since this is clearly negative, so are all multiples, and we have that $P_n = 0$ for all n. Therefore again X has $\kappa = -\infty$.

Since $H^2 = 2$ (the surface has degree 2), we see that $K^2 = 8$.

Any two smooth quadrics are isomorphic (there is only one nondegenerate quadratic form up to isomorphism over $\mathbf{C}$) and we may take $xy = zw$ (in the homogeneous coordinates $[x : y : z : w]$ of 3-space) as a defining equation for X. We see easily that $X \cong \mathbf{P}^1 \times \mathbf{P}^1$ by the map sending $([a : b], [c : d])$ to $[x = ac : y = bd : z = ad : w = bc]$. Therefore the Künneth formulas give that

$$\begin{aligned} H^0(\mathbf{Z}) &\cong \mathbf{Z} \\ H^1(\mathbf{Z}) &= 0 \\ H^2(\mathbf{Z}) &\cong \mathbf{Z}^2 \\ H^3(\mathbf{Z}) &= 0 \\ H^4(\mathbf{Z}) &\cong \mathbf{Z}. \end{aligned}$$

Hence

$$b_1 = b_3 = 0, \quad b_2 = 2, \quad e = 4.$$

Again $q = p_g = 0$ and $\chi = 1$; this time $h^{1,1} = 2$. In the $\mathbf{P}^1 \times \mathbf{P}^1$ representation, we have the obvious horizontal curve G and the vertical curve F; these generate $H^2(\mathbf{Z})$ and the Picard group, with $F^2 = G^2 = 0$ and $(F \cdot G) = 1$. The hyperplane class (as a quadric surface) is $H = F + G$, hence $K = -2F - 2G$.

Riemann-Roch now gives that

$$\chi(rF + sG) = ((rF + sG) \cdot (r + 2)F + (s + 2)G)/2 + \chi = (r + 1)(s + 1)$$

which represents the number of monomials of a bihomogenous polynomial of bidegree (r, s). The Genus Formula for a curve C linearly equivalent to $rF + sG$ is

$$p_a(rF + sG) = (r - 1)(s - 1).$$

Projection of the quadric X from a point p on X gives a birational map from X to $\mathbf{P}^2$; this map blows down the two lines through p and blows up p. In particular, X is a rational surface.

2.3 Ruled Surfaces

A *ruled surface* is a surface birational to $X = C \times \mathbf{P}^1$ for some curve C of genus g. This decomposition implies that we can construct a meromorphic 2-form on X by taking a 1-form on C and wedging with a 1-form on $\mathbf{P}^1$. Therefore a canonical divisor on X is gotten by taking a canonical divisor on C (which therefore has degree $2g - 2$) and pulling back via the first projection, plus taking a canonical divisor on $\mathbf{P}^1$ (which is -2 points) and pulling back via the second projection. If we denote by F the $\mathbf{P}^1$ fiber and by C the C fiber, then *numerically* (that is, in $H^2(\mathbf{Z})$) we have that $K = (2g-2)F-2C$. No multiple of this divisor can be linearly equivalent to a positive divisor; its intersection with the curve F is strictly negative. Therefore again $\kappa = -\infty$ for this surface, and hence for any ruled surface.

The Künneth decomposition gives that

$$\begin{aligned} H^0(\mathbf{Z}) &\cong \mathbf{Z} \\ H^1(\mathbf{Z}) &= 2g \\ H^2(\mathbf{Z}) &\cong \mathbf{Z}^2 \\ H^3(\mathbf{Z}) &= 2g \\ H^4(\mathbf{Z}) &\cong \mathbf{Z}. \end{aligned}$$

Hence

$$b_1 = b_3 = 2g, \quad b_2 = 2, \quad e = 4 - 4g.$$

Now $q = g$, $p_g = 0$, $\chi = 1 - g$, and $h^{1,1} = 2$; the Neron-Severi group is generated by F and C, with $F^2 = C^2 = 0$ and $(F \cdot C) = 1$. Also $K^2 = 8 - 8g$. Note that now if $g > 0$ there is a continuous part to the Picard group, coming from the Jacobian of C. The 1-forms on X are all pulled back from the g linearly independent 1-forms on C, which is why $q = g$.

2.4 Elementary Transformations

Suppose that X is ruled surface, birational to $C \times \mathbf{P}^1$, with $g(C) > 0$. Then as noted in the previous lecture X may be obtained by blowing up $C \times \mathbf{P}^1$ a finite number of times, then blowing down some (-1)-curves. Now every (-1)-curve is a rational curve, and therefore cannot dominate C if $g > 0$. Therefore these (-1)-curves must all map to points under the first projection to C; in other words, they must lie in fibers of this first projection $\pi : C \times \mathbf{P}^1 \to C$. We conclude that this map π to C is also defined on X, and so X comes also with a projection to C, which is hence natural.

The prototype for this phenomenon is to choose any point of X, contained in a smooth fiber F, and blow that point up, obtaining the exceptional curve E. Then the proper transform of F on the blowup also is a (-1)-curve, and can therefore be blown down. This produces a new surface, and this operation is called an *elementary transformation* of the ruled surface. We note that elementary transformations preserve essentially all discrete invariants.

Again assuming $g > 0$, any (-1)-curve on X must live in a fiber of the map π to C. Therefore if every fiber is a smooth curve (the so-called *geometrically ruled* case) then since fibers have self-intersection 0 (not -1) there can be no (-1)-curves on X, and X is a minimal surface. The converse is also true: if X is minimal, then every fiber of π is smooth.

The basic theorem relating minimal ruled surfaces with $q > 0$ is the following.

Theorem 2.7 *Let X be a minimal ruled surface over a curve C of genus $g > 0$. Then X is obtained from the product surface $C \times \mathbf{P}^1$ by a finite number of elementary transformations.*

Since elementary transformations preserve all discrete invariants, we therefore know these invariants for all minimal ruled surfaces over positive genus curves.

We remark that there is not a unique minimal surface birational to any ruled surface, but infinitely many: just perform an elementary transformation on one to get another.

2.5 Hirzebruch Surfaces

If X is a ruled surface with $q = 0$, then X is a rational surface. In this case the ruling on X may not be unique, especially if X is not minimal. However the concept of the elementary transformation still exists; let us apply it iteratively to $\mathbf{P}^1 \times \mathbf{P}^1$ and see what we get.

Define $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$. Choose a horizontal section G_0 (here $G_0^2 = 0$) and perform an elementary transformation, blowing up a point on G_0 and blowing down the proper transform of the fiber through p. The formulas give that the proper transform of G_0 is a curve G_1 with self-intersection -1; the ruling as noted above is still preserved. Call this surface $\mathbf{F}_1$. The section G_1 and the fiber F of the ruling generate the Picard group of $\mathbf{F}_1$.

Recursively, we will have a surface $\mathbf{F}_n$, ruled over $\mathbf{P}^1$, with a section G_n of self-intersection $-n$ which, together with the fiber F, generates the Picard group. Perform an elementary transformation, blowing up a point on G_n, and blowing down the fiber; the result is a ruled surface $\mathbf{F}_{n+1}$. The proper transform of G_n is a curve G_{n+1} of self-intersection $-n-1$, which is a section of the ruling; it and the fiber generate the Picard group.

The classification of minimal ruled surfaces with $q = 0$ is now at hand:

Theorem 2.8 *Let X be a minimal ruled surface with $q = 0$. Then X is either $\mathbf{P}^2$ or $\mathbf{F}_n$ for some $n \geq 0$, $n \neq 1$. The surface $\mathbf{F}_1$ is the blowup of $\mathbf{P}^2$ at one point.*

These surfaces $\mathbf{F}_n$ are commonly called the *Hirzebruch* ruled surfaces.

