Topics on interpolation problems in Algebraic Geometry

C. Bocci R. Miranda

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Introduction

These are the expanded and detailed notes of the lectures given by the authors during the school and workshop entitled “Polynomial Interpolation and Projective Embeddings”, held at the Politecnico di Torino during the period September 15 - 20, 2003.

The second author gave five lectures of length one hour each. He attempted to cover the basic facts in interpolation problems in algebraic geometry. Given the extensiveness of the subject, it was not possible to go into great detail in every proof. The first author gave two exercise sessions where he made examples and exercises and he introduced some topics that were complementary to the standard lectures.

We believe that these notes can be a valuable addition to the literature and give new details and new points of view of the subject that cannot be found in other expository work.

In the expository Section 1 we introduce the origin of the subject. We first focus our attention on the planar case and on the Segre and Harbourne-Hirschowitz Conjectures. We discuss the state of the art introducing several results and different techniques.

In Sections 2 and 3 we focus on one of these techniques based on the results of Lorentz and Lorentz [42] and others, which is related to a detailed study of the interpolation matrix.

Although the technique can be used more broadly, we will present the main ideas by concentrating on the study of linear systems in two variables with prescribed multiple base points, i.e., Hermite interpolation in two variables.

In Section 4 we will explain the essential features of a particular specialization technique introduced by Ciliberto and the second author in [22].

Although related closely to other specializations, the new feature is that the degeneration is not of sets of points, but, instead, we degenerate the surface where these points live. The idea is based on a degeneration method used by Z. Ran ([48]) to study enumerative problems on singular curves and consists in degenerating the plane to a reducible surface. The restriction of the limit linear system to the components of the surface are hopefully easier to understand than the system that one begins with.

In Section 5 we will explain some interesting applications of the previous degeneration technique. In particular, we will present some Lemmas that permit one to obtain information on the system by simply working with the degenerated system. At the end of the Section there will be some examples to illustrate and better understand these results.

In Section 6 we introduce a new topic for the interpolation problems: the concept of special effect varieties. These varieties are the main subject of the

Ph.D. thesis of the first author. During the workshop he gave a communication about his recent results and we present this summary in the notes as a sixth lecture.

Both authors want to thank the main organizers of this School/Workshop, Gianfranco Casnati, Silvio Greco, Nadia Chiarli, Roberto Notari and Maria Luisa Spreafico. We are also most grateful to the participants with their mathematical discussions and communications that made for a very interesting and productive week in the lovely city of Torino.

1 Lecture One: Overview

1.1 Interpolation Problems

This section is dedicated to an overview on linear systems with base points and their relationship with polynomial interpolation.

Let us start with the following naive problem: fix points $\{P_i\}$ and values $\{c_i\}$; find f such that $f(P_i) = c_i$ for each i.

The first question we can pose is "from where do we take the function f?". Let us consider the case when f is a polynomial; to be specific, let us take $f \in V_d$, with $V_d = \{\text{polynomials of degree } \leq d\}$. Even in this case the nature of the solution depends on the **number of variables**.

In one variable, the classical polynomial interpolation theory of functions in numerical analysis and statistics gives that a single-variable polynomial $f \in K[x]$ of degree d over a field K is uniquely determined by $d+1$ distinct points $P_0, \ldots, P_d$ on the affine line $\mathbb{A}^1_K$ and a set of values $c_i \in K, i = 0, \ldots, d$ such that $f(P_i) = c_i$ for each $i = 0, \ldots, d$. This is essentially due to the nonsingularity of the Vandermonde matrix.

We can generalize this problem slightly by asking not only for values of the function, but also for values of derivatives. Specifically, we can fix distinct points $z_0, \ldots z_n$ and positive integers m_i, $i = 0, \ldots, n$ such that $m_1 + \cdots + m_n = d+1$ and set the values of the derivatives:

$$f^{(j)}(z_i) = w_{i,j}, \quad i = 1, \ldots, n, \quad j = 0, \ldots, m_i - 1.$$

Again it is a standard exercise to show that one finds a unique polynomial $f(x)$ satisfying the previous conditions, for any desired values $\{w_{i,j}\}$. This is a linear problem in the vector space of polynomials $\{f\}$ of degree d, and if we set all values $w_{i,j}$ equal to zero we obtain the corresponding homogeneous linear problem, where we are seeking polynomials with values and derivatives equal to zero. In one variable, this linear problem has full rank, and the only solution to the homogeneous problem is the identically zero polynomial. We note that the only requirement is that the points $\{z_i\}$ be distinct; in particular it is not necessary that they be general in any way.

The situation for $r \geq 2$ variables is quite different. A polynomial of degree at most d in r variables $f \in K[x_1, \ldots, x_r]$ depends on $N_{r,d}+1 = \binom{d+r}{r}$ parameters. Suppose that we fix n points P_i in the r-dimensional affine space $\mathbb{A}^r_K$ and integers $m_1, \ldots, m_n$ such that

$$\sum_{i=1}^{n} \binom{m_i+r-1}{r} = N_{r,d} + 1.$$

We can then impose that $D^{(j)} f(P_i) = 0$, $i = 1, \ldots, n$, $j = 0, \ldots, m_i - 1$, where $D^{(k)}$ is any derivative of order k. (This is the homogeneous linear problem.) In analogy with the one-variable case, we can ask if the only polynomial satisfying these conditions is identically zero; in several variables, there is as yet no answer to this problem in this generality.

Going back to our starting problem, it is possible to incorporate derivatives in a more general way. Define the set

$$D_d = \{\text{constant coefficient differentiable operators of order} \leq d\}.$$

If P is any point, then the mapping

$$\begin{array}{ccc} D_d \times V_d & \rightarrow & k \\ (L, f) & \mapsto & L(f)(P) \end{array}$$

is a perfect pairing. If we fix distinct points P_i and, for each i, fix a subspace $A_i \subseteq D_d$, we can pose the following problem: determine all $f \in V_d$ such that for each i, $L(f)(P_i) = 0$ for all $L \in A_i$. We denote by $\mathcal{L}_d(-\sum A_i P_i)$ the (projectived) subspace of polynomials verifying the previous condition, i.e.

$$\mathcal{L}_d(-\sum A_i P_i) = \{f \in V_d \text{ such that } L(f)(P_i) = 0, \forall L \in A_i, \forall i\}$$

minus zero, modulo scalars.

Example 1.1. If $A_i =< I >$ for each i, then we are not asking for any derivatives; we are asking only for values. This case is called *Lagrange Interpolation*.

Example 1.2. If $A_i = D_{m_i-1}$ for each i, then we are asking that all derivatives of order at most $m_i - 1$ are zero at P_i. Thus the coefficient of the Taylor expansions are zero up through order $m_i - 1$. This is a condition on the multiplicity of the polynomial at the point P_i; in particular it means that $\text{mult}_{P_i}(f) \geq m_i$. The corresponding interpolation problem is called *Hermite Interpolation*. The corresponding linear system is denoted by $\mathcal{L}_d(-\sum m_i P_i)$. This kind of interpolation is very important because it does not depend on the choice of coordinates.

Example 1.3. Assume A_i is spanned by "monomials" $\frac{\partial^{a_1+a_2+\cdots+a_r}}{\partial x_1^{a_1} \partial x_2^{a_2} \ldots \partial x_r^{a_r}}$. The corresponding interpolation problem is called *Birkhoff Interpolation*.

Example 1.4. Let C be a smooth curve through a point P and consider a polynomial f. We can ask $f_{|C}$ vanishes to order $\geq k$ at P on C. It is not difficult to see that this can be expressed with a particular k-dimensional subspace A_i as above. We refer to this type of problem as *Curvilinear Interpolation*.

Example 1.5. The theory of splines can in some ways be considered as a generalization of interpolation problems. For splines, one considers collections of polynomials $(f_1, f_2, \ldots, f_k) \in V \times \cdots \times V$, and imposes interpolation-type conditions on these collections. For example we can ask that

$$f_1(P_1) = f_2(P_1) \qquad \text{and} \qquad f_2(P_2) = f_3(P_3) \qquad \ldots$$

1.2 The Dimension Problem

The main question we can pose in all previous examples is: what is the dimension of $\mathcal{L}_d(-\sum A_i P_i)$? As a first step we can define the **virtual dimension** of the system $\mathcal{L}_d(-\sum A_i P_i)$ as

$$\nu(\mathcal{L}_d(-\sum A_i P_i)) := \dim(\mathcal{L}_d) - \sum \dim A_i.$$

(Note that this is the projective dimension; in particular we have that $\dim(\mathcal{L}_d) = N_{r,d} = \binom{d+r}{r} - 1$.) This formula simply represents the expectation that each additional condition imposed by the space A_i will drop the dimension of the space by one. In other words, this formula will be true if all of the conditions imposed by the A_i are independent.

This number can be negative: in this case we expect that $\mathcal{L}_d(-\sum A_i P_i)$ is empty. We can then define the **expected dimension** of $\mathcal{L}_d(-\sum A_i P_i)$ as

$$\epsilon(\mathcal{L}_d(-\sum A_i P_i)) := \max\{\nu(\mathcal{L}_d(-\sum A_i P_i)), -1\};$$

here we take the convention that the empty projective space has dimension equal to -1.

Remark 1.6. It is important to observe that the dimension (and all other phenomena) of the previous system depends in a critical way on the position of points. Consider, for example, a two-variable polynomial f of degree 5 vanishing at 8 points on a line l. The dimension of the space of quintics in two variables is 20, so that expected dimension of this linear system is $20 - 8 = 12$. However if f vanishes at the first 6 of the points, then by Bezout's Theorem f vanishes all along l, and therefore vanishes at all 8 of the points. Hence the conditions imposed by the vanishing at the final 2 points are not independent of the first 6 conditions; indeed, the dimension of this space is $20 - 6 = 14$, which is exactly the dimension of the space of residual quartics.

The "reason" this phenomenon has occured is that the points are related geometrically in an obvious way. To avoid this, we assume that the points P_i are in *general position*. This notion of general position means something different for every interpolation problem.

1.3 The Interpolation Matrix

To be more precise, we introduce a matrix M associated to the interpolation problem, called the *interpolation matrix*. This matrix has columns indexed by a basis $\{f_k\}$ for the vector space V from which we are drawing our polynomials and has rows doubly-indexed by the points P_i and a basis $D_{i,j}$ for the interpolation conditions A_i for each i. We form the entries of the interpolation matrix M by applying the $D_{i,j}$'s to the f_k's, and evaluating at P_i; specifically, the entry in row $(i, D_{i,j})$ and column f_k is $D_{i,j}(f_k)(P_i)$.

It is clear that the subspace of polynomials satisfying the interpolation problem may be identified with the kernel of multiplication by M. Therefore the dimension problem is equivalent to the computation of the rank of the interpolation matrix.

Now if we take the points P_i to have undetermined coordinates, then the various minors of M become polynomials in these coordinates. The largest size (say $s \times s$) minor of M which is not identically zero determines the rank of M, for values of the coordinates of P_i which makes at least one $s \times s$ minor nonzero. This condition (that at least one of these minors be nonzero) gives a Zariski open subset of the set parametrizing n points in r-space, and determines the precise notion of "general position" for this particular interpolation problem.

Remark 1.7. Hermite interpolation, expressed via imposing multiplicities to polynomials at given points, lends itself also to working with homogeneous polynomials and points in projective space. In this way the multiplicity conditions give a homogeneous ideal in the homogeneous coordinate ring of projective space; the graded pieces (as we vary the degree d) represent the $\mathcal{L}_d$ interpolation problems. In this way the interpolation problem is equivalent to the study of the Hilbert function of the given ideal. Other commutative algebra tools now may come into play, and more complicated problems related to this ideal (such as determining generators, syzygies, ranks of multiplication maps, etc.) are of great interest also.

The Hermite interpolation problem for polynomials of degree d having multiplicities m_i at n points in general position will give a space which we will denote by $\mathcal{L}_d(m_1, \ldots, m_n)$. This notation is convenient when we do not want or need to refer to the particular positions of the n points. If there are repetitions in the multiplicities, these might be denoted using superscripts; for example, $\mathcal{L}_d(m^n)$ means the linear system of polynomials of degree d having multiplicity m at n general points.

1.4 Special Linear Systems

As explained at the beginning of the section, the Hermite interpolation problem has full rank in one variable. Thus the above-mentioned questions are all relatively easy in $\mathbb{P}^1$. However when we consider $\mathbb{P}^2$, there is still much unknown. Here, as we will see later, we have some precise conjectures. From now on, we will assume that we are working with Hermite interpolation in two variables.

A naive conjecture would be that, for points in general position, every Hermite interpolation problem leads to a linear system which always has the expected dimension: all Hermite conditions are linearly independent at distinct points. This turns out to be false, as the following example shows.