2.6 The Classification Theorem

The fundamental classification theorem can now be stated.

Theorem 2.9 *Every algebraic surface with $\kappa = -\infty$ is ruled.*

Every rational surface (birational to $\mathbf{P}^2$) is automatically ruled, since $\mathbf{P}^2$ is birational itself to $\mathbf{P}^1 \times \mathbf{P}^1$. Sometimes this is explicitly stated: every surface with $\kappa = -\infty$ is rational or ruled.

There is a wonderful refinement of this in the case of rational surfaces, namely the Castelnuovo criterion:

Theorem 2.10 *An algebraic surface is rational if and only if $q = P_2 = 0$.*

The Castelnuovo criterion says that if there are no holomorphic 1-forms on X ($q = 0$) and no holomorphic 2-fold 2-forms (forms locally expressible as $f(z,w)(dz \wedge dw)^{\otimes 2}$) then the surface is rational. Of course if $P_2 = 0$ then certainly the geometric genus $P_1 = p_g = 0$ (which indicates that there are no holomorphic 2-forms). It was an open question in the latter part of the last century whether the criterion could be weakened to $q = p_g = 0$; this was shown to be false by Enriques, and we will see an example in the next lecture.

One also has the following criterion of Enriques:

Theorem 2.11 *An algebraic surface has $\kappa = -\infty$ if and only if $P_4 = P_6 = 0$.*

2.7 Scrolls and Rational Normal Scrolls

A ruled surface embedded in projective space such that the fibers of the ruling are straight lines is called a *scroll*. No doubt the most important family of scrolls are the *rational normal scrolls*, which we now describe.

Recall the construction of a *rational normal curve*, which is $\mathbf{P}^1$ embedded into $\mathbf{P}^d$ by all monomials of degree d: $[x : y] \mapsto [x^d : x^{d-1}y : \cdots : xy^{d-1} : y^d]$. Call this map ν_d; it is simply the Veronese map of degree d for $\mathbf{P}^1$. Its image is the rational normal curve, of degree d in $\mathbf{P}^d$. Note that this is the minimal degree for a nondegenerate curve in $\mathbf{P}^d$; if one cuts a curve of degree e with a general hyperplane, one will obtain e points spanning $\mathbf{P}^{d-1}$, so that $e \geq d$.

To build a *rational normal scroll* in $\mathbf{P}^{a+b+1}$, choose two complementary linear spaces of dimension a and b, and in each put a rational normal curve, the image of the line under ν_a and ν_b respectively. Form a surface $S_{a,b}$ by taking the union of all of the lines joining $\nu_a(p)$ to $\nu_b(p)$ as p varies in $\mathbf{P}^1$.

Proposition 2.12 *Assume that $1 \leq a \leq b$. The surface $S_{a,b}$ is nondegenerate, and is isomorphic to $\mathbf{F}_n$, where $n = b - a$. The rational normal curve of degree a is the curve G_{b-a}. The hyperplane divisor on $S_{a,b}$ is $G_n + bF$; the degree of $S_{a,b}$ is $a + b$.*

The case of $S_{1,1}$ is that of a smooth quadric in $\mathbf{P}^3$.

The limiting case where we take $a = 0$ is simply a cone over a rational normal curve of degree b.

2.8 Surfaces of Minimal Degree

Suppose that $C \subset \mathbf{P}^n$ is a nondegenerate curve. As we noted above, cutting C with a general hyperplane H we will find d points in H which span H. Therefore $d \geq n$: any nondegenerate curve in $\mathbf{P}^n$ has degree at least n. The curves of minimal degree are exactly the rational normal curves.

We can iterate this argument for surfaces, and we find that the minimal degree for a nondegenerate surface in $\mathbf{P}^n$ is $n-1$: the hyperplane section will be a nondegenerate curve in $\mathbf{P}^{n-1}$, and will therefore have degree at least $n-1$. This is the case for the rational normal scrolls $S_{a,b}$ introduced above: they have degree $a+b$ and lie in $\mathbf{P}^{a+b+1}$.

Another example of a surface of minimal degree is the double Veronese, defined to be the image of $\mathbf{P}^2$ in $\mathbf{P}^5$ under the map sending $[x:y:z]$ to $[x^2:y^2:z^2:xy:xz:yz]$. This has degree 4 (two conics intersect in four points).

These turn out to be all the examples!

Proposition 2.13 *Let X be a surface of degree $n-1$ in $\mathbf{P}^n$. Then X is either a rational normal scroll $S_{a,b}$ or the Veronese surface in $\mathbf{P}^5$.*

2.9 The Cubic Surface

Let X be a smooth cubic surface in $\mathbf{P}^3$. By using the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}[-3H] \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_X \to 0$$

we see that $q = p_g = 0$. By adjunction we have that $K_X = -H$, which is negative, so all plurigenera are zero and X is a rational surface. $K^2 = H^2 = 3$, so X is either a 6-fold blowup of the plane or a 5-fold blowup of an $\mathbf{F}_n$. In fact the former is true:

Proposition 2.14 *Let X be a smooth cubic surface in $\mathbf{P}^3$. Then X is a 6-fold blowup of $\mathbf{P}^2$, and the embedding of X into $\mathbf{P}^3$ is obtained by the linear system of cubic curves through the 6 blown-up points. There are 27 lines on X, coming from the 6 exceptional curves, the 15 planar lines through 2 of the blown-up points, and the 6 planar conics through 5 of the blown-up points.*

There are whole books written about the 27 lines on the cubic surface: each meets 10 others in a beautiful configuration.

2.10 Del Pezzo Surfaces

The cubic surface does not have minimal degree, but has degree one more than minimal. This is true of all the surfaces obtained by choosing 6 or fewer points in the plane, and embedding the plane into $\mathbf{P}^n$ by the linear systems of cubic curves through the points. If there are r points, the resulting surface is a surface of degree $9-r$ in $\mathbf{P}^{9-r}$, called a *Del Pezzo surface.*

Classifying surfaces of low degree has been fun for centuries.

Theorem 2.15 *Let X be a smooth surface of degree k in $\mathbf{P}^k$. Then X is either a Del Pezzo surface or is the projection into $\mathbf{P}^4$ of the Veronese surface.*

If we blow up $r = 0$ points, then the Del Pezzo surface of degree 9 in $\mathbf{P}^9$ is simply the triple Veronese image of the plane. The $r = 1$ case is the Hirzebruch surface $\mathbf{F}_1$ embedded in $\mathbf{P}^8$ via the linear system $2G_1 + 3F$. The $r = 6$ case is, as we noted above, the cubic surface in $\mathbf{P}^3$. The $r = 5$ case is a complete intersection in $\mathbf{P}^4$ of type $(2, 2)$, that is, the intersection of two quadrics. The cases $r \leq 4$ are not complete intersection surfaces.

2.11 Curves on Rational Surfaces

Suppose we choose points $p_1, \ldots, p_n$ on $\mathbf{P}^2$ and consider the linear system of all curves of degree d having a point of multiplicity at least m_i and p_i for each i. The dimension of the linear system of curves of degree d is $d(d + 3)/2$; imposing a point of multiplicity m gives $m(m + 1)/2$ conditions. Therefore the expected dimension of the linear system is

$$d(d+3)/2 - \sum_i m_i(m_i+1)/2$$

when this quantity is at least -1.

Of course when the points are is special position the conditions imposed by the multiplicities may not be independent; the first example is when all the points lie on a line, forcing the line to be a component of every curve in the system if the multiplicities are high enough. However one can hope that the expected dimension is the correct dimension if one chooses the points generically; but this is an open problem.

Finally let me mention a related question; when does such a linear system embed the blowup of the plane at the n points into $\mathbf{P}^4$? As an exercise, check the case of quartics double at one point, and passing through 7 others; or quartics through 10 points. Can you find all such linear systems which work for generic points?

Smooth surfaces in $\mathbf{P}^4$ are difficult to construct in general, and have only been classified for very low degrees (up to about 10). It is known that the degree of a smooth rational surface in $\mathbf{P}^4$ is bounded (Ellingsrud, Peskine), but the current bound (about 250, proved by Braun and Floystad) is much higher than the highest degrees known to exist (15, after work by Alexander, Aure, Okonek, Ranestad, Serrano, Decker, Ein, Schreyer, Popescu, ...).

3 Surfaces with Kodaira Dimension Zero

3.1 Abelian Surfaces

Let X be a complex torus, obtained by choosing a rank four $\mathbf{Z}$-lattice L inside $\mathbf{C}^2$ and forming the quotient surface $X = \mathbf{C}^2/L$. If z and w are coordinates on $\mathbf{C}^2$, they descend to give local coordinates on X, which are well-defined up to the translations in L. Therefore the 1-forms dz and dw are well-defined, as is the 2-form $dz \wedge dw$, globally. These generate the spaces of holomorphic 1- and 2-forms, and we see that therefore $q = 2$ and $p_g = 1$;

$\chi = 0$. The Euler number $e = 0$, since topologically X is a product of four circles; indeed, the simplicial cohomology is

$$\begin{aligned} H^0(\mathbf{Z}) &\cong \mathbf{Z} \\ H^1(\mathbf{Z}) &\cong \mathbf{Z}^4 \\ H^2(\mathbf{Z}) &\cong \mathbf{Z}^6 \\ H^3(\mathbf{Z}) &\cong \mathbf{Z}^4 \\ H^4(\mathbf{Z}) &\cong \mathbf{Z} \end{aligned}$$

by the Künneth formula, and so the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

We see here our first example of a delicate Neron-Severi group analysis; the lattice $H^2(\mathbf{Z})$ may intersect the subspace $H^{1,1} \subset H^2(\mathbf{C})$ in any rank, from 0 to 4. If the rank is zero, then there are no ample divisors on X, and X is not an algebraic surface. Here's a challenge for your understanding: construct lattices L for which the Picard number $\rho(X)$ is a given integer between 0 and 4 (I believe all are possible).

In any case, the 2-form $dz \wedge dw$ has no zeroes or poles, so that the canonical divisor K is trivial. Therefore $P_n = 1$ for every n, and $\kappa = 0$. A complex torus X which is algebraic is called an *abelian surface*.

3.2 $K3$ Surfaces

Let X be a quartic surface in $\mathbf{P}^3$. By adjunction we have $K = 0$, and by analyzing the ideal sheaf sequence one easily sees that $q = 0$ and $p_g = 1$, so that $\chi = 2$. Since $K = 0$, surely $K^2 = 0$, so that $e = 24$ and the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Again there is a subtle interplay between $H^{1,1}$ and $H^2(\mathbf{Z})$, and the Neron-Severi group can have any rank between 1 and 20. The quartic is simply connected.

If X is a complete intersection of type $(2, 3)$ in $\mathbf{P}^4$, or of type $(2, 2, 2)$ in $\mathbf{P}^5$, similar analyses lead to the same invariants as above.

Definition 3.16 *A* K3 surface *is a (minimal) compact complex surface X with $q = 0$ and $K = 0$.*

With these assumptions, $p_g = 1$, and X is simply connected, with the above Hodge numbers. A $K3$ surface is an algebraic surface only if its Picard number is at least one; there are non-algebraic $K3$ surfaces (with Picard number 0). There are even non-algebraic $K3$ surfaces with Picard number 1, but all of whose divisor classes have $D^2 = 0$. If a $K3$ surface has a divisor D with $D^2 > 0$, then it is algebraic.

3.3 Kummer surfaces

Another easy family of examples of $K3$ surfaces are obtained as quotients. If A is a complex torus of dimension two, then consider the quotient $X = A/\{\pm I\}$, called a *Kummer surface.* (There are sixteen ordinary double points on X, coming from the 16 points fixed under $\{\pm I\}$, but these singularities resolve in a nice way.) Note that no 1-form on A is invariant under $\{\pm I\}$, but the 2-form is, so certainly $q = 0$ and $p_g = 1$ for X; moreover the 2-form (which has no zeroes or poles on A) descends to one with no zeroes or poles on X, so $K = 0$ too. Thus X is a $K3$ surface.

One finds Kummer surfaces in all of the families of $K3$ surfaces of genus g. For example, in the $g = 2$ case of double covers of the plane branched along a sextic curve, one obtains a Kummer surface if one takes the sextic curve to be six lines all tangent to a given conic. The complex torus which this surface is a quotient of is the Jacobian of the hyperelliptic curve of genus two obtained by taking the double cover of the conic branched at the six points of tangency.

3.4 $K3$ surfaces in Projective Space

Counting parameters, one sees that quartics depend on 19 moduli (there are 35 coefficients, with the 16-dimensional group of 4×4 matrices acting). Similarly the complete intersections (of type $(2,3)$ and of type $(2,2,2)$) mentioned above all depend on 19 moduli. However the space of first-order deformations $H^1(\Theta)$ is 20-dimensional for any $K3$ surface; one sees the non-algebraic $K3$ surfaces lurking around every corner. In fact, the moduli space of all $K3$ surfaces is 20-dimensional, with a countable number of 19-dimensional families representing algebraic $K3$ surfaces.

What are these 19-dimensional families? If a $K3$ surface X contains a curve C of genus $g \geq 3$, then C tends to be very ample on X, and by Riemann-Roch the linear system $|C|$ will embed X into projective space as a smooth surface of degree $2g - 2$ in $\mathbf{P}^g$. The condition that X contain such a curve is simply a condition on how the lattice $H^2(\mathbf{Z})$ intersects $H^{1,1}$, and so in the 20-dimensional moduli space there are these 19-dimensional families, one for each g, of projective algebraic surfaces of degree $2g - 2$ in $\mathbf{P}^g$. Such a $K3$ surface is called a "$K3$ surface of genus g".

There is even a degenerate case of genus 2: a $K3$ surface of degree 2 in $\mathbf{P}^2$ is a double cover of the plane, branched along a smooth sextic curve. Note that sextics depend on 28 coefficients, and subtracting the 9 dimensions for the group of 3×3 matrices acting on the plane, we see the 19 moduli again for these surfaces.

3.5 Curves on $K3$ Surfaces

Given a $K3$ surface of genus g, hence of degree $2g-2$ in $\mathbf{P}^g$, its general hyperplane section C is a smooth curve of degree $2g-2$ in $\mathbf{P}^{g-1}$; indeed, by adjunction $H|_C$ is the canonical class on C, so C is a canonical curve (embedded by its canonical system).

The number of moduli of such curves is clearly $19+g$; 19 for the $K3$ surface and g for the choice of hyperplane. This is less than the number $3g-3$ of moduli for canonical curves in general, so when $g \geq 12$ there is no chance that every canonical curve could be the hyperplane section of a $K3$ surface.