Example 1.8. Consider the system of conics $\mathcal{L}_2$, and impose two double points P_1, P_2. The notation for this system would be: $\mathcal{L}_2(-2P_1-2P_2)$ or $\mathcal{L}_2(2^2)$. Since any double point imposes three conditions to a curve in $\mathbb{P}^2$, we obtain

$$\epsilon(\mathcal{L}_2(2^2)) = \nu(\mathcal{L}_2(2^2)) = 5 - 3 - 3 = -1$$

and we expect that the system is empty. But if $f(x, y, z) = 0$ is the homogeneous linear polynomial defining the line through P_1 and P_2, then $f(x, y, z)^2$ is a nonzero conic double at P_1 and P_2. This conic exists for any two distinct points P_1 and P_2, and in particular for general points; therefore $\dim(\mathcal{L}_2(2^2)) = 0 > -1 = \epsilon(\mathcal{L}_2(2^2))$. This system does *not* have the expected dimension. Moreover note that f^2 is singular (has multiplicity two) at every point of the line.

Example 1.9. We have the same phenomenon with $\mathcal{L}_4(-\sum_{i=1}^{5} 2P_i) = \mathcal{L}_4(2^5)$ which is the linear system of quartics with five general double points. This system has expected dimension -1, but, if $q(x, y, z)$ is the polynomial of the conic through the points P_i's, then q^2 is a quartic double at every point of the conic, in particular at the five (general) points $P_1, \ldots, P_5$. This system does *not* have the expected dimension.

Note that in the first example, if we blow up the plane at the two points, the line in question becomes a (-1)-curve on the blowup, and the corresponding linear system on the blowup consists of this (-1)-curve, with multiplicity two. Similarly, in the second example, if we blow up the five points, the conic becomes a (-1)-curve, and the corresponding linear system becomes this curve, with multiplicity two.

These two examples show that the naive conjecture, that every such linear system in the plane has the expected dimension, is false.

Definition 1.10. *A system $\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_n)$ is* **special** *if its dimension is larger than the expected dimension:*

$$\dim(\mathcal{L}) > \epsilon(\mathcal{L});$$

otherwise $\mathcal{L}$ is said to be **non-special**.

Consider the blow-up $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of the plane $\mathbb{P}^2$ at $P_1, \ldots, P_n$ and let E_i, $i = 1, \ldots, n$ be the exceptional divisors corresponding to the blow-up of the points $P_i, i = 1, \ldots, n$. If we denote by H the pull-back of a general line of $\mathbb{P}^2$ via π, then we can write the strict transform of the system $\mathcal{L} := \mathcal{L}_d(\sum_{i=1}^{n} m_i P_i)$ as the complete linear system $\tilde{\mathcal{L}} = |dH - \sum_{i=1}^{n} m_i E_i|$. In the future, if confusion cannot arise, we will indicate both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ by $\mathcal{L}$.

Note that on the blowup, $\nu(\mathcal{L}) = \frac{\mathcal{L}\cdot(\mathcal{L}-K)}{2}$.

By Riemann–Roch, remembering that $h^2(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}) = 0$, we obtain

$$\dim(\mathcal{L}) = h^0(\tilde{\mathbb{P}}^2, \mathcal{L}) - 1 = \nu(\mathcal{L}) + h^1(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}). \tag{1.11}$$

Hence

$$\mathcal{L} \text{ is non-special if and only if } h^0(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}) \cdot h^1(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}) = 0. \tag{1.12}$$

Using this cohomological reformulation, it is not hard to compute that if such a linear system $\mathcal{L}$ has a (-1)-curve E on the corresponding blowup with $\mathcal{L} \cdot E \leq -2$, then E occurs as (at least) a double fixed component of $\mathcal{L}$, and if $\mathcal{L}$ is not empty, it must be special. If this happens, we will call the linear system (-1)*-special.*

1.5 Conjectures In Two Variables

Going back to the conjectures for special systems in $\mathbb{P}^2$, B. Segre was the first author who claimed that speciality is related to the non-reducedness of the general curve of the given linear system with general multiple base points.

Conjecture 1.13 ((SC) B. Segre, 1961). *If a linear system of plane curves with general multiple base points $\mathcal{L}_{2,d}(-\sum_{i=1}^{n} m_i P_i)$ is special, then its general member is non-reduced, i.e. the linear system has, according to Bertini's theorem, some multiple fixed component.*

In 1987, Gimigliano [34] studied several examples of special linear systems on $\mathbb{P}^2$ and made the previous Conjecture more precise.

Conjecture 1.14 ((GC) A. Gimigliano, 1987). *Consider a linear system of plane curves with general multiple base points $\mathcal{L}_d(-\sum_{i=1}^{n} m_i P_i)$. Then one has the following possibilities:*

- *(i) the system is non-special and its general member is irreducible;*
- *(ii) the system is non-special, its general member is non-reduced, reducible, its fixed components are all rational curves, except for at most one (this may occur only if the system has dimension 0), and the general member of its movable part is either irreducible or composed of rational curves in a pencil;*
- *(iii) the system is non-special of dimension 0 and consists of a unique multiple elliptic curve;*
- *(iv) the system is special and it has some multiple rational curve as a fixed component.*

This conjecture, in the case of special systems, was made more precise by the following conjecture given separately by B. Harbourne and A. Hirschowitz.

Conjecture 1.15 ((HHC) Harbourne–Hirschowitz, 1989). *A linear system of plane curves $\mathcal{L} := \mathcal{L}_d(-\sum_{i=1}^{n} m_i P_i)$ with general multiple base points is special if and only if it is $(-1)-$special, i.e. it contains some multiple rational curve of self-intersection -1 in the base locus.*

The last conjecture we want to mention is related to the homogeneous case, i.e. when $m_1 = m_2 = \cdots = m_n = m$.

Conjecture 1.16 ((NC) Nagata, 1960). $\mathcal{L}_d(m^n)$ *is empty as soon as* $n \geq 10$ *and* $d \leq \sqrt{n} \cdot m$

Recently, Ciliberto and Miranda, in [24], proved the following implications

$$\begin{array}{ccc} SC & \Longleftrightarrow & HHC \\ \Downarrow & & \\ NC & & \end{array}$$

Although the conjectures are still unproved it is important to note that, in more than a century of research, all known special systems are consistent with them.

1.6 Results To Date

We now mention some results on these conjectures, in particular on the conjecture of Harbourne and Hirschowitz.

The first case we treat is $\mathcal{L}_d(1^n)$, that is, when all points have multiplicity one. In this case we are asking for polynomials of degree d that simply vanish at the points. It is easy to see that this always has the expected dimension. One argues by induction on the number of points n; the statement is clearly true for $n = 0$. Assume it is true for $n-1$, and consider the system $\mathcal{L}_d(1^n)$, which is a subsystem of the system $\mathcal{L}_d(1^{n-1})$. These two systems, unless they are both empty, have expected dimensions which are different by one, and we must show that indeed $\mathcal{L}_d(1^n)$ has dimension one less. This is equivalent to showing that it is a proper subsystem of $\mathcal{L}_d(1^{n-1})$. It will be if we choose the n-th point so that it is not a base point of the system. This is true if the points are in general position. This proves the following:

Theorem 1.17 (Multiplicity One Theorem). *If the points $P_i, \ldots, P_n$ are in general position, then the dimension of $\mathcal{L}(-\sum_{i=1}^n P_i)$ is equal to the expected dimension.*

Consider now the linear systems $\mathcal{L}_{2,d}(-\sum_{i=1}^n m_i P_i)$. When the number of points n is less than or equal to 9, the anticanonical class $-\tilde{K}$ is effective on the blowup $\tilde{\mathbb{P}}^2$ and we can use vanishing theorems (Kodaira's and Kawamata-Vieweg's or Mumford's and Franchetta-Ramanujam's on the specific case of surfaces) to establish that $h^1(\tilde{\mathcal{L}}, \tilde{\mathbb{P}}^2) = 0$ and use (1.12). In this way one proves the following result already known to Castelnuovo and later rediscovered by several authors:

Theorem 1.18 (Castelnuovo, 1891; Nagata, 1960; Gimigliano 1986; Harbourne, 1986). *The Harbourne–Hirschowitz Conjecture holds for all linear systems with $n \leq 9$ general multiple base points.*

The second simple case is the one with only double points, i.e $m_1 = \cdots = m_n = 2$. This case was examined by several authors, e.g. Campbell, Palatini, Terracini. More recently, Arbarello and Cornalba used an approach based on an infinitesimal deformation technique consisting in moving the base points of the system and computing the first order deformation of a curve which moves keeping its singularities. In general, in these deformation techniques, one tries to show that if there exists $C \in \mathcal{L}$ with isolated singularities, then $H^1(\mathcal{L}) = 0$ (and conclude that $\mathcal{L}$ is non-special). In order to do this one tries to interpret this H^1 as an obstruction space to deforming $C \in \mathcal{L}$ and to prove that every element of H^1 occurs as an obstruction. In essence one tries to construct a map

$$\{\text{ Deformations of } P_i\} \xrightarrow{\text{obstruction to moving } C} H^1(\mathcal{L})$$

and show that it is onto; the existence of C for points in general position allows one to claim that it is also zero, and one deduces that the $H^1 = 0$. Using this general idea, Arbarello and Cornalba proved the following:

Theorem 1.19 (Arbarello–Cornalba, 1981). *Consider* $\mathcal{L} := \mathcal{L}_d(2^n)$. *Assume:*

(i) $\frac{d(d+3)}{2} \geq 3n$, *i.e.* $\nu(\mathcal{L}) \geq 0$;

(ii) $\binom{d-1}{2} \geq n$, *i.e.* $g_{\mathcal{L}} \geq 0$.

Then $\mathcal{L}$ *is non-special, and a general* $C \in \mathcal{L}$ *is irreducible, with nodes at the imposed general double points* $P_1, \dots P_n$, *and smooth elsewhere, except for* $\mathcal{L}_6(2^9)$ *which is a double cubic.*

Another result by (slightly different) deformation techniques is the following

Theorem 1.20 (A. Bruno, 1998). $\mathcal{L} = \mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ *is non-special if* $\nu(\mathcal{L}) \geq 0$ *and* $g_{\mathcal{L}} \geq 0$ *and the general curve has ordinary* m_i*-tuple points at* P_i, $i = 1, \dots, n$.

Although the hypothesis is rather strong, the main tool in Bruno's proof is the use of moduli space of curves, of stable reduction, and of the theory of limit linear system that is a really new idea in this setting.

A different way to attack the problem is to argue by *degeneration*. In this technique, we specialize the base points of the linear system so as to make it easier to compute the dimension of the system. Since the dimension of $\mathcal{L}(-\sum_{i=1}^n m_i P_i)$ is upper semicontinuous in the position of the points, it is enough to find a particular set of points $Q_1, \dots Q_h$ such that $\mathcal{L}(-\sum_{i=1}^n m_i Q_i)$ is non-special to conclude that also the general system $\mathcal{L}(-\sum_{i=1}^n m_i P_i)$ is non-special. In general, we try to put the points P_i in a special enough position that we can compute the dimension, but not so special that the dimension will rise.

Example 1.21. Consider, for example, the system $\mathcal{L}_5(2^7)$. Its expected dimension is $20 - 7 \cdot 3 = -1$; in other words, we expect that this system (of quintics with seven general double points) is empty. Put three of the seven points on a line l. In this case, by Bezout's Theorem, every element in $\mathcal{L}$ contains the line. Then one has

$$\mathcal{L} = l + \mathcal{L}_4(2^4, 1^3)$$

and so the dimension is equal to the dimension of the system $\mathcal{L}_4(2^4, 1^3)$ (where the three simple points are collinear). Now if C is a conic through the four points that appear with multiplicity 2 in $\mathcal{L}_4(2^4, 1^3)$ and through one of the three points with multiplicity 1, one has

$$C \cdot \mathcal{L}_4(2^4, 1^3) = 8 - 8 - 1 = -1$$

and therefore C must be a base curve of the system. Since there are three such curves, we see that this system has a sextic in its base locus; but it only has degree four. We conclude that the system must be empty.

Unfortunately, very often, convenient particular positions of the points do increase the dimension of the system. Therefore this technique has its limitations.

In [35], Hirschowitz was able to improve this degeneration technique, introducing the Horace Method (*la méthode d'Horace*). This technique is not only applicable to the planar case, but can be used on every projective variety.