In fact there is an obstruction, whose analysis is due to J. Wahl (see [W]). His idea is to assume that C lies on a $K3$ surface, and to then degenerate the $K3$ surface X to the cone Y over C. This would be a non-trivial deformation of the cone; however the relevant deformation space of the cone is the cokernel of the Gaussian map

$$\phi : \bigwedge^2 H^0(C, K_C) \longrightarrow H^0(C, 3K_C)$$

defined by $\phi(f dz \wedge g dz) = (f'g - fg')(dz)^3$. If the Gaussian map for C is surjective, then the cokernel is trivial, the deformation space is trivial, and the deformation of the cone Y to X cannot exist; so C could not lie on any such X.

It is known that ϕ is surjective for the general curve of genus $g = 11$ and $g \geq 13$, using a degeneration argument; see [CHM]. It is conjectured that C lies on a $K3$ surface if and only if the Gaussian map ϕ for C is not surjective.

3.6 Enriques Surfaces

In the latter part of the last century it had been conjectured that a surface with $q = p_g = 0$ must be rational; no counterexamples were known. Castelnuovo's rationality criterion (that $q = P_2 = 0$) is stronger, since $P_2 = 0$ implies $P_1 = p_g = 0$. It was Enriques who finally settled this question and constructed non-rational surfaces with $q = p_g = 0$, which are named after him.

One of the simplest constructions is given in [GH]. Let S be the quartic $K3$ surface in $\mathbf{P}^3$ defined by $x^4 + y^4 = z^4 + w^4$. Let σ be the automorphism sending $[x : y : z : w]$ to $[x : iy : -z : -iw]$, which has order four, and acts on S. The quotient of S by the cyclic group generated by σ is an algebraic surface X with invariants $p_g = q = 0$ and $2K = 0$, but $K \neq 0$; K is torsion in $H^2(\mathbf{Z})$. The Hodge diamond for X is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and so $h^{1,1} = b_2 = 10$, and the Neron-Severi group has rank $\rho = 10$.

Definition 3.17 *An algebraic surface with $q = p_g = 0$ and $2K = 0$ is an* Enriques surface*.*

In fact the torsion in the cohomology is reflected by torsion in the fundamental group: $\pi_1(X) \cong \mathbf{Z}/2$. The universal (double) cover of X is a $K3$ surface; in our example above, the universal cover is the quotient of the original quartic S by the involution σ^2, which is a $K3$ surface again; the further quotient by σ gives the Enriques surface.

Since $2K = 0$ but $K \neq 0$, we have that the plurigenera P_n are either 0 or 1, depending on the parity of n. Hence $\kappa = 0$ for an Enriques surface.

Projective models of Enriques surfaces form a fascinating subject; the reader may wish to consult [CD] for lots of information. Probably the most famous is the realization of an Enriques surface as a sextic surface in $\mathbf{P}^3$ which has 6 double lines, arranged as in the edges of a tetrahedron!

The generic Enriques surface can be constructed as follows. Take six quadratic forms Q_1, Q_2, Q_3, R_1 ,R_2, R_3 each in three variables. Form the complete intersection surface Y of type $(2, 2, 2)$ in $\mathbf{P}^5$ as the zeroes of the three polynomials

$$Q_i(x, y, z) + R_i(u, v, w) \quad (i = 1, 2, 3)$$

where $[x : y : z : u : v : w]$ are the homogeneous coordinates in $\mathbf{P}^5$. For general choices of the forms, Y will be a $K3$ surface, and the involution σ which changes the sign of u, v, and w (leaving x, y, and z alone) acts on Y without fixed points. The quotient Y/σ is an Enriques surface.

Every Enriques surface contains elliptic fibrations, which we will discuss in the next lecture; this is an alternative avenue to pursue in constructing them.

3.7 Hyperelliptic/Bielliptic Surfaces

Enriques surfaces come naturally as fixed-point-free quotients of $K3$ surfaces, and are the only such surfaces. One might ask whether similar quotients can be taken of abelian surfaces, to obtain other surfaces with a torsion canonical class. This is the case; let us briefly go over the construction.

It turns out that the abelian surface, in order to support the right kind of automorphism, must be a product of elliptic curves. So take $S = E \times F$, with E and F elliptic. E may always be taken to be arbitrary, $E = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_1)$. Write $F = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_2)$. We have the following seven group actions:

1. τ_2 arbitrary, $\phi(e, f) = (e + 1/2, -f)$; $G =< \phi >\cong \mathbf{Z}/2$.

2. $\tau_2 = \omega$ where $\omega^3 = 1$, $\phi(e, f) = (e + 1/3, \omega f)$; $G =< \phi >\cong \mathbf{Z}/3$.

3. $\tau_2 = i$, where $i^2 = -1$, $\phi(e, f) = (e + 1/4, if)$; $G =< \phi >\cong \mathbf{Z}/4$.

4. $\tau_2 = \omega$, where $\omega^3 = 1$, $\phi(e, f) = (e + 1/6, -\omega^2 f)$; $G =< \phi >\cong \mathbf{Z}/6$.

5. τ_2 arbitrary, $\phi(e, f) = (e + 1/2, -f)$, $\psi(e, f) = (e + \tau_1/2, f + 1/2)$; $G =< \phi, \psi >\cong \mathbf{Z}/2 \times \mathbf{Z}/2$.

6. $\tau_2 = \omega$, $\phi(e, f) = (e + 1/3, \omega f)$, $\psi(e, f) = (e + \tau_1/3, f + (1 - \omega^2)/2)$; $G =< \phi, \psi >\cong \mathbf{Z}/3 \times \mathbf{Z}/3$.

7. $\tau_2 = i$, $\phi(e,f) = (e+1/4, if)$, $\psi(e,f) = (e+\tau_1/2, f+(1+i)/2)$; $G = <\phi,\psi> \cong \mathbf{Z}/4 \times \mathbf{Z}/2$.

This classification goes back to Bagnera and DeFranchis from 1907.

In each of these cases the quotient $X = (E \times F)/G$ is an algebraic surface with $p_g = 0$ and $q = 1$. In every case the canonical class is torsion, of order 2, 3, 4, 6, 2, 3, and 4 respectively. Therefore the plurigenera are all either 0 or 1, and are 1 for all multiples of 12 in every case, so $\kappa = 0$. Since they are covered by abelian surfaces without branching, the Euler numbers are all 0. Therefore the Hodge diamond must be

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

and so $b_2 = h^{1,1} = 2$. The Neron-Severi group has rank two, with elements easily described by descending the elliptic curves.

As is the case with the Enriques surfaces, all hyperelliptic surfaces have a pencil of elliptic curves (with the above description this pencil is inherited from E).

These surfaces are beginning to be called "bielliptic" as in Beauville's book [Be].

3.8 The Classification Theorem

Theorem 3.18 *Suppose that X is an algebraic surface with $\kappa = 0$. Then X is either abelian, $K3$, Enriques, or hyperelliptic.*

So we have seen them all!

4 Surfaces with Kodaira Dimension One

4.1 Elliptic Surfaces

Suppose that the Kodaira dimension κ of a surface X is one, and consider an n-canonical map for X; this map cannot be onto a surface, else the growth rate of the plurigenera would be two, by applying Riemann-Roch to the multiples of the hyperplane class of the image (which pull back to multiples of nK). Indeed, the map for large n will be onto a curve only, and the fibers of the map will be curves on X which do not meet the canonical class. If F is such a fiber, then $F^2 = 0$ (since it is the fiber of a map) and $(F \cdot K) = 0$ (since it is the fiber of the canonical map). Hence by the Genus Formula we see that F has genus one. We are thereby led to consider the following situation.