Using a refined version of the Horace Method, (the so-called *differential Horace Method*, see [6]), Alexander and Hirschowitz were able to prove the following asymptotic result:

Theorem 1.22 (Alexander–Hirschowitz, 1998). *Given any projective, reduced variety X and an ample line bundle $\mathcal{L}$ on it, there is a function $d(m)$ such that if $m_i < m$, $i = 1, \dots, n$, and $d > d(m)$ then $\mathcal{L}^{\otimes d}(-\sum_{i=1}^n m_i P_i)$ is non-special.*

The prototype for results of this sort is the following theorem of Hirschowitz ([36]):

Theorem 1.23. *The system $\mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ in $\mathbb{P}^2$ is non-special as soon as*

$$\left\lceil \frac{(d+3)^2}{4} \right\rceil > \sum_{i=1}^{n} \binom{m_i + 1}{2}.$$

Returning to the general Harbourne–Hirschowitz Conjecture, we mention some other recent results.

If we pass to the quasi-homogeneous case, i.e. all m_i's equal to m except one, the Harbourne–Hirschowitz Conjecture is proved for $m = 2, 3$ by Ciliberto and Miranda [22] and for $m = 4$ by Siebert and (independently) Laface (see [38]).

The following theorem is due to T. Mignon in his thesis ([46], [47]) and it is based on the use of the Horace method:

Theorem 1.24 (T. Mignon, 1998). *Let $\mathcal{L} = \mathcal{L}_{2,d}(-\sum_{i=1}^n m_i P_i)$. Then:*

(i) if $m_i \leq 4$ then the Harbourne–Hirschowitz Conjecture 1.15 holds;

(ii) if $g_{\mathcal{L}} \leq 4$ and $\nu(\mathcal{L}) \geq 0$ then the Harbourne–Hirschowitz Conjecture 1.15 holds;

(iii) if $m_i \leq 3$, $d \geq 33$, $\nu(\mathcal{L}) \geq 0$ and $g_{\mathcal{L}} \geq 0$ then the Harbourne–Hirschowitz Conjecture 1.15 holds.

Recently, S. Yang was able to generalize part (i) of the previous result to $m_i \leq 6$. She uses a combination of the Ciliberto–Miranda degeneration with a particular specialization of the points on a fixed line with a fixed point (see [52]).

Another result to mention is the following

Theorem 1.25 (L. Evain, 1998). *$\mathcal{L}_d(m^n)$ is never special if n is of the form $n = 4^k$.*

The same result was obtained in [11] by A. Buckley and M. Zompatori using a degeneration technique; moreover they proved the same statement for the case $n = 9^k$, and for products of powers of 4 and 9.

Recently, Ciliberto and the second author, using a particular degenerations technique (which we will describe in sections 4 and 5) were able to prove the following (see [23]):

Theorem 1.26 (Ciliberto–Miranda, 1998). *The Harbourne–Hirschowitz Conjecture holds in the quasi-homogeneous cases $\mathcal{L}_d(m_0, m^n)$, $m \leq 3$ and in the homogeneous cases $\mathcal{L}_d(m^n)$, $m \leq 12$.*

Another result of Ciliberto and Miranda in [22] is the full classification of homogeneous $(-1)-$special systems.

Theorem 1.27 (Classification of homogeneous $(-1)-$special systems). *The only homogeneous linear systems $\mathcal{L}_d(m^n)$ which are $(-1)-$special are:*

$$\mathcal{L}_d(m^2) \text{ with } m \leq d \leq 2m-2$$

$$\mathcal{L}_d(m^3) \text{ with } \tfrac{3m}{2} \leq d \leq 2m-2$$

$$\mathcal{L}_d(m^5) \text{ with } 2m \leq d \leq \tfrac{5m-2}{2}$$

$$\mathcal{L}_d(m^6) \text{ with } \tfrac{12m}{5} \leq d \leq \tfrac{5m-2}{2}$$

$$\mathcal{L}_d(m^7) \text{ with } \tfrac{21m}{8} \leq d \leq \tfrac{8m-2}{3}$$

$$\mathcal{L}_d(m^8) \text{ with } \tfrac{48m}{17} \leq d \leq \tfrac{17m-2}{6}$$

For homogeneous systems, with all multiplicities equal, the Harbourne – Hirschowitz conjecture is then equivalent to stating that the only such systems that are special are on the above list. In particular, the conjecture implies that for nine or more points, there are no special homogeneous systems.

Remark 1.28. The way Ciliberto and Miranda degenerate the systems can be seen as a way to degenerate the set of points $P_1, \ldots, P_n$ by putting b of them on a line, and letting the line split from the curves of the linear system k times. This approach seems to be very systematic and, in [26] C. Ciliberto, F. Cioffi, R. Miranda and F. Orecchia applied a more refined computational algebra approach to improve the bound $m \leq 12$. In particular they have been able to work out a computer program to test the Harbourne–Hirschowitz Conjecture for $\mathcal{L}_d(m^n)$ and to prove it for $m \leq 20$.

1.7 Higher Dimensions and Waring's Problem

As we said in the first section, the general problem of computing the dimension of a system with imposed multiple points can be formulated in any dimension and for any ambient variety X, not only in the plane. But, unfortunately, just in the simplest case of $X = \mathbb{P}^r$, $r \geq 3$ very little is known. In this setting we fix notation and define $\mathcal{L}_{r,d} := |\mathcal{O}_{\mathbb{P}^r}(dH)|$.

The most important result is a theorem due to Alexander and Hirschowitz which classifies the special linear systems with imposed double points $\mathcal{L}_{r,d}(2^n)$.

Theorem 1.29 (Alexander–Hirschowitz, 1996). *The system $\mathcal{L}_{r,d}(2^n)$ is non-special unless r, d, and n are in one of the columns of the following table:*

r	*any*	2	3	4	4
d	2	4	4	4	3
n	$2,\dots,r$	5	9	14	7

The original proof of this theorem requires the Horace method, and occupies a whole series of papers [1], [2], [3], [4], [5]. Another proof, somewhat shorter, was recently given by K. Chandler in [19]. She still used the Horace method but in a particularly efficient way, specializing part of the points to a hyperplane.

Let us analyze the systems in Theorem 1.29. It is very easy to show that linear systems $\mathcal{L}_{r,2}(2^n)$ with $2 \le n \le r$ are special. We know that every quadric hypersurface is defined by a quadratic polynomial, which, if we homogenize, can be considered as a quadratic form in $r+1$ variables. This in turn can be considered as a symmetric matrix Q of size $r+1$. We can choose coordinates so that the first $r+1$ points (if there are that many) occur at the "coordinates points" whose homogeneous coordinates correspond to the standard basis vectors, i.e. the points $(1,0,0,\dots,0)$, $(0,1,0,\dots,0)$, etc. For the quadric hypersurface to have multiplicity at least two at $(1,0,0,\dots,0)$, the first row (and column) of the matrix Q must be zero. This clearly imposes $r+1$ conditions, as we expect. However, for the quadric to have multiplicity at least two at the second point $(0,1,0,\dots,0)$, the second row and column of Q must be zero. If the first row and column are already zero, the first entries of the second row and column are automatically zero, so there are only r additional entries that must be zero. Hence the second point imposes only r conditions and the dimension of $\mathcal{L}_{r,2}(2^n)$ is one larger than the expected. This phenomenon continues until there are $r+1$ points, in which case the matrix Q is all zero and there are no nontrivial quadratic polynomials satisfying the conditions: if $n \ge r+1$ then $\mathcal{L}_{r,2}(2^n)$ is empty as we expect.

The cases $\mathcal{L}_{2,4}(2^5)$, $\mathcal{L}_{3,4}(2^9)$ and $\mathcal{L}_{4,4}(2^{14})$ are similar. The first one is already treated in example 1.9. For $\mathcal{L}_{3,4}(2^9)$ and $\mathcal{L}_{4,4}(2^{14})$ we observe that both have expected dimension -1, but there is an element given by the double of the quadric respectively in $\mathbb{P}^3$ and $\mathbb{P}^4$ through the 9 and the 14 points.

Finally consider the system $\mathcal{L}_{4,3}(2^7)$. Its virtual dimension is -1 whereas it is not empty. In fact there is an unique rational normal quartic curve C_4 through 7 general points $P_1,\dots,P_7$ in $\mathbb{P}^4$. Let X be the first secant variety of C, i.e. $X := \mathrm{Sec}_1(C)$. Then X is a cubic hypersurface and it is singular along C; therefore it is singular at $P_1,\dots,P_7$. Thus X sits in $\mathcal{L}_{4,3}(2^7)$.

More recently, some results on the higher dimension case are given by Bocci ($\mathbb{P}^r$ and $\mathbb{P}^{r_1}\times\cdots\times\mathbb{P}^{r_t}$), Catalisano, Geramita and Gimigliano ($\mathbb{P}^{r_1}\times\cdots\times\mathbb{P}^{r_t}$) and De Volder, Laface and Ugaglia ($\mathbb{P}^3$).

Let $Q = [q_o : \cdots : q_r] \in \mathbb{P}^r$. We define the differential operator $\Delta_Q = \sum q_i \frac{\partial}{\partial x_i}$. Moreover, given a set of n points $\{Q_i\}$, we define

$$A_d(\sum Q_i) = \{\sum M_i \Delta_{Q_i}^{d-1}, \deg(M_i) = 1\}.$$

Thus we have a pairing

$$\begin{array}{ccc} \{\text{differential operators of degree} \le d\} & \longleftrightarrow & \{\text{polynomials of degree } \le d\} \\ \cup & & \cup \\ A_d(\sum Q_i) & & \mathcal{L}_d(-\sum 2Q_i) \end{array}$$

By Terracini's Lemma, with this pairing, $A_d(\sum Q_i)$ and $\mathcal{L}_d(-\sum 2Q_i)$ annihilate each other.

If we let W be the d-Veronese variety of $\mathbb{P}^r$, the space $A_d(\sum Q_i)$ can be identified with the tangent space to the n-secant variety to W (at the point corresponding to the n points Q_i). Therefore information about the dimension of $\mathcal{L}_d(\sum 2Q_i)$ will give information about the dimension of this secant variety. In particular, when this secant variety is the whole space, then the general form of degree d can be written as a sum of n pure d-th powers of linear forms. This is a version of Waring's Problem for Forms, and is a beautiful application of the Alexander–Hirschowitz theorem.

Recently, in [13], Carlini proposed and interesting generalization of Waring's problem.

2 Lecture Two: The Matrix Approach I

2.1 Visualizing Birkhoff Interpolation

In the previous lecture we did not mention the results of Lorentz and Lorentz [42] and others, which use a different technique based on a detailed study of the interpolation matrix. We will now present the essential features of this approach.

Although the technique can be used more broadly, we will present the main ideas by concentrating on the study of linear systems in two variables with prescribed multiple base points, i.e., Hermite interpolation in two variables. We recall that a polynomial f has multiplicity at least m at a point P if all derivatives of f, up through order $m-1$, are zero at P.

The technique for Hermite interpolation however immediately leads to considerations of the more general Birkhoff interpolation, where one considers a general set of derivatives (which may be considered as "monomial" differential operators) to be zero at a given point. Using suitable coordinates centered at the point in question, this means that the polynomial is contained in an ideal generated by monomials, which defines a zero-dimensional scheme. A graphical representation of such an ideal is often useful, where one uses the lattice of all monomials (in the first quadrant of the plane) and indicates those monomials which are not in the ideal.

If, for example, the ideal is given by $< y^2, x^2y, x^4 >$, one may visualize it as follows:

$$\begin{array}{c|cccccc}
\vdots & & & & & & \\
y^4 & & & & & & \\
y^3 & & & & & & \\
y^2 & & & & & & \\
y^1 & \bullet & \bullet & & & & \\
y^0 & \bullet & \bullet & \bullet & \bullet & & \\
\hline
 & x^0 & x^1 & x^2 & x^3 & x^4 & \dots
\end{array}$$

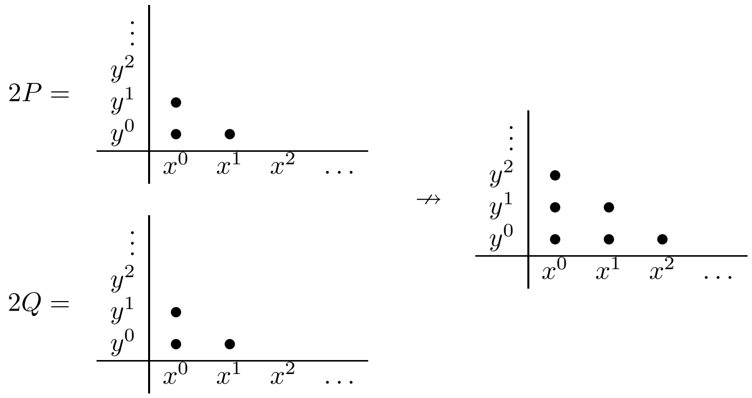
One may ask two natural questions about these zero-dimensional schemes:

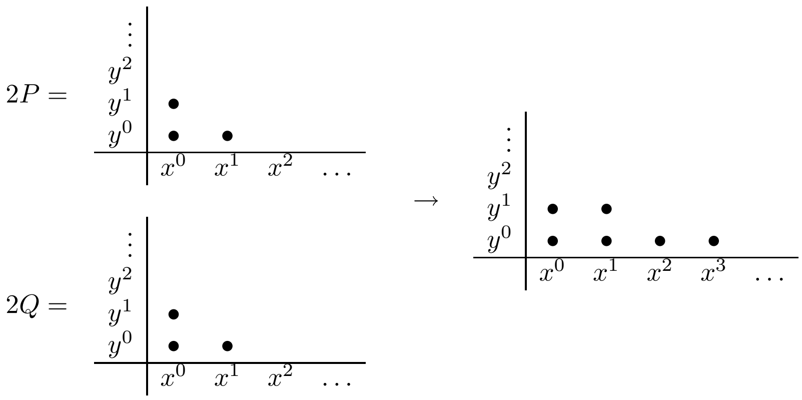
1) How do these zero-dimensional schemes degenerate ?

2) How do these zero-dimensional schemes collide?

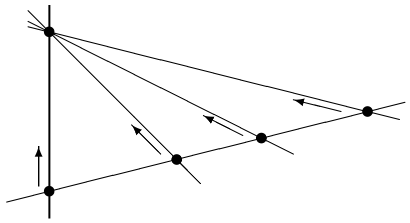
If the movement is along an axis, the answer is obtained by just "stacking" the monomials. In other words, if an ideal I_1 has the monomials $\{x^iy^j \mid 0 \leq j < r_i\}$ not in it, and a second ideal I_2 has the monomials $\{x^iy^j \mid 0 \leq j < s_i\}$ not in it, then the flat limit of these two zero-dimensional schemes, if they approach each other along the y-axis, has the monomials $\{x^iy^j \mid 0 \leq j < r_i + s_i\}$ not in it. (This is a relatively easy exercise which we encourage the reader to attempt.)

This stacking algorithm however is not the only type of collision that can occur with such monomial ideals, and the question of what actually is possible to obtain as a flat limit is quite delicate. For example, if one stacks two ordinary double points, one obtains the tacnode ideal generated by $\{y^2, x^2y, x^4\}$ drawn above, which has the six monomials $\{1, y, x, xy, x^2, x^3\}$ not in it. Are there other possibilities for the collision of two double points? Flatness requires that the codimension of the ideals must be preserved; since a double point ideal has codimension three, the collision of two double points must have codimension six. Is a triple point (which also has codimension six) possible? We will see later that the collision of two double points can not be a triple point, even though both have codimension six: a triple point scheme is not the (flat) limit of two double point schemes.





Another limit that is not so obvious is the following one, of four multiple points on a line approaching a single point:



We can ask for example what is the limit here as the points come together.

2.2 The Interpolation Matrix (revisited)

Going back to our interpolation problem, fix a vector space V of bivariate polynomials which are spanned by monomials $x^i y^j$ indexed by a set of lattice points S. This means that a typical polynomial in V has the form

$$f(x,y) = \sum_{(i,j)\in S} a_{i,j} \cdot x^i y^j$$

where the numbers $a_{i,j}$ are the coefficients of the term $x^i y^j$.

Birkhoff interpolation (at a point $P = (x_P, y_P)$) would impose some (monomial) differential operators A at P, i.e. $L(f(P)) = 0$, for $L \in A$. Therefore A is spanned by monomial differential operators, e.g.

$$\Delta = \frac{\partial^{a+b}}{\partial x^a \partial y^b}.$$

Define

$$T_m = \{(r,s) | r \geq 0, s \geq 0, r+s < m\}$$

to be the "triangle" of derivatives orders at most $m-1$. Thus, with Hermite interpolation, for any i, $A_i = T_{m_i}$ for some integer m_i. We can write $f \in V$ as a row-column product:

$$f = \sum a_{i,j} x^i y^j = \begin{pmatrix} \dots & x^i y^j & \dots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{i,j} \\ \vdots \end{pmatrix}.$$

To analyze the condition that $\Delta(f)(P) = 0$, one takes the row vector of monomials

$$\begin{pmatrix} \dots & x^i y^j & \dots \end{pmatrix},$$

applies Δ, and evaluates at P:

$$\begin{pmatrix} \dots & \Delta(x_P^i y_P^j) & \dots \end{pmatrix}.$$

This gives a new row, and finally we ask the product

$$\begin{pmatrix} \dots & \Delta(x_P^i y_P^j) & \dots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{i,j} \\ \vdots \end{pmatrix}$$

has to be zero.

The generalization of this process lead us to the definition of the *Matrix of the Interpolation Problem*. For that, consider

1. an integer n, the number of points;

2. for each i, $1 \leq i \leq n$, a point $P_i := (x_i, y_i)$;

3. For each i, $1 \leq i \leq n$, a finite set A_i of (monomial) differential operators.

From now on we assume that $\dim(V) = \sum_{i=1}^{n} \dim(A_i)$ so that we have a square linear problem. Denote by $\mathfrak{A} := \{A_i\}_{i=1}^{n}$. Let us denote the matrix corresponding to the linear system by $M_S(A_1, \ldots, A_n)$ or $M_S(\mathfrak{A})$ and its determinant by $D_S(\mathfrak{A}) = \det(M_S(\mathfrak{A}))$. The columns of $M_S(\mathfrak{A})$ are indexed by the monomials $x^j y^k$ in V. The rows of $M(\mathfrak{A})$ are doubly indexed by i and then by single partial differential operators in A_i. Thus the $(i, r, s) - (j, k)$ entry of $M_S(\mathfrak{A})$ is given by

$$\frac{\partial^{r+s} x^j y^k}{\partial x^r \partial y^s}(P_i) = r!s! \binom{j}{r} \binom{k}{s} x_i^{j-r} y_i^{k-s}.$$

The interpolation matrix itself is then

$$M_S(\mathfrak{A}) = \begin{array}{r} A_1\{ \\ \\ A_2\{ \\ \\ A_3\{ \\ \\ \vdots \end{array} \left| \begin{array}{c} \hline P_1 \\ \hline P_2 \\ \hline P_3 \\ \hline \vdots \\ \hline \end{array} \right| .$$

Note that if the coordinates of the P_i's are undetermined, then $D_S(\mathfrak{A})$ is a polynomial in 2n variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. For consistency we will order the monomials and the derivatives in degree lexicographic order. Every vector $(\ldots, a_{i,j}, \ldots)$ such that

$$M_S(\mathfrak{A}) \begin{pmatrix} \vdots \\ a_{i,j} \\ \vdots \end{pmatrix} = 0$$

is the vector of coefficients of a polynomial satisfying the condition of the given interpolation problem, i.e.

$$\ker M_S(\mathfrak{A}) = \{f \in V | L(f)(P_i) = 0 \quad \forall L \in A_i, \forall i\}.$$

Example 2.1. Consider the system $\mathcal{L}_2(2^2)$. We obtain

$$\begin{array}{c} \\ (x_1, y_1) \\ \\ \\ (x_2, y_2) \\ \\ \end{array} \begin{array}{|cccccc|} \hline 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 0 & 1 & 0 & 2x_1 & y_1 & 0 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 \\ \hline 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 0 & 1 & 0 & 2x_2 & y_2 & 0 \\ 0 & 0 & 1 & 0 & x_2 & 2y_2 \\ \hline \end{array} \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ a_{0,1} \\ a_{2,0} \\ a_{1,1} \\ a_{0,2} \end{pmatrix} = 0.$$

Definition 2.2. The interpolation problem is called

1. *regular* if the determinant $D_S(\mathfrak{A})$ is a non-zero constant, i.e., for any set of points $\{P_i\}$ there is no nonzero polynomial in V satisfying the interpolation conditions;

2. *almost regular* or *generically non-special* if the determinant $D_S(\mathfrak{A})$ is a non-constant polynomial in the x_i's and y_i's, i.e., for a general set of points, there is no nonzero polynomial in V satisfying the interpolation conditions;

3. *singular* if the determinant $D_S(\mathfrak{A})$ is identically zero, i.e., there is always a nonzero polynomial $P \in V$ satisfying the interpolation conditions.

The determinant $D_S(\mathfrak{A})$ of an interpolation problem is in general a polynomial in the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. To prove that the interpolation problem is non-singular one must show that this determinant is not identically zero.

2.3 Derivatives of D and Shifts

It suffices to show that a *derivative* of the determinant is not identically zero; this leads us to analyze such derivatives in more detail. Let us introduce the following notation, for any matrix M whose entries are functions of a variable x:

$$\frac{\partial(p)}{\partial x} M := \text{matrix obtained by applying } \frac{\partial}{\partial x} \text{to the } p\text{th row in the matrix } M.$$

Remark 2.3. The product rule for derivatives implies that, if M is a square matrix,

$$\frac{\partial}{\partial x}(\det M) = \sum_p \det(\frac{\partial(p)}{\partial x} M)$$

where the sum is taken over all rows p of M.

From now we denote $D_S(\mathfrak{A})$ simply by D. Applying the previous remark to our interpolation matrices, we obtain

$$\frac{\partial}{\partial x_i} D_S(\mathfrak{A}) = \sum_{(r,s)\in A_i} det\left(\frac{\partial(i,r,s)}{\partial x_i} M_S(\mathfrak{A})\right), \tag{2.4}$$

recalling that the rows are indexed by triples (i, r, s). We note that the sum is taken over only those rows in the A_i part because the other rows do not involve the variable x_i, and hence the derivatives all vanish as does the determinant. The similar equation holds for taking y_i derivatives also.

Next we note that apply $\frac{\partial}{\partial x_i}$ to the rows is the same thing as replacing partial derivatives in A_i by partial derivatives with one additional x-derivative.

$$\begin{array}{ccc} \text{row} & \longrightarrow & \frac{\partial^{a+b}}{\partial^a x \partial^b y} \\ \downarrow & & \downarrow \\ \frac{\partial}{\partial x} - \text{row} & \longrightarrow & \frac{\partial^{a+b+1}}{\partial^{a+1} x \partial^b y} \end{array}$$

Thus we can write

$$\frac{\partial(i, r, s)}{\partial x_i} M_S(A_1, \dots, A_i, \dots, A_n) = M_S(A_1, \dots, A_i^*, \dots, A_n)$$

where A_i^* is the result of replacing the (r, s) lattice point with the $(r + 1, s)$ lattice point: we are simply taking one more derivative with respect to x_i. The equation above is actually only true up to a possible re-ordering of the rows in the i-th part, since we may have to re-order to put the A_i^* rows in degree lex ordering. If $(r + 1, s)$ is already in A_i, this leads to a matrix with two identical rows, whose determinant is therefore zero. In addition, if $(r + 1, s)$ is no longer in the *lower closure* of S, (that is, the set of lattice points to the left and below some lattice point of S), we will have a matrix with an identically zero row whose determinant is therefore zero. Similarly, the derivatives with respect to y_i leads to replacing the (r, s) point in A_i with $(r, s + 1)$. This leads us to the following concept; let A be the set of lattice points, such as one of the sets A_i of derivative orders as above.

Definition 2.5. A **right shift** of A moves a point $(r, s) \in A$ to the position $(r+1, s)$. An **up shift** moves a point $(r, s) \in A$ to the position $(r, s+1)$. A right or up shift gives a *collision* if the resulting lattice point $(r + 1, s)$ or $(r, s + 1)$ is already in A. A right or up shift gives an *exit* if the resulting lattice point leaves the lower closure of the set S.

Example 2.6. Consider shifts of the set $A_1 = T_2 = \{(0, 0), (1, 0), (0, 1)\}$. Let us index these by $1 = (0, 0)$, $2 = (1, 0)$, and $3 = (0, 1)$. Consider $\frac{\partial}{\partial x_1}$ and apply it respectively to the elements $2, 3 \in A_1$. These right shifts can be visualized as follows:

$$\begin{array}{c|cccc} \vdots & & & & \\ y^2 & & & & \\ y^1 & 3 & & & \\ y^0 & 1 & \boxed{2} & & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array} \quad \xrightarrow{\frac{\partial}{\partial x_1}} \quad \begin{array}{c|cccc} \vdots & & & & \\ y^2 & & & & \\ y^1 & 3 & & & \\ y^0 & 1 & & \boxed{2} & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array}$$

$$\begin{array}{c|cccc} \vdots & & & & \\ y^2 & & & & \\ y^1 & \boxed{3} & & & \\ y^0 & 1 & 2 & & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array} \quad \xrightarrow{\frac{\partial}{\partial x_1}} \quad \begin{array}{c|cccc} \vdots & & & & \\ y^2 & & & & \\ y^1 & & \boxed{3} & & \\ y^0 & 1 & 2 & & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array}$$

We note that the right shift of the element 1 of A_1 gives a collision (with the element 2) and need not be considered in the determinant formula.