Definition 4.19 *Let X be an algebraic surface. An* elliptic fibration *on X is a holomorphic map $f : X \to C$ where C is a curve, such that the general fiber of f is a smooth curve of genus one. An* elliptic surface *is a surface with a given elliptic fibration.*

The main classification result for surfaces with $\kappa = 1$ is:

Theorem 4.20 *Every algebraic surface with $\kappa = 1$ is elliptic.*

The converse is not true; there are elliptic surfaces which are rational, ruled, abelian, and $K3$; and every Enriques and hyperelliptic surface is elliptic. However every elliptic surface has $\kappa \leq 1$.

4.2 Jacobian Surfaces and Weierstrass Equations

If F is a curve of genus one, then F is isomorphic to its Jacobian, once a point (which serves as the origin of the group law on F) has been chosen. A curve of genus one together with a chosen point is called an *elliptic curve*, or a *Jacobian elliptic curve.*

Every elliptic curve can be written in the Weierstrass form

$$y^2 = x^3 + Ax + B$$

where A and B are numbers such that $4A^3 + 27B^2 \neq 0$ (this is a smoothness condition). The chosen point is the point at infinity.

Taking our cue from this, we can try to form a family of elliptic curves $\{F_t\}$ by letting A and B vary with t, obtaining

$$y^2 = x^3 + A(t)x + B(t).$$

In this way one obtains an elliptic surface, with the property that the chosen point also varies with t, forming a *section* S_0 (called the zero-section of the fibration). If t varies on an algebraic curve, and A and B are rational functions, then this describes an algebraic elliptic surface.

There is no problem when A and/or B has a pole: one can "clear denominators" at will, since the same family of curves is obtained using the equation

$$y^2 = x^3 + Af^4x + Bf^6$$

where f is any holomorphic function. (Just change coordinates, replacing y by yf^3 and x by xf^2, then divide through by f^6.) So locally the family is always definable with regular coefficients.

One sees readily now that there is actually a line bundle L (the conormal bundle of the zero-section S_0) such that A and B are global sections of L^4 and L^6 respectively; the discriminant $D = 4A^3 + 27B^2$ is a section of L^{12}, and vanishes at a point p of the curve exactly when the fiber over p is not smooth.

An elliptic surface defined in this way with a Weierstrass equation is called a *Jacobian* elliptic surface; moreover an elliptic surface is a Jacobian surface if and only if the elliptic fibration has a section.

4.3 Kodaira's Classification of Singular Fibers

In the 1950's, in his fundamental papers [K] on surfaces, Kodaira realized the importance of elliptic surfaces in the whole picture and proved the first classification theorems. One of the most fascinating and useful results was his classification of the possible singular fibers of an elliptic fibration; to this day it represents the only complete classification of degenerations which one can remember (degenerations of curves of genus two are also classified, but the list is quite long).

Kodaira's list of singular fibers:

Notation	Description
I_0	smooth elliptic curve
I_1	rational curve with one node
I_2	two lines meeting at two points
I_n, $n \geq 3$	n lines meeting in a cycle
II	rational curve with one cusp
III	two lines simply tangent at one point
IV	three concurrent lines
I_0^*	five lines, one meeting the other four (as in $\tilde{D}_4$)
I_n^*	$n+5$ lines, meeting as in $\tilde{D}_{n+4}$
IV^*	7 lines, meeting as in $\tilde{E}_6$
III^*	8 lines, meeting as in $\tilde{E}_7$
II^*	9 lines, meeting as in $\tilde{E}_8$
${}_mI_n$	like I_n, but all components with multiplicity m

By the word "line" in the above table, I mean a smooth rational curve. These diagrams (the extended Dynkin diagrams) mentioned above are

$\tilde{D}_4$:

$$\begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 1 & - & 2 & - & 1 \\ & & | & & \\ & & 1 & & \end{array}$$

$\tilde{D}_{n+4}$:

$$\begin{array}{ccccccccc} 1 & - & 2 & - & \dots & - & 2 & - & 1 \\ & & | & & & & | & & \\ & & 1 & & & & 1 & & \end{array}$$

$\tilde{E}_6$:

$$\begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 2 & - & 1 \\ & & & & | & & & & \\ & & & & 2 & & & & \\ & & & & | & & & & \\ & & & & 1 & & & & \end{array}$$

$\tilde{E}_7$:

$$\begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 3 & - & 2 & - & 1 \\ & & & & & & | & & & & & & \\ & & & & & & 2 & & & & & & \end{array}$$

$\tilde{E}_8$:

$$\begin{array}{ccccccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 4 & - & 2 \\ & & & & & & & & & & | & & & & \\ & & & & & & & & & & 3 & & & & \end{array}$$

The numbers each stand for a single $\mathbf{P}^1$ component, of self-intersection -2; the value of the number is the multiplicity of that component in the fiber. So for example in the I_0^* case of the $\tilde{D}_4$ diagram, the central component C_0 has multiplicity two, while the other components $C_1, \ldots, C_4$ have multiplicity one, and the fiber (as a divisor on the surface) is $2C_0 + C_1 + C_2 + C_3 + C_4$.

A smooth elliptic curve may always be written in the plane as a smooth cubic curve, and many of these degenerate fibers can also be seen there. For example, I_1 is a nodal cubic curve, II is a cuspidal cubic curve, I_2 is a conic plus non-tangent line, III is a conic plus a tangent line, I_3 are three lines in a triangle, and IV is three concurrent lines.

4.4 The J-function

Two elliptic curves in Weierstrass form $y^2 = x^3 + Ax + B$ are isomorphic if and only if their J-invariants

$$J = \frac{4A^3}{4A^3 + 27B^2}$$

are equal. For an elliptic surface $f : X \to C$, the J-function becomes a function on the base curve C. When there are singular fibers J may have a pole, and so one usually considers it as a map $J : C \to \mathbf{P}^1$. However J may not have a pole, even at singular fibers; for surfaces in Weierstrass form the local behaviour of J is determined by the order of A, B, and $D = 4A^3 + 27B^2$. We have the following table (where $a = \mathrm{order}(A)$, $b = \mathrm{order}(B)$, and $d = \mathrm{order}(D)$):

Name	a	b	d	J	mult(J)
I_0	0	0	0	$\neq 0,1,\infty$	-
I_0	$a \geq 1$	0	0	0	$3a$
I_0	0	$b \geq 1$	0	1	$2b$
$I_n, n \geq 1$	0	0	n	∞	n
I_0^*	2	3	6	$\neq 0,1,\infty$	-
I_0	$a \geq 3$	3	6	0	$3a-6$
I_0	2	$b \geq 4$	6	1	$2b-6$
$I_n, n \geq 1$	2	3	n+6	∞	n
II	$a \geq 1$	1	2	0	$3a-2$
III	1	$b \geq 2$	3	1	$2b-3$
IV	$a \geq 2$	2	4	0	$3a-4$
IV^*	$a \geq 3$	4	8	0	$3a-8$
III^*	3	$b \geq 5$	9	1	$2b-9$
II^*	$a \geq 4$	5	10	0	$3a-10$

Note that in the above table, we do not consider $a \geq 4$ and $b \geq 6$ together; if this happens, we can divide (locally) by t^4 and t^6 respectively and get an isomorphic family, and continue doing this until either $a \leq 3$ or $b \leq 5$.

Note that if one has a local paramter t, and one replaces A by At^2 and B by Bt^3, the J-function is left invariant. However this operation (called a "quadratic twist") does change the nature of the singular fiber: for example, it switches an I_n fiber to an I_N^* fiber, and vice-versa, and a glance at the table shows that it also switches the types II and IV^*, III and III^*, and IV and II^*. Therefore knowledge of the J-function does not determine the elliptic surface locally: not even the type of the singular fiber is determined completely.