Using this notation, (2.4) becomes

$$\frac{\partial}{\partial x_i} D_S(A_1, \dots, A_i, \dots, A_n) = \sum_{\substack{\text{right shifts} \\ A_i^* of A_i \\ \text{without collision} \\ \text{or exit}}} \mu D_S(A_1, \dots, A_i^*, \dots, A_n) \quad (2.7)$$

where $\mu = \pm 1$ depends on the particular right shift and comes from the possible re-ordering of the rows as noted above.

2.4 Higher-Order Derivatives and Iterated Shifts

We want to understand higher-order derivatives of the determinant, which leads to iterated shifts. Number each lattice point of the set A_i, using as the index set the integers $1, \dots, |A_i|$. For any given index ℓ, denote by $(r(\ell), s(\ell))$ that element of A_i. A right (respectively up) shift of the ℓ-th element of A_i simply increments $r(\ell)$ (respectively $s(\ell)$) by one, and will be denoted by R_ℓ (respectively U_ℓ). Applying a right shifts (of the elements indexed by $\ell_1, \dots, \ell_a$, with duplication allowed) followed by b up shifts (of the elements indexed $m_1, \dots, m_b$, duplicates allowed) to A_i will be denoted by

$$U_{m_b} \cdots U_{m_1} R_{\ell_a} \cdots R_{\ell_1} A_i$$

and will be called an *iterated* (a, b)*-shift* of A_i. Such an operation has a *collision* if at any stage of the process, one of the separate $a + b$ shifts do.

With this notation, (2.7) applied $a + b$ times gives

$$\frac{\partial^b}{\partial y_i^b} \frac{\partial^a}{\partial x_i^a} D_S(A_1, \dots, A_i, \dots, A_n) =$$
$$\sum_{\substack{(\underline{m}, \underline{\ell}) \in A_i^b \times A_i^a \\ \text{giving iterated } (a,b)\text{-shifts} \\ \text{without collisions} \\ \text{or exits}}} \mu_{(\underline{m}, \underline{\ell})} D_S(A_1, \dots, U_{m_b} \cdots U_{m_1} R_{\ell_a} \cdots R_{\ell_1} A_i, \dots, A_n) \quad (2.8)$$

where $\mu_{(\underline{m}, \underline{\ell})} = \pm 1$ comes from possible re-orderings of the rows.

Re-organize the sum above based on the final set resulting from the various iterated shifts gives:

$$\frac{\partial^b}{\partial y_i^b}\frac{\partial^a}{\partial x_i^a} D_S(A_1,\ldots,A_i,\ldots,A_n) =$$

$$\sum_{\substack{\text{final positions}\\ A_i^*}} \quad \sum_{\substack{(\underline{m},\underline{\ell})\in A_i^b\times A_i^a\\ \text{giving iterated } (a,b)\text{-shifts}\\ \text{without collisions or exits}\\ \text{ending with} A_i^*}} \mu_{(\underline{m},\underline{\ell})} D_S(A_1,\ldots,A_i^*,\ldots,A_n)$$

$$= \sum_{\substack{\text{final positions}\\ A_i^*}} \epsilon(A_i^*) D_S(A_1,\ldots,A_i^*,\ldots,A_n) \qquad (2.9)$$

where

$$\epsilon(A_i^*) = \sum_{\substack{(\underline{m},\underline{\ell})\in A_i^b\times A_i^a\\ \text{giving iterated } (a,b)\text{-shifts}\\ \text{without collisions or exits}\\ \text{ending with} A_i^*}} \mu_{(\underline{m},\underline{\ell})}$$

is an integer.

We say that the shifts resulting in a given A_i^* are *non-cancelling* if $\epsilon(A_i^*) \neq 0$.

Example 2.10. Consider $(2,2)$-shifts of the set $A = T_2 = \{(0,0),(1,0),(0,1)\}$. Let us index these by $1 = (0,0)$, $2 = (1,0)$, and $3 = (0,1)$.

$$\begin{array}{c|cccc}
\vdots & & & & \\
y^2 & & & & \\
y^1 & 3 & & & \\
y^0 & 1 & 2 & & \\
\hline
 & x^0 & x^1 & x^2 & \ldots
\end{array}$$

One possible final position is the set $A^* = \{(2,0),(1,1),(0,2)\}$. There are only four iterated shifts without collisions leading to this final position: $U_3U_1R_1R_2$, $U_1U_3R_1R_2$, $U_1U_1R_3R_2$, and $U_1U_1R_2R_3$. All four end up with 2 in the position $(2,0)$. The first two have 3 in position $(0,2)$ and 1 in position $(1,1)$, while the last two have these reversed.

$$\begin{array}{c|cccc}
\vdots & & & & \\
y^2 & & & & \\
y^1 & 3 & & & \\
y^0 & 1 & 2 & & \\
\hline
 & x^0 & x^1 & x^2 & \ldots
\end{array}
\quad \xrightarrow{U_3U_1R_1R_2} \quad
\begin{array}{c|cccc}
\vdots & & & & \\
y^2 & 3 & & & \\
y^1 & & 1 & & \\
y^0 & & & 2 & \\
\hline
 & x^0 & x^1 & x^2 & \ldots
\end{array}$$

$$\begin{array}{c|cccc}
\vdots & & & & \\
y^2 & & & & \\
y^1 & 3 & & & \\
y^0 & 1 & 2 & & \\
\hline
 & x^0 & x^1 & x^2 & \ldots
\end{array}
\quad \xrightarrow{U_1U_3R_1R_2} \quad
\begin{array}{c|cccc}
\vdots & & & & \\
y^2 & 3 & & & \\
y^1 & & 1 & & \\
y^0 & & & 2 & \\
\hline
 & x^0 & x^1 & x^2 & \ldots
\end{array}$$

$$\begin{array}{c|cccc} \vdots & & & & \\ y^2 & & & & \\ y^1 & 3 & & & \\ y^0 & 1 & 2 & & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array} \xrightarrow{U_1U_1R_3R_2} \begin{array}{c|cccc} \vdots & & & & \\ y^2 & 1 & & & \\ y^1 & & 3 & & \\ y^0 & & & 2 & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array}$$

$$\begin{array}{c|cccc} \vdots & & & & \\ y^2 & & & & \\ y^1 & 3 & & & \\ y^0 & 1 & 2 & & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array} \xrightarrow{U_1U_1R_2R_3} \begin{array}{c|cccc} \vdots & & & & \\ y^2 & 1 & & & \\ y^1 & & 3 & & \\ y^0 & & & 2 & \\ \hline & x^0 & x^1 & x^2 & \dots \end{array}$$

Therefore two of the μ's are equal to 1 and two are equal to -1, and the resulting ϵ is zero. Therefore this is a *cancelling* set of shifts.

3 Lecture Three: The Matrix Approach II

3.1 Coalescence

Suppose $D = D(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a polynomial in these $2n$ variables. Denote by $\mathrm{coal}(D)$ the polynomial obtained by coalescing the variables (x_1, y_1) with the variables (x_2, y_2): this is essentially setting $x_2 = x_1$ and $y_2 = y_1$:

$$\mathrm{coal}(D) = D(x_1, x_1, x_3, \ldots, x_n, y_1, y_1, y_3, \ldots, y_n).$$

Note that coal is a linear operation.

Let us apply this to the determinants of the interpolation matrices that we are considering. Coalescing the first two variables in $D_S(\mathfrak{A})$ exactly means that we are requiring the second set of derivatives A_2 to vanish at the first point $P_1 = (x_1, y_1)$. If there is overlap between A_1 and A_2, then the interpolation matrix will have two identical rows after this coalescence. Otherwise, we simply have the union of A_1 and A_2 at the first point. This proves the following:

Lemma 3.1.

$$\mathrm{coal}\left(D_S(\mathfrak{A})\right) = \begin{cases} \pm D_S\big(A_1 \cup A_2, A_3, \ldots A_n\big), & \textit{if } A_1 \cap A_2 = \emptyset; \\ 0, & \textit{if } A_1 \cap A_2 \neq \emptyset. \end{cases}$$

Applying coalescence to both sides of (2.9) gives the following, using Lemma 3.1.

$$\begin{aligned} \mathrm{coal}\, \frac{\partial^b}{\partial y_2^b} \frac{\partial^a}{\partial x_2^a} D_S(A_1, A_2, \ldots, A_n) &= \\ &= \sum_{\substack{\text{final positions} \\ A_2^* \text{ such that} \\ A_1 \cap A_2^* = \emptyset}} \pm \epsilon(A_2^*) D_S(A_1 \cup A_2^*, \ldots, A_n) \end{aligned} \tag{3.2}$$

3.2 Minimal Shifts

Our goal is to reduce the sum in the above formula to a single determinant. We observe that a and b are determined by the final position A_2^*. In particular we can give the following

Definition 3.3. (a, b) is a *minimal shift with final position* A_2^* for the pair (A_1, A_2) if A_2^* is the unique final position for an (a, b)-shift of A_2 for which $A_1 \cap A_2^* = \emptyset$ and $\epsilon(A_2^*) \neq 0$.

Corollary 3.4. *Suppose (a, b) is a minimal shift for (A_1, A_2) with final position A_2^*. Then*

$$\mathit{coal}\left(\frac{\partial^{a+b}}{\partial x_2^a \partial y_2^b} D_S(\mathfrak{A}) \right) = \epsilon D_S(A_1 \cup A_2^*, A_3 \ldots, A_n).$$

for some nonzero constant ϵ.

The main application we have for this is the following.

Proposition 3.5. *Suppose that (a, b) is a minimal shift for (A_1, A_2) with final position A_2^*. Suppose that the interpolation problem for S and $\mathfrak{A}' = \{A_1 \cup A_2^*, \ldots, A_n\}$ is non-singular. Then the original interpolation problem for S and $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ is non-singular.*

The goal is ultimately to reduce to the following situation, using the above Proposition.

Proposition 3.6. *Suppose S is lower closed. Then the determinant $D_S(A_1)$ with $A_1 = S$ is nonzero.*

Example 3.7. Consider A_1 and A_2 both equal to $\{1, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. If we apply a $(5, 0)$-shift and coalesce P_1 and P_2, then the only final position for A_2^* which is disjoint from A_1 is $A_2^* = \{\frac{\partial^2}{\partial x^2}, \frac{\partial^3}{\partial x^3}, \frac{\partial^2}{\partial x \partial y}\}$. Thus

$$\text{coal}\left(\frac{\partial^5}{\partial x_1^5} D_S(A_1, A_2, \ldots, A_n)\right) = \epsilon D_S(A_1 \cup A_2^*, A_3 \ldots, A_n).$$

for some nonzero constant ϵ, where $A_1 \cup A_2^*$ is $\{1, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^3}{\partial x^3}\}$.

3.3 Shifts of T_1 and T_2

In this section we first suppose that $A_2 = T_1 = \{(0, 0)\}$. With only one element in A_2, the only condition for (a, b) to be a minimal shift is that the resulting A_2^* (which is $\{(a, b)\}$ of course) is disjoint from A_1. Hence it is a trivial matter, if $A_2 = T_1$, to apply Proposition 3.5 and simply add the lattice point (a, b) to A_1.

Next we suppose that $A_2 = T_2 = \{(0, 0), (1, 0), (0, 1)\}$. This case is already quite a bit more complicated. The following lemma, proved in [30], at least gives us good information about when an (a, b) shift of T_2 is non-cancelling:

Lemma 3.8. *The final position A_2^* for initial position $A_2 = T_2$ is cancelling if and only if the three elements of A_2^* are collinear.*

The proof of this is a rather involved computation of the contributions to the ϵ factor.

We have already seen an example of this in Example 2.10 in the previous Lecture.

We note that if the final position A_2^* has two lattice points on one row and one on another, then they cannot be collinear, and the shift will be non-cancelling. This is the primary type of shift that is necessary to consider in most applications.

When is a shift of T_2 minimal? This question is equivalent to asking: when is a final position A_2^* (which is disjoint from A_1) unique, given the numbers a and b of right and up shifts, respectively? This is an easier question to address in most circumstances, and we simply note the following.

Suppose that S is lower closed, and that $A_1 \subset S$ is also lower closed. Suppose that the first k lowest rows of A_1 are equal to the first k lowest rows of S, and that the $k + 1$-st row of A_1 is not equal to the $k + 1$-st row of S.