However this quadratic twist is the *only* way that two Jacobian elliptic surfaces can have the same J-function, at least locally: the precise result is that a Jacobian elliptic fibration is determined by the J-function up to quadratic twists.

For more on the basic theory of elliptic surfaces, the reader may consult [Mi], which is an elementary introduction concentrating on the Jacobian elliptic surfaces.

4.5 Multiple Fibers and Logarithmic Transforms

The existence of the multiple fibers is somwhat counter-intuitive: how can an elliptic curve degenerate to a double one? There is no real contradiction, and actually examples are not too hard to come by. Take a plane sextic curve S with 9 double points; this has geometric genus 1, and can be constructed by embedding any elliptic curve in $\mathbf{P}^5$ via a linear system of degree 6 and generically projecting to a plane. Through these nine points there also lies a cubic curve C. Consider the pencil generated by $2C$ (which is a sextic) and S. This pencil has nine base points (the nine double points) since these are the only points at which C and S meet. Blowing these nine points up one obtains an elliptic surface over $\mathbf{P}^1$, and the fiber corresponding to $2C$ is a fiber of multiplicity two (of type ${}_2I_0$ if C is smooth).

Indeed, this pattern continues: one may construct fibers of multiplicity m by taking a S to be a curve of degree $3m$ with 9 points of multiplicity m, and the cubic C through the nine points, and looking at the pencil generated by S and mC.

There can only be a finite set of singular fibers, and in particular only finitely many multiple fibers. Therefore in a punctured neighborhood of a multiple fiber (that is, take a neighborhood and delete the fiber itself), the surface looks like a surface without any multiple fiber, with a well-defined J-function.

Now take an elliptic fibration $f : X \to C$, and a point $0 \in C$, and a small neighborhood Δ of 0 in C. Consider the pre-image $f^{-1}(\Delta - \{0\})$; for small Δ, f restricts to this pre-image and gives an elliptic fibration with no singular fibers.

The punctured disc is unfortunately not a small neighborhood of any point, so the J-function does not quite determine the elliptic fibration over $\Delta - \{0\}$. To be brief, there is also the monodromy of the two real generators of H_1 going around 0. The theorem for punctured discs is that the elliptic fibration is determined by the J-function and the monodromy.

Therefore one can do "surgery" at will: take out this neighborhood and glue in another, with the same J-function and monodromy on the punctured disc of course, but with a different singular fiber.

Now comes the surprise: a multiple fiber has the same J-function and monodromy as a multiplicity one fiber! Therefore this surgery is possible, and we can replace any ${}_mI_n$ (where J has a pole of order n) with any other ${}_kI_n$ (where J also has a pole of order n). This operation is called a "logarithmic transform", and can radically change the nature of the elliptic surface.

4.6 Numerical Invariants

Let $f : X \to C$ be a minimal elliptic surface, with no (-1)-curves lying in the fibers of f. The canonical class K does not meet the general fiber, and is therefore supported at fibers. In fact, there is a canonical bundle formula

$$K = f^*(K_C + D) + \sum_i (m_i - 1)F_i$$

where the F_i are the multiple fibers (F_i having multiplicity m_i) and D is some non-negative divisor on C. (If f is a Jacobian fibration then D is the conormal bundle of the zero-section of f.) This formula implies in particular that $K^2 = 0$ for an elliptic surface.

All one-forms are pulled back from C, unless X is the product surface $C \times E$ for some elliptic curve E. Therefore $q = g(C)$ (unless X is a product, in which case $q = g(C) + 1$). The value of p_g depends on the multiplicities and the divisor D of course.

Suppose for example that we start with the product surface $X = \mathbf{P}^1 \times E$. This is ruled, and has Kodaira dimension $\kappa = -\infty$. We have $K = -2F$, numerically, where F is a fiber of the elliptic structure, namely the projection to $\mathbf{P}^1$. Now start making logarithmic transforms; this will begin to add terms of the form $(m - 1)F_i$, and for enough terms, we can switch K to a positive divisor eventually. In particular, we can change the Kodaira dimension!

Suppose that we take the previous example of $\mathbf{P}^1 \times E$ and only perform a *single* logarithmic transform. A relatively straightforward compution of the homology of this space shows that it has $b_1 = 1$; thus it is not even an algebraic surface! (For an algebraic surface $b_1 = 2q$ and is always even.)

As an example of a rational elliptic surface, take two cubic curves in the plane and blow up the nine base points of the pencil they generate; this blowup maps to the parameter line

$\mathbf{P}^1$ of the pencil, with general fiber a general cubic curve in the pencil, exhibiting its elliptic structure.

Of course, if E is an elliptic curve, then $E \times \mathbf{P}^1$ is both ruled (over E) and elliptic (over $\mathbf{P}^1$).

Any product $E \times E'$ with both E and E' elliptic is an elliptic abelian surface.

A quartic surface in $\mathbf{P}^3$ containing a line is a $K3$ surface which is elliptic; the pencil of hyperplanes through the line cuts the surface in the line plus a residual cubic curve, which exhibits the elliptic fibration.

If one takes a rational elliptic surface and performs two logarithmic transformations, creating two double fibers, one gets an Enriques surface. Moreover *every* Enriques surface can be obtained in this way; every Enriques surface is elliptic, with infinitely many elliptic fibrations.

The hyperelliptic surfaces are obviously elliptic by their definition, and in fact carry two elliptic structures.

To finally see an example of an elliptic surface with $\kappa = 1$, consider the Weierstrass equation

$$y^2 = x^3 + A_{4n}(t)x + B_{6n}(t),$$

where A and B are polynomials of degree $4n$ and $6n$ respectively. This is the general Weierstrass form for a Jacobian elliptic surface over $\mathbf{P}^1$. For $n = 0$ we have the product surface $E \times \mathbf{P}^1$. For $n = 1$ the surface is a rational elliptic surface, and can be described by a pencil of cubic curves as mentioned above. For $n = 2$ the surface is a $K3$ surface. For $n \geq 3$ the surface has $\kappa = 1$; the canonical class $K = (n-2)F$.

4.7 Classification Questions

The classification problem for elliptic surfaces is considered to be "solved". Let me explain the sense in which this is true.

The first step is to classify all Jacobian elliptic surfaces. As noted above, these are all described by a line bundle L on the base curve C, and two sections A of L^4 and B of L^6; then the Weierstrass equation

$$y^2 = x^3 + Ax + B$$

defines the surface. So up to understanding curves, line bundles, and sections of powers of the bundle, we consider this to be manageable.

The second step is to realize that, using logarithmic transformations, one can add or subtract multiple fibers on a given elliptic surface at will. Therefore one needs only to classify the elliptic surfaces without any multiple fibers; all others are obtained from these by performing logarithmic transformations.

Finally there is the problem of classifying elliptic surfaces with no multiple fibers, but without any section (so that it is not a Jacobian surface). A glance at the list of possible singular fibers shows that for any non-multiple fiber, there is at least one component of multiplicity one, and hence a *local* section. So every such surface is *locally* isomorphic to a Jacobian surface, and the classification problem becomes a cohomological one, as does every classification of objects locally like a given one. For more details, the reader may consult the Shafarevich seminar notes [S2].

5 Surfaces of General Type

5.1 Positivity of K^2, e, χ

We have now introduced all of the surfaces of special type, and finally we turn to the surfaces of general type, that is, surfaces with Kodaira dimension $\kappa = 2$. By Riemann-Roch, one has a formula for the higher plurigenera

$$P_n = \dim H^0(nK) = (nK)(nK - K)/2 + \chi = \chi + n(n-1)K^2/2$$

for $n \geq 2$, using Kodaira vanishing for $H^1(nK)$ (this is assuming the surface is minimal). Therefore since for $\kappa = 2$ the plurigenera grow quadratically, we must have

$$K^2 > 0$$

for a minimal surface of general type. In the Mori minimal model point of view this property, if K is nef, becomes the definition of general type.

Slightly more technical is the positivity of the Euler number

$$e > 0.$$

Therefore by Noether's inequality $12\chi = K^2 + e$ we have that

$$\chi = 1 - q + p_g > 0$$

or, in other words, $p_g \geq q$.

5.2 Noether's Inequality

About 100 year ago Max Noether proved the inequality

$$p_g \leq 2 + K^2/2$$

for a minimal surface of general type. The basic idea is as follows; assume that a general member $C \in |K|$ is irreducible, and recall that by adjunction we have

$$(K + C)|_C = K_C.$$

On the one hand we know that

$$h^0(K_C) = g(C) = (C^2 + CK)/2 + 1 = K^2 + 1;$$

but a general fact says that in this situation we have $h^0((K+C)|_C) \geq h^0(K|_C) + h^0(C|_C) - 1 = 2h^0(K|_C) - 1$. Now from the sequence

$$0 \to \mathcal{O} \to K \to K|_C \to 0$$

we see that $h^0(K|_C) \geq p_g - 1$, so that $h^0((K+C)|_C) \geq 2p_g - 3$. Hence $K^2 + 1 \geq 2p_g - 3$, leading to Noether's inequality.

More care is necessary in the case that the linear system $|K|$ does not have an irreducible member, but this is the main line of the argument.

5.3 The Bogomolov-Miyaoka-Yau Inequality

The positivity of the Euler number e leads immediately to the inequality

$$K^2 \leq 12\chi$$

by Noether's formula. This has been sharpened to the Bogomolov-Miyaoka-Yau inequality

$$K^2 \leq 9\chi$$

which is much deeper and 100 years younger than the previous inequalities.

5.4 Geography

The four inequalities

$$K^2 \geq 1, \quad \chi \geq 1, \quad K^2 \geq 2\chi - 6, \quad \text{ and } K^2 \leq 9\chi$$

give a region in the first quadrant of the (χ, K^2) plane, and the classification questions concerning surfaces of general type center primarily on understanding the particular surfaces with K^2 and χ in this region. It is a theorem of Ulf Persson [P] that all values with $K^2 \leq 8\chi$ actually occur; there exist surfaces of general type in this sub-region of the region of invariants.

Note that the signature τ of the intersection form on $H^2(\mathbf{R})$ is

$$\tau = (K^2 - 2e)/3 = (K^2 - 2(12\chi - K^2))/3 = K^2 - 8\chi$$

so that these surfces constructed by Persson have negative signature. Surfaces with positive signature are in general much harder to construct.

5.5 Complete intersections

Suppose that X is a complete intersection in $\mathbf{P}^n$ of type $(d_1, d_2, \ldots, d_{n-2})$. Then

$$K = (\sum_{i=1}^{n-2} d_i - n - 1)H$$

so that as soon as $\sum d_i \geq n + 2$ the surface is of general type. Since we should always take $d_i \geq 2$, the only exceptions to this are the complete intersections of the following types:

$n = 3:$	(2)	quadric surface, rational
$n = 3:$	(3)	cubic surface, rational
$n = 3:$	(4)	quartic surface, $K3$
$n = 4:$	$(2,2)$	quartic Del Pezzo surface, rational
$n = 4:$	$(2,3)$	$K3$ surface of genus 4
$n = 5:$	$(2,2,2)$	$K3$ surface of genus 5.

All other complete intersections are of general type.

For hypersurfaces, that is, surfaces of degree d in $\mathbf{P}^3$, one has

$$\chi = 1 + \frac{(d-1)(d-2)(d-3)}{6}$$

while $K = (d-4)H$ so that

$$K^2 = (d-4)^2 d.$$

Hence, asymptotically for large d, one has $K^2 \approx 6\chi$.

A similar but more complicated analysis for general complete intersections yields the following:

Proposition 5.21 *If X is a complete intersection surface of general type, then $K^2 \leq 8\chi$. Moreover asymptotically, $K^2 \approx 8\chi$ as the degrees grow.*

5.6 Double Covers

A *double cover* of the plane is a surface defined affinely by an equation

$$z^2 = f(x, y)$$

where f is a polynomial of even degree (otherwise the covering is branched also at the line at infinity). Note that z is a square root of f, and therefore numerically, if f has degree $2d$, then z has degree d. Note that

$$2z dz = df$$

so that $dz = df/z$.

Globalizing this is a straight-forward matter. Passing to a general divisor B on a general surface Y, suppose that $B \equiv 2L$ for some divisor L (this is the analogue of the "even degree" assumption). Then a double cover $\pi : X \to Y$ exists, branched over the curve B; the local formula that $dz = df/z$ as above, where f is a local equation for B, leads to the global formula that

$$K_X = \pi^*(K_Y + L).$$

The Euler number of the covering is easy to calculate; let us illustrate this for $Y = \mathbf{P}^2$. Assume that the branch curve B is smooth of degree $2d$, so that also its preimage $R \subset X$ is smooth. Then

$$\begin{aligned} e(X) &= e(X - R) + e(R) \\ &= 2e(\mathbf{P}^2 - B) + e(B) \\ &= 2e(\mathbf{P}^2) - e(B) \\ &= 6 - (2 - 2g(B)) \\ &= 4 + (2d-1)(2d-2) = 4d^2 - 6d + 6. \end{aligned}$$

Since $K^2 = \pi^*((d-3)H)$, and since $H^2 = 1$ and π^* doubles intersections, then

$$K^2 = 2(d-3)^2 = 2d^2 - 12d + 18.$$

Therefore

$$\chi = \frac{1}{12}(e + K^2) = \frac{1}{12}(6d^2 - 18d + 24) = \frac{1}{2}(d^2 - 3d + 4).$$

Therefore we see that, asymptotically, double covers of $\mathbf{P}^2$ only gives surfaces with $K^2 \approx 4\chi$.

5.7 Horikawa's Analysis of Surfaces on the Noether Line

Let us consider minimal surfaces of general type whose invariants lie on the Noether line $K^2 = 2\chi - 6$. For these surfaces $q = 0$, and $p_g = \chi - 1$. The canonical map has total degree K^2 of course, but Horikawa in the 1970s (see [Ho2]) found that in fact, in almost all cases this map is a double cover onto a surface of degree $\chi - 3$ in $\mathbf{P}^{\chi-2}$. This is a surface of minimal degree, and therefore is either a rational normal scroll or the Veronese surface in $\mathbf{P}^5$. Therefore these surfaces are constructed rather easily as double covers of very well-understood surfaces.

If one takes the double cover of $\mathbf{P}^2$ branched over a curve of degree 8 or 10, then the formulas above give invariants

$$\begin{aligned} B \text{ of degree } 8: &\quad \chi = 4; \quad K^2 = 2 \\ B \text{ of degree } 10: &\quad \chi = 7; \quad K^2 = 8 \end{aligned}$$

which are on the Noether line; for degrees 12 or larger, they are not.

Let us consider also the double covers of $\mathbf{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ branched over a curve of bidegree $(6, 2d)$, that is, a curve B linearly equivalent to $6G+(2d)F$, where G and F are the horizontal and vertical rulings. Then $L = 3G + dF$, and $K_{\mathbf{F}_0} = -2G - 2F$, so that on the double cover X we have

$$K^2 = 2(G + (d-2)F)^2 = 4d - 8.$$

A similar computation as the above one for $\mathbf{P}^2$ leads to

$$\chi = 2d - 1$$

so that for any $d \geq 3$ we have a minimal surface of general type on the Noether line.