Lemma 3.9. *Suppose that there are at least two elements of S in the $k+1$-st row which are not in A_1, and at least one element of S in a higher row that is not in A_1. Then the shift of T_2 placing $(0,0)$ and $(1,0)$ into the first two elements of the $k+1$-st row of S which are not in A_1, and which places $(0,1)$ in the first element of the next higher row of S which has an element not in A_1, is a minimal shift.*

Lemma 3.10. *Suppose that there is exactly one element of S in the $k+1$-st row which is not in A_1, and in the next higher row of S that has elements not in A_1, there are at least two elements of S that are not in A_1. Then the shift of T_2 placing $(0,0)$ into the final element of the $k+1$-st row of S which is not in A_1, and which places $(0,1)$ and $(1,0)$ in the first two elements of the next higher row of S which has the two elements not in A_1, is a minimal shift.*

To prove the above two lemmas, the readers need only convince themselves that these final positions are the unique ones disjoint from A_1 for $A_2 = T_2$; for this we use up shifts first, then right shifts.

Example 3.11. We give here a visualization of the final position A_2^*.
Suppose A_1 fills completely the lattice indexing monomials in V untill the row k and it has some element at the row $k+1$. In the next figures we indicate the elements in A_1 with a bullet $\bullet$ while the elements in $A_2 = T_2$ are marked with $\circ_j$.
As a first case, we suppose there are at least two free boxes in the $(k+1)^{st}-$row.

$$
\begin{array}{r|ccccccccccc}
\vdots & & & & & & & & & & & \\
k+1 & \bullet & \dots & \bullet & & & & & & & & \\
k & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & & & \\
\vdots & & & & & & & & & & & \\
3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & \\
2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & \\
1 & \circ_3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \\
0 & \circ_1 & \circ_2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

Then we start with all the up shifts and we reach the position

$$
\begin{array}{r|ccccccccccc}
\vdots & & & & & & & & & & & \\
k+2 & \circ_3 & & & & & & & & & & \\
k+1 & \circ_1 & \circ_2 & \bullet & & & & & & & & \\
k & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & & & \\
\vdots & & & & & & & & & & & \\
3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & \\
2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & \\
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \\
0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

After that, we move to right, performing all the right shifts:

$$
\begin{array}{r|ccccccccccc}
\vdots & & & & & & & & & & & \\
k+2 & \circ_3 & & & & & & & & & & \\
k+1 & \bullet & \dots & \bullet & \circ_1 & \circ_2 & & & & & & \\
k & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & & & \\
\vdots & & & & & & & & & & & \\
3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & \\
2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & \\
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \\
0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

and this is the final position for A_2^*.
We suppose now there is only one free box in the $(k+1)^{st}-$row.

$$
\begin{array}{r|ccccccccccc}
\vdots & & & & & & & & & & & \\
k+1 & \bullet & \bullet & \dots & \bullet & & & & & & & \\
k & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & & & \\
\vdots & & & & & & & & & & & \\
3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & \\
2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & \\
1 & \circ_3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \\
0 & \circ_1 & \circ_2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

Again we start with all the up shifts and we reach the position

$$
\begin{array}{r|ccccccccccc}
\vdots & & & & & & & & & & & \\
k+2 & \circ_3 & \circ_2 & & & & & & & & & \\
k+1 & \circ_1 & \bullet & \dots & \bullet & & & & & & & \\
k & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & & & \\
\vdots & & & & & & & & & & & \\
3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & \\
2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & \\
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \\
0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

After that, we move to right, performing all the right shifts:

$$
\begin{array}{r|ccccccccccc}
\vdots & & & & & & & & & & & \\
k+2 & \circ_3 & \circ_2 & & & & & & & & & \\
k+1 & \bullet & \bullet & \dots & \bullet & \circ_1 & & & & & & \\
k & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & & & \\
\vdots & & & & & & & & & & & \\
3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & & \\
2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & & \\
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \\
0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

and this is the final position for A_2^*.

3.4 The Paving Strategy

Let us now specialize the interpolation problem and assume the following: S is a lower-closed set, A_1 is a lower-closed set, and all A_i sets for $i \geq 2$ are either T_1 or T_2. A strategy for proving that such an interpolation problem is nonsingular is to perform a minimal shift to A_2 and coalesce it with A_1, thereby reducing the number of A_i sets. If we can reduce to the case where there is only one such set, A_1, then $A_1 = S$ and the problem is nonsingular by Proposition 3.6.

The minimal shifts of T_1 and of T_2 will be all of the type introduced above. In particular, those of T_2 will be to place two elements on one row and one on another, systematically filling up the rows of S from the bottom up.

We refer to this strategy as "paving" the set S by the sets A_i.

Example 3.12. Consider the system of conics with two double points, namely the linear system $\mathcal{L}_2(2^2)$. In this case we have $S = T_3$, and $A_1 = A_2 = T_2$; S has six elements, and both of the A_i's have three elements. We see that in this case, the paving strategy introduced above fails, since the three elements in $S - A_1$ (namely $(2,0)$, $(1,1)$, and $(0,2)$) are collinear, and the shift placing A_2 into these elements as a final possible position A_2^* is cancelling, by Lemma 3.8. We are not particularly surprised that the paving strategy has failed in this case, since we know that the interpolation problem is indeed singular!

To continue the discussion, it is useful to illustrate the paving strategy by visualizing the set S (representing the basis of the underlying vector space V) and the successive increasingly larger sets A_1 (obtained by coalescing the next A_i set in its turn) as the algorithm using the paving strategy proceeds.

For example, Consider now the system of cubics with three double points and one simple point; we have $\dim(V) = \sum \dim(A_i) = 10$, with $S = T_4$, $A_1 = A_2 = A_3 = T_2$ and $A_4 = T_1$:

$$
S = \begin{array}{c|cccc}
3 & \bullet & & & \\
2 & \bullet & \bullet & & \\
1 & \bullet & \bullet & \bullet & \\
0 & \bullet & \bullet & \bullet & \bullet \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

$$
\text{first } A_1 = \begin{array}{c|cccc}
3 & & & & \\
2 & & & & \\
1 & \circ & & & \\
0 & \circ & \circ & & \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
\qquad
\text{second } A_1 = \begin{array}{c|cccc}
3 & & & & \\
2 & & & & \\
1 & \circ & \circ & & \\
0 & \circ & \circ & \circ & \circ \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

$$
\text{third } A_1 = \begin{array}{c|cccc}
3 & & & & \\
2 & \circ & \circ & & \\
1 & \circ & \circ & \circ & \\
0 & \circ & \circ & \circ & \circ \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
\qquad
\text{fourth } A_1 = S = \begin{array}{c|cccc}
3 & \circ & & & \\
2 & \circ & \circ & & \\
1 & \circ & \circ & \circ & \\
0 & \circ & \circ & \circ & \circ \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

In this case the paving strategy has succeeded. We note that in creating the second A_1, we have executed a $(5,0)$ minimal shift of the T_2, moving two elements into the first row (filling it up) and one element into the second row. In creating the third A_1, we have executed a $(2,4)$ minimal shift of the T_2, placing one element into the second row (filling it up) and two elements into the third row (filling it up also). Finally in creating the fourth A_1, we have executed a $(0,3)$ minimal shift (namely three up shifts) of the T_1, placing the element into the fourth row at the top, and paving the entire set S.

It is more efficient to encode all of this a bit more simply as follows:

$$
S = \begin{array}{c|cccc}
3 & 4 & & & \\
2 & 3 & 3 & & \\
1 & 1 & 2 & 3 & \\
0 & 1 & 1 & 2 & 2 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

Here the numbers filling in the elements of S represent which A_i contribute to the final paving of that particular element. Another way of saying this is that the n-th A_1 in the paving strategy algorithm consists of the union of all of the elements labeled with integers between 1 and n.

Example 3.13. In a similar way, if we consider the system of quartics with five double points, we again get stuck, in a similar way to the case of $\mathcal{L}_2(2^2)$. Indeed, applying the paving strategy as above, the fourth A_1 is

$$
\text{fourth } A_1 = \begin{array}{c|ccccc}
4 & \bullet & & & & \\
3 & 4 & \bullet & & & \\
2 & 4 & 4 & \bullet & & \\
1 & 1 & 2 & 3 & 3 & \\
0 & 1 & 1 & 2 & 2 & 3 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

and in order to finish paving S, we would need to make a minimal shift into the final three collinear elements, which is not possible by Lemma 3.8.

Example 3.14. The paving strategy works very well for the system $\mathcal{L}_5(2^7)$ of quintics with seven double points:

$$\begin{array}{c|cccccc}
5 & 7 & & & & & \\
4 & 7 & 7 & & & & \\
3 & 5 & 6 & 6 & & & \\
2 & 4 & 5 & 5 & 6 & & \\
1 & 1 & 2 & 3 & 4 & 4 & \\
0 & 1 & 1 & 2 & 2 & 3 & 3 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}$$

This proves that $\mathcal{L}_5(2^7)$ is non-special.

3.5 Proof of the Double Points Theorem in Dimension Two

The matrix approach that we are presenting here is powerful enough to prove that, except for conics and quartics with two and five double points respectively, all linear systems with simple and double points are non-special.

Theorem 3.15 (Alexander–Hirschowitz for $\mathbb{P}^2$). *Suppose that $d \geq 5$ and $S = \{(i,j)|i+j \leq d\}$ (i.e. S represents all monomials of degree $\leq d$) and all A_i are T_2 or T_1. Then $D_S(\mathfrak{A}) \not\equiv 0$ and the linear system $\mathcal{L}_d(1^r, 2^s)$ is non-special whenever $r + 3s \leq (d+2)(d+1)/2$.*

The proof is simply an analysis of the paving strategy, proving that it does work and in this case, with $d \geq 5$, it never requires the three collinear elements for a minimal shift.

One uses the strategy of always moving two elements on the lowest unfilled row, and one on the next, whenever possible; if there is only one element left on the lowest unfilled row, one shows that there are at least two elements on the next unfilled row. This is relatively simple as long as there are more than four unfilled rows left, since using this algorithm, the lowest unfilled row fills up twice as fast as the next unfilled row, and hence when it does get near the end (with only zero or one element left) there are at least two unfilled elements in that next row.

Thus the first remark to make is that one get "near the top" using this algorithm without any problems. The next remark is that, when we have exactly filled the sixth-to-last row, there is at least one element filled in the fifth-to-last row. This again follows from the considerations above: one cannot simultaneously exactly fill the seventh-to-last and the sixth-to-last rows.

Finally one simply checks by hand that if there is at least one element filled in the fifth-to-last row, one can finish the paving of S from that point on. Since this fifth-to-last row has only five elements in it total, there are really only four cases to check:

One element in fifth-to-last row:

$$\begin{array}{r|ccccc} d & 6 & & & & \\ d-1 & 4 & 5 & & & \\ d-2 & 3 & 4 & 4 & & \\ d-3 & 1 & 2 & 3 & 3 & \\ d-4 & \bullet & 1 & 1 & 2 & 2 \\ \hline & d-4 & d-3 & d-2 & d-1 & d \end{array}$$

Two elements in fifth-to-last row:

$$\begin{array}{r|ccccc} d & 5 & & & & \\ d-1 & 4 & 4 & & & \\ d-2 & 3 & 3 & 4 & & \\ d-3 & 1 & 2 & 2 & 3 & \\ d-4 & \bullet & \bullet & 1 & 1 & 2 \\ \hline & d-4 & d-3 & d-2 & d-1 & d \end{array}$$

Three elements in fifth-to-last row:

$$\begin{array}{r|ccccc} d & 4 & & & & \\ d-1 & 4 & 4 & & & \\ d-2 & 2 & 3 & 3 & & \\ d-3 & 1 & 2 & 2 & 3 & \\ d-4 & \bullet & \bullet & \bullet & 1 & 1 \\ \hline & d-4 & d-3 & d-2 & d-1 & d \end{array}$$

Four elements in fifth-to-last row:

$$\begin{array}{r|ccccc} d & 5 & & & & \\ d-1 & 3 & 4 & & & \\ d-2 & 2 & 3 & 3 & & \\ d-3 & 1 & 1 & 2 & 2 & \\ d-4 & \bullet & \bullet & \bullet & \bullet & 1 \\ \hline & d-4 & d-3 & d-2 & d-1 & d \end{array}$$

These four simple computations finish the proof of the Theorem.