These examples, and others built in the same way from the higher $\mathbf{F}_n$'s, classify all surfaces on the Noether line, except for a few surfaces with low invariants. These surfaces are being called the *Horikawa surfaces.*

5.8 Surfaces on the BMY line

As indicated above, it is quite difficult to construct surfaces with positive index, and especially difficult to construct surfaces whose invariants satisfy the Bogomolov-Miyaoka-Yau equality $K^2 = 9\chi$. About 25 years ago Mumford in [Mu] made a complicated construction involving lifting a singular configuration of surfaces over a characteristic two field to characteristic zero, and produced an example of a surface with $\chi = 1$ and $K^2 = 9$. Hirzebruch, about 13 years ago, produced relatively simple constructions involving covers of the plane, of degree larger than two, branched over unions of lines (see [Hi]). That is, these examples were surfaces defined by equations of the form

$$z^n = \prod_{i=1}^{k} L_i(x, y)$$

where the L_i's are linear. One needs the lines in a very special position for the resulting surface to lie on the BMY line. The first example is to take four general points, then take the six lines through any two; with $n = 5$, one obtains (as a quintic cover of the plane) a surface with $K^2 = 9 \cdot 5^4$ and $e = 3 \cdot 5^4$ so $\chi = 5^4$ and we are on the BMY line.

Another example is to take a smooth cubic curve, and consider the nine flex points; take the corresponding nine lines in the dual plane, and take the quintic cover. Here $K^2 = 333 \cdot 5^6$!

There are only finitely many such constructions known, though. Constructions of surfaces with $K^2 = 9\chi$ is still an active and interesting area of current research. It is known that every such surface is a quotient of the 2-ball by a group acting freely and properly discontinuously. For each value of χ, there are only finitely many minimal surfaces of general type up to isomorhism with $K^2 = 9\chi$; these surfaces are rigid (i.e., no deformations).

5.9 Quintic Surfaces

A quintic surface in $\mathbf{P}^3$ has $K = H$, so $K^2 = H^2 = 5$; $q = 0$ and $p_g = 4$ so $\chi = 5$ too.

Another example of a surface with these invariants is the double cover of $\mathbf{F}_0$ branched along a curve of bidegree $(6, 8)$, with two quadruple points lying on a single fiber of the ruling on $\mathbf{F}_0$. Without the singularities, we saw above that this surface has $\chi = 7$ and $K^2 = 8$. However the singularities drop the invariants; in this case Horikawa shows that χ drops by 2 and K^2 by three. Therefore the resulting surface actually has $\chi = K^2 = 5$ as does the quintic.

Horikawa (see [Ho1]) proved that every surface with $K^2 = \chi = 5$ was either a quintic or a double cover of this type, or a degeneration of one of these; there are two components to the moduli space of surfaces with these invariants. These two components meet; there are common degenerations of these surfaces.

5.10 Godeaux Surfaces

Very few other surfaces are completely classified. As an embarrassing example, let us consider surfaces with $K^2 = 1$ and $p_g = 0$, the smallest possible set of invariants for minimal surfaces of general type. These surfaces are historically important; after Enriques' example of the Enriques surface, disproving the conjecture that if $p_g = q = 0$ then the surface was rational, it was not known that a surface of general type could be constructed with $p_g = q = 0$ until Godeaux in the 1930's gave the following example.

Take the Fermat quintic surface $x^5 + y^5 + z^5 + w^5 = 0$ in $\mathbf{P}^3$. Let G be the cyclic group of order 5, acting on $\mathbf{P}^3$ by letting the generator send a point $[x : y : z : w]$ to $[x : \zeta y : \zeta^2 z : \zeta^3 w]$ (where ζ is a primitive fifth root of unity). This acts on the quintic, freely. The quotient X is a compact surface of general type, and it is easy to see that there are no G-invariant holomorphic 1-forms or 2-forms on the quintic; hence $p_g = q = 0$ for the quotient X. The canonical class on X lifts to that on the quintic, and since $K^2 = 5$ for the quintic, we have $K^2 = 1$ for X. The fundamental group is $\mathbf{Z}/5$.

A minimal surface of general type with $p_g = q = 0$ and $K^2 = 1$ is called a *Godeaux surface* honoring this example. Godeaux surfaces are not classified; it is not known how many components the moduli space has.

Miles Reid in [Rd] has shown that Godeaux surfaces have at most an order five torsion subgroup T of the Picard group, and has classified those with T of order at least 3.

5.11 Rational Double Points (DuVal Singularities) and Canonical Models

Suppose that a surface contains a (-2)-curve, that is, a smooth rational curve C isomorphic to $\mathbf{P}^1$, with self-intersection -2. By the adjunction formula, we have

$$-2 = 2g(C) - 2 = C^2 + (C \cdot K) = -2 + (C \cdot K)$$

so that $(C \cdot K) = 0$.

Suppose that X is a minimal surface of general type, and consider a pluricanonical map for X, given by the sections of nK for some large n. Since X is of general type, the image of this map will be a surface; however since $(C \cdot nK) = 0$, any (-2)-curve C will be contracted by this map.

The contraction of a (-2)-curve creates a singular point on the image. More generally, if one has any connected collection of (-2)-curves on X, they are all contracted, to a single singular point on the image. Any connected configuration of curves which is contracted must be have a negative definite intersection form by the Hodge Index Theorem, and this restricts greatly the possible configurations; they are

A_n:

$$\begin{array}{ccccccccc} 1 & - & 1 & - & \dots & - & 1 & - & 1 \end{array}$$

D_n:

$$\begin{array}{ccccccccc} 1 & - & 2 & - & \dots & - & 2 & - & 1 \\ & & | & & & & & & \\ & & 1 & & & & & & \end{array}$$

E_6:

$$\begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 2 & - & 1 \\ & & & & | & & & & \\ & & & & 2 & & & & \end{array}$$

E_7:

$$\begin{array}{ccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 3 & - & 2 \\ & & & & & & | & & & & \\ & & & & & & 2 & & & & \end{array}$$

E_8:

$$\begin{array}{ccccccccccccc} 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 4 & - & 2 \\ & & & & & & & & | & & & & \\ & & & & & & & & 3 & & & & \end{array}$$

The singular points which are created are the most important in the singularity theory of surfaces; they are called the *rational double points*, or the *DuVal singularities*. They are relatively ubiquitous in the theory of surfaces, although we have not mentioned them up until this point. The reader can see their relationship with the singular fibers of elliptic surfaces; the numbers which appear above are the multiplicities of the corresponding curve in the inverse image of the maximal ideal of the singular point in the smooth resolution X. An interesting account may be found in [D].

These singularities, as we have seen, cannot be avoided in the pluricanonical images of a surface of general type. However these are the only singularities which appear.

Theorem 5.22 *Let X be a minimal surface of general type. Then for any $n \geq 5$, the pluricanonical map given by the linear system $|nK|$ maps X birationally to a surface Y, contracting all configurations of (-2)-curves to rational double points. This map is an isomorphism away from the (-2)-curves.*

The statement that $5K$ is enough to embed any surface X is due to Bombieri [Bo], and there is much more detailed information concerning surfaces which can be embedded by lower multiples of K. An alternate approach to this was given by Reider in [Rr].

The image Y is independent of n, and is called the *canonical model* of X. It can be described rather succintly as

$$Y = \operatorname{Proj} \bigoplus_{n=0}^{\infty} H^0(nK_X).$$

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