Example 3.16. One can use this method to study the problem in $\mathbb{P}^1 \times \mathbb{P}^1$ instead $\mathbb{P}^2$. This time S is given by $d_1 \times d_2$ boxes, arranged in a rectangle; these are the monomials in the complete linear systems on $\mathbb{P}^1 \times \mathbb{P}^1$.

$$S = \begin{array}{r|cccc|} \hline d_1 & & & & \\ \vdots & & & & \\ 1 & & & & \\ 0 & & & & \\ \hline & 0 & 1 & \dots & d_2 \end{array}$$

Assume $d_1 = 2$. If $A_1 = A_2 = T_2$ we have

$$\begin{array}{r|ccccc|} \hline 2 & 2 & 2 & & & \\ 1 & 1 & 2 & & & \\ 0 & 1 & 1 & & & \\ \hline & 0 & 1 & 2 & \dots & d_2 \end{array}$$

Thus two double points fill exactly two columns. If d_2 is even, we have an odd number of columns (i.e. $d_2 + 1$) and the system is special. In Dent's thesis the case of rectangular S is analyzed more completely; see also [30]

The matrix approach presented here has several variations, and has been used to prove that several classes of interpolation problems are non-special. See [43] for a recent survey.

Dent, in her thesis ([29]), has used the method to prove the Alexander-Hirschowitz Theorem for three variables.

It would be a wonderful project to systematically relate these matrix approach methods for studying interpolation problems, to degeneration techniques coming from a more standard algebro-geometric approach, using upper semicontinuity.

4 Lecture Four: Degenerations of the Plane

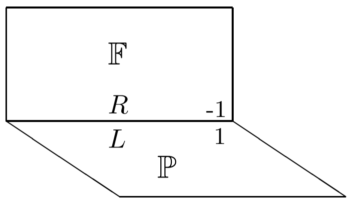
4.1 Blowing up the Trivial Family of Planes

We already spoke, in Lecture One, about various specialization techniques to attack the interpolation problem. Since the dimension of a system with imposed multiple points is upper-semicontinuous in the position of the fat points $\{Z_i\}$, we can consider a degeneration of Z_1 and Z_2 to a suitable $Z_1 \cup Z_2$. In this way one reduces the study of $H^0(X_S, \mathcal{L} \otimes \mathcal{I}_{Z_1} \otimes \mathcal{I}_{Z_2} \otimes \cdots \otimes \mathcal{I}_{Z_n})$ to the study of $H^0(X_S, \mathcal{L} \otimes \mathcal{I}_{Z_1 \cup Z_2} \otimes \mathcal{I}_{Z_3} \otimes \cdots \otimes \mathcal{I}_{Z_n})$.

In the next two lectures we will explain the essential features of a particular specialization technique introduced by Ciliberto and Miranda in [22].

Although related closely to other specializations, the new feature is that the degeneration is not of sets of points, but, instead, we degenerate the surface where these points live. The idea is based on a degeneration method used by Z. Ran ([48]) to study enumerative problems on singular curves and consists in degenerating the plane to a reducible surface. The restriction of the limit linear system to the components of the surface are hopefully easier to understand than the system that we began with.

In more detail, let Δ be a complex disc around the origin. We consider the trivial family of planes which is the product $V = \mathbb{P}^2 \times \Delta$, with its two projections $p_1 : V \to \mathbb{P}^2$ and $p_2 : V \to \Delta$. We denote the fiber over $t \in \Delta$ by $V_t = p_2^{-1}(t) = \mathbb{P}^2 \times \{t\}$. Consider a line $L \subset V_0$ and blow it up to obtain a threefold X with maps $f : X \to V$, $\pi_1 = p_1 \circ f : X \to \mathbb{P}^2$ and $\pi_2 = p_2 \circ f : X \to \Delta$. The map π_2 is a flat family of surfaces over Δ: for $t \neq 0$, $X_t = V_t = \mathbb{P}^2$, while, for $t = 0$, X_0 is the union of the proper transform $\mathbb{P}$ of V_0 (which is again isomorphic to $\mathbb{P}^2$) and of the exceptional divisor $\mathbb{F}$ of the blow-up (which is isomorphic to a Hirzebruch surface $\mathbb{F}_1$). They are joined transversally along a curve R which is a line L in $\mathbb{P}$ and is the exceptional curve on $\mathbb{F}$.



4.2 The Triple Point Formula

Note that we have $(R^2)_{\mathbb{F}} + (R^2)_{\mathbb{P}} = 0$. This is a special case of the so-called Triple Point Formula for double curves of the special fiber of a degeneration of surfaces. In fact, let X be a smooth 3-fold with a map $\pi : X \to \Delta$, whose general fiber is a smooth surface and whose central fiber X_0 is the union $\cup A_i$ of smooth A_i meeting transversally along smooth curves $R_{ij} \subset A_i$. The Triple Point Formula states that with this situation, one has:

$$(R_{ij}^2)_{A_i} + (R_{ij}^2)_{A_j} = -(\text{numbers of triple points on } R_{ij}). \qquad (4.1)$$

Proof. This is rather elementary intersection theory on the threefold X. First note that $A_i^2 \cdot A_j$ can be viewed as taking the self-intersection of the surface A_i, then restricting to A_j. Restriction is a homomorphism of the intersection product, and so we can restrict A_i to A_j first (obtaining R_{ji}) and then take the self-intersection. Hence

$$(A_i^2 \cdot A_j)_X = (R_{ji}^2)_{A_j}.$$

Now consider $A_i \cdot A_j \cdot X_0$. On the one hand this is zero, since $X_0 \equiv X_t$ and, for $t \neq 0$, X_t is disjoint from the A_i's. Using that $X_0 = \sum_k A_k$, we have

$$0 = A_i \cdot A_j \cdot X_0 = A_i \cdot A_j \cdot \sum_k A_k = A_i^2 \cdot A_j + A_j^2 \cdot A_i + \sum_{k \neq i,j} A_i \cdot A_j \cdot A_k.$$

The sum on the right is the number of triple points on R_{ij}. This and the identification above of $A_i^2 \cdot A_j$ proves the Triple Point Formula (4.1). Q.E.D

4.3 The Degeneration of the Linear System

We pass now to analyzing the invertible sheaf on X_0. The Picard group of X_0 is the fibered product of Pic($\mathbb{P}$) (generated by $\mathcal{O}(1)$) and Pic($\mathbb{F}$) (generated by the class H of a line and the class R of the exceptional divisor) over Pic(R). Since $H \cdot R = 0$ and $R \cdot R = -1$, we have

$$\mathcal{O}_{\mathbb{F}}(H)_{|R} \cong \mathcal{O}_R$$

and

$$\mathcal{O}_{\mathbb{F}}(R)_{|R} \cong \mathcal{O}_R(-1).$$

Hence if χ is a line bundle on X_0 given by $\chi_{\mathbb{P}}$ and $\chi_{\mathbb{F}}$, in order that the restrictions to R agree, one must have $\chi_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(d)$ and $\chi_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(cH - dR)$ for some c and d; we denote this line bundle on X_0 by $\chi(c, c-d)$. In particular for any d and k, $\chi(d,k)|_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(d-k)$ and $\chi(d,k)|_{\mathbb{F}} = \mathcal{O}_{\mathbb{F}}(dH - (d-k)R)$.

Let $\mathcal{O}_X(d)$ be the line bundle $\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(d))$. If $t \neq 0$ then the restriction to X_t is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d)$ and the restriction to X_0 is the line bundle $\chi(d,0)$. Since the normal bundles of $\mathbb{P}$ and $\mathbb{F}$ on X are respectively $-L$ and $-R$ we have $\mathcal{O}_X(\mathbb{P})_{|\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-1)$ and $\mathcal{O}_X(\mathbb{P})_{|\mathbb{F}} = \mathcal{O}_{\mathbb{F}}(R)$.

Consider now the line bundle $\mathcal{O}_X(d,k) := \mathcal{O}_X(d) \otimes \mathcal{O}_X(k\mathbb{P})$; from the previous discussion we have

$$\begin{array}{rcl} \mathcal{O}_X(d,k)_{|X_t} \cong & \mathcal{O}_{\mathbb{P}^2}(d) & t \neq 0 \\ \mathcal{O}_X(d,k)_{|X_0} \cong & \chi(d,k) & \end{array}$$

and we therefore have that all line bundles $\chi(d,k)$ on X_0 are flat limits of line bundles $\mathcal{O}_{\mathbb{P}^2}(d)$ on the general fiber X_t in this degeneration.

Fix a positive integer n and a non-negative integer $b \leq n$. We now consider $n-b$ general points $P_1, \ldots P_{n-b}$ in $\mathbb{P}$ and b general points $P_{n-b+1}, \ldots P_n$ in $\mathbb{F}$. These points are viewed as limits of n general points $P_{1,t}, \ldots, P_{n,t}$ on X_t. Our goal is to understand, on X_t, the linear system $\mathcal{L}_d(-\sum_{i=1}^n m_i P_{i,t})$.

To this end we now consider the system $\mathcal{L}(k,b)$ on X_0 which is formed by divisors on $|\chi(d,k)|$ with the prescribed multiplicities at the points P_i, $i = 0,\dots,n$. Then $\mathcal{L}(k,b)$ can be regarded as a flat limit on X_0 of the desired system $\mathcal{L}_d(-\sum_{i=1}^n m_i P_{i,t})$ and we call this a $(k,b)-$*degeneration* of the linear system. We can observe that $\mathcal{L}(k,b)$ restricts to $\mathbb{P}$ as $\mathcal{L}_{d-k}(-\sum_{i=1}^{n-b} m_i P_i)$ and to $\mathbb{F}$ as a system of the form $\mathcal{L}_d(-(d-k)Q_0 - \sum_{i=n-b+1}^n m_i P_i)$ where Q_0 is a point in $\mathbb{P}^2$ at which we blow up to obtain $\mathbb{F}$. (Here we are viewing the surface $\mathbb{F}$ as a blowup of the plane, and the corresponding line bundle on $\mathbb{F}$ as a linear system of the same form as the others we are considering.)

We note that the restricted system on R in which they must agree is given by $\mathcal{O}_R(d-k)$.

4.4 The Computation of the Limit Linear System

A global section of $\mathcal{L}(k,b)$ is a section on $\mathbb{P}$ of $\mathcal{L}_{d-k}(-\sum_{i=1}^{n-b} m_i P_i)$ and a section on $\mathbb{F}$ of $\mathcal{L}_d(-(d-k)Q_0 - \sum_{i=n-b+1}^n m_i P_i)$ which agree on the intersection curve R. In other words, $H^0(X_0, \mathcal{L}(k,b))$ is the fiber product of

$$H^0(\mathbb{P}, \mathcal{L}_{d-k}(-\sum_{i=1}^{n-b} m_i P_i))$$

and

$$H^0(\mathbb{F}, \mathcal{L}_d(-(d-k)Q_0 - \sum_{i=n-b+1}^{n} m_i P_i))$$

over the restriction to R, which is $H^0(R, \mathcal{O}_R(d-k))$.

If we denote by l_0 the dimension of $\mathcal{L}(k,b)$, by semicontinuity we have

$$l_0 \geq \dim(\mathcal{L}_d(-\textstyle\sum_{i=1}^n m_i P_i)) \geq \epsilon(\mathcal{L}_d(-\textstyle\sum_{i=1}^n m_i P_i)).$$

Therefore we have the following

Lemma 4.2. *If*

$$l_0 = \epsilon(\mathcal{L}_d(-\sum_{i=1}^n m_i P_i))$$

then $\mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ *is non-special.*

This is the basis of the approach: given the degree d and the multiplicities m_i, choose an appropriate k and b and make a computation of the limit dimension l_0. If that limit dimension is equal to the expected dimension of the system, then it is non-special.

This method has the capability of reducing the problem for the original linear system into two easier linear systems on $\mathbb{P}$ and $\mathbb{F}$, of the same general sort. On $\mathbb{P}$ the degree has gone down (if $k > 0$) and the number of points is less (if $b > 0$); on $\mathbb{F}$ the number of points is less (if $b < n$). We can hope to determine their dimensions by a process of induction, and complete the analysis of the limit dimension by a transversality argument which allows the easy computation of the relevant fibered product of systems.

4.5 Results from this Method

This was the approach taken by Ciliberto and Miranda in [22] and [23], which resulted in the proof of the following theorem:

Theorem 4.3. *For any m_0 and any $m \leq 3$ the Harbourne–Hirschowitz Conjecture holds in the quasi-homogeneous cases $\mathcal{L}_d(m_0, m^n)$. For any $m \leq 12$ this Conjecture holds in the homogeneous cases $\mathcal{L}_d(m^n)$.*

Later this has been extended via a more efficient computer algebra component, by Ciliberto, Cioffi, Miranda, and Orrechia [26], and we now have:

Theorem 4.4. *The Harbourne–Hirschowitz Conjecture holds in the homogeneous cases $\mathcal{L}_d(m^n)$ for $m \leq 20$.*

One reason that a computer algebra technique needed to be developed for this is that unfortunately the degeneration procedure as described above, by computing the space of global sections of the limiting linear system, does not work in all cases, even for these low multiplicities. For example, to study the Dixmier example $\mathcal{L}_{19}(6^{10})$ worked out by Hirschowitz with the Horace method, there are no integers k and b which have the limit bundle having no global sections, as expected. For these cases, which are finite in number for any fixed m, the result above relied on a separate computer algebra computation.

In the next Lecture we will describe a recently developed technique which seems to offer some promise to avoid the computer algebra methods and to give a more systematic approach.

5 Lecture Five: Refined Matching Conditions

5.1 The Fiber Product Condition

As noted in the previous lecture, a section of the relevant limit line bundle $\mathcal{L}_0$ on the reducible surface $X_0 = \mathbb{P} \cup \mathbb{F}$ is a section over $\mathbb{P}$ and a section over $\mathbb{F}$ which agree on the double curve R. In other words, if we denote by $\mathcal{L}_{\mathbb{P}}$ the line bundle on $\mathbb{P}$ and by $\mathcal{L}_{\mathbb{F}}$ the line bundle on $\mathbb{F}$, one has natural restriction maps

$$\rho_{\mathbb{P}} : H^0(\mathbb{P}, \mathcal{L}_{\mathbb{P}}) \longrightarrow H^0(R, \mathcal{O}_R(d-k))$$

and

$$\rho_{\mathbb{F}} : H^0(\mathbb{F}, \mathcal{L}_{\mathbb{F}}) \longrightarrow H^0(R, \mathcal{O}_R(d-k))$$

and the global sections of the limit line bundle may be identified with the fiber product

$$\begin{aligned} H^0(X_0, \mathcal{L}_0) :=& \{(\alpha, \beta) \in H^0(\mathbb{P}, \mathcal{L}_P P) \times H^0(\mathbb{F}, \mathcal{L}_F) \\ & \mid \rho_{\mathbb{P}}(\alpha) = \rho_{\mathbb{F}}(\beta) \text{ in } H^0(R, \mathcal{O}_R(d-k))\}. \end{aligned}$$

Thinking in terms of divisors on the two surfaces, this condition means that if we have a divisor $A \in |\mathcal{L}_{\mathbb{P}}|$ on $\mathbb{P}$ and $B \in |\mathcal{L}_{\mathbb{F}}|$ on $\mathbb{F}$, in order that they patch together to give a divisor on X_0, we must have that $A|_R = B|_R$ as divisors on the curve R. For example, if A is tangent to R at a point $r \in R$, so that $A|_R$ contains r with some multiplicity, then $B|_R$ must also contain r with that multiplicity, which implies B must have some tangency (at least) with R at r.

However it could be the case that A has a singularity at r, which is not distinguishable from a tangency, when one only looks at the restriction to R. For example, if A has a triple point at r, with no tangent in the direction of R, then $A|_R$ will contain the divisor $3r$, and this will then force B to have a flexed tangent at r along R. It will not force B to have a triple point though.

Should the possible "extra" singularity of the divisor A have an effect on the divisor B, if we assume that the union $A \cup B$ is a limit of curves in the general surface? This is a relevant hypothesis, for the following reason.

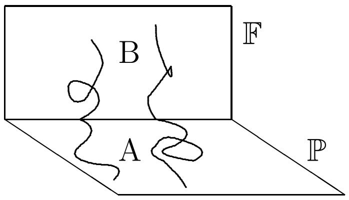
All of the dimension problems in interpolation theory that we have been considering can be reduced to proving that a certain linear system is in fact empty. Indeed, if the expected dimension is $e > -1$, then adding $e+1$ simple base points to the linear system will result in a linear system which is expected to be empty; if it is, then the original linear system will have the correct (expected) dimension also.

5.2 Refined Matching Conditions

Now suppose that we are trying to prove that a certain linear system $\mathcal{L}$ is empty, using the degeneration method. We fix the geometric part of the degeneration (namely the number of points b that go to the $\mathbb{F}$ surface) and assume on the contrary that it is not. Thus there will exist a family of curves $\mathcal{C} \to \Delta$ such that C_t is an "unexpected" curve in X_t and C_0 is the curve in the central fiber, i.e., C_0 is the union of a curve A in $\mathbb{P}$ and a curve B in $\mathbb{F}$. Then C_0 must be a divisor for one of the limit bundles, i.e. there must exist an integer k for which A and B are divisors in the corresponding bundles on $\mathbb{P}$ and $\mathbb{F}$, and which agree on R (as sections of $\mathcal{O}_R(d-k)$).

Thus it is enough to show that for every k there are no sections in the fiber product which are different from zero in both factors, *and which could be limits of curves C_t in the general fiber.*

From now, assume that C_0 is given by $A+B$, where $A \subset \mathbb{P}$ and $B \subset \mathbb{F}$.



What we have been able to show is that if B has a *multiple component* which is a (-1)-curve in $\mathbb{F}$, and which meets R at a point r, then A must have a *singularity* at the point r. Precisely, we have been able to show the following:

Lemma 5.1. *Suppose $E \subset \mathbb{F}$ is a (-1)-curve meeting R at a point $r \in R$, and $E \cdot B = -\sigma < 0$. Then B contains E as a component with multiplicity σ, and $\mathrm{mult}_r(A) \geq \sigma$.*

The idea of the proof is to blow up the threefold X along E and analyze the proper transform $\mathcal{C}'$ of the family $\mathcal{C}$. For details see [25].

5.3 Cremona Transforms and the Three-Point Lemma

In studying linear systems of plane curves with general multiple base points, the opportunity of applying a Cremona transformation of the plane is available at any time, and may indeed be useful in certain situations to reduce the degree or otherwise make the system more amenable to analysis. Let $\mathcal{L}$ be the system of plane curves of degree d with base points of multiplicity $m_1, \ldots, m_n$. Compute $s = m_1 + m_2 + m_3$; performing a Cremona transform based at these three points results in a linear system of curves of degree $2d - s$ and with base points of multiplicity $m_1 - s + d, m_2 - s + d, m_3 - s + d, m_4, \ldots, m_n$. In particular if $s > d$ then both the degree and the first three multiplicities will drop under the Cremona transform.

Instead of applying a Cremona transformation to the system, let us degenerate it, putting the three points on the $\mathbb{F}$ surface. If we are interested in showing that the general system is empty, applying the technique explained above, we must show that for any integer k, there is no limit curve $A + B$ possible in the $(k, 3)$-degeneration. It is not hard to see that the only k that needs to be checked is $k = s - d$, so let us focus on this case.

Let P_1, P_2, P_3 be the three points on $\mathbb{F}$ with multiplicity m_1, m_2, m_3 such that $s = m_1 + m_2 + m_3 > d$. Let F_1, F_2, and F_3 be the corresponding fibers of the ruling of $\mathbb{F}$ through the three points. The linear system on $\mathbb{F}$ that any limit curve B must belong to is the system $|dH - (2d-s)R - m_1P_1 - m_2P_2 - m_3P_3|$. We note that if F is the class of the ruling, we have $H \cdot F = R \cdot F = 1$. Therefore $B \cdot F_i = s - d - m_i = m_j + m_k - d$. We may assume that this is negative, else the line joining P_j and P_k must split off the system anyway, and we would have reduced the degrees and multiplicities to consider already. Hence we can apply Lemma 5.1 and conclude that the curve A on the $\mathbb{P}$ surface must have a point of multiplicity $d - m_j - m_k$ at the point $R \cap F_i$.

In order to show that there are no such limits, we are therefore put into a position of showing that the system on $\mathbb{P}$, namely curves of degree $d - k = 2d - s$, with $n-3$ points of multiplicity $m_4, \ldots, m_n$, has no divisors A *with three additional points of multiplicity* $d - m_1 - m_2$*,* $d - m_1 - m_3$*, and* $d - m_2 - m_3$, lying on the line R.

This is exactly the numerology of applying a Cremona transformation to the original system! The possible advantage to doing this instead of applying a Cremona transformation is that any geometric relationships between the other points would be preserved intact with this operation (e.g., if some subset of the others are not in fact general, but lie on a line), while applying a Cremona transformation would spoil this.

We call this the Three-Point Lemma.

Lemma 5.2 (Three-Point Lemma). *Suppose that the first three multiplicities of the original system $\mathcal{L}$ are m_1, m_2, and m_3, and set $s = m_1+m_2+m_3$. Suppose that the virtual dimension of this system is negative. In order to show that it is in fact empty, it suffices to show that the system obtained by replacing the degree d by $2d-s$, and by replacing these first three multiplicities by $d-m_1-m_2$, $d-m_1-m_3$, and $d-m_2-m_3$, and by enforcing that these three points are collinear, is empty.*

5.4 The Four-Point Lemma

In a similar way we can try to use four points on $\mathbb{F}$, and make a similar analysis.

Again set $s = m_1+m_2+m_3+m_4$, assume that $s > d$, and write $s-d = 2t+e$, with $e = 0, 1$. Again it is not hard to see that the relevant k to analyze is $k = t+e$ (i.e. we drop the degree on $\mathbb{P}$ by $t+e$). The four fibers through the four points on $\mathbb{F}$ have intersection number with B equal to $-(m_i - t - e)$, and there is a fifth (-1)-curve (namely the conic through the four points and the point blown up to R) which has intersection number with B equal to $-t$. Using Lemma 5.1, we therefore have the following.

Lemma 5.3 (Four-Point Lemma). *Suppose that the first four multiplicities of the original system $\mathcal{L}$ are m_1, m_2, m_3, and m_4, and set $s = m_1+m_2+m_3+m_4$. Write $s - d = 2t + e$ as above, with $e = 0$ or 1. Assume that $t + e \leq d$ and $t + e \leq m_i$ for $i = 1, \ldots, 4$. Suppose that the virtual dimension of this system is negative. In order to show that it is in fact empty, it suffices to show that the* *system obtained by replacing the degree d by $d-t-e$, by replacing these first four multiplicities m_i by m_i-t-e, by adding one additional point of multiplicity t, and by enforcing that these five points are collinear, is empty.*

It turns out that the virtual dimension of this reduced system is exactly the same as the virtual dimension of the original system.

5.5 The Five-Point Lemma

One can continue in this vein; let us present one more case, that of putting five points on the surface $\mathbb{F}$.

Again set $s = m_1 + m_2 + m_3 + m_4 + m_5$, assume that $s > d$, and write $s - d = 2t + e$, with $e = 0, 1$. Again it is not hard to see that the relevant k to analyze is $k = t + e$ (i.e. we drop the degree on $\mathbb{P}$ by $t + e$). The five fibers through the five points on $\mathbb{F}$ have intersection number with B equal to $-(m_i - t - e)$. Using Lemma 5.1, we therefore have the following.

Lemma 5.4 (Five-Point Lemma). *Suppose that the first five multiplicities of the original system $\mathcal{L}$ are m_1, m_2, m_3, m_4, and m_5, and set $s = m_1 + m_2 + m_3 + m_4 + m_5$. Write $s - d = 2t + e$ as above, with $e = 0$ or 1. Assume that $t + e \leq d$ and $t + e \leq m_i$ for $i = 1, \ldots, 5$. Suppose that the virtual dimension of this system is negative. In order to show that it is in fact empty, it suffices to show that the system obtained by replacing the degree d by $d-t-e$, by replacing these first five multiplicities m_i by $m_i - t - e$, and by enforcing that these five points are collinear, has the expected dimension.*

In the Five-Point Lemma, the virtual dimension of the system may go up; if it becomes non-negative, it is necessary to show that it has the expected dimension (not of course that it is empty).

5.6 Examples

Using the Three-, Four-, and Five-Point Lemmas, which incorporate the more refined matching conditions as explained above, we can handle several cases which were only possible using the computer algebra packages.

We have focused our attention on homogeneous systems with ten points, which is the first case of interest. Those with virtual dimension equal to -1 should be the hardest to prove are actually empty. The number theory to determine which degrees and multiplicities give virtual dimension -1 for ten points was worked out by A. Malone in her Master's degree paper [44]. The smallest one has $d = 19$ and $m = 10$, and the next smallest has $d = 38$ and $m = 12$. The third smallest has $d = 174$ and $m = 55$. The methods presented here are sufficient to handle the first two, but not the latter one.

The first such example was the one that Dixmier proposed, and that we referred to before; Hirschowitz was successful in using the Horace Method with this system, but we could not provide a proof using the original version of (k, b)-degenerations of the plane, with the naive matching conditions.

Example 5.5. Consider the linear system $\mathcal{L}_{19}(6^{10})$ of curves of degree 19 with ten general points of multiplicity six. The virtual dimension of this system is -1, and so we expect the system to be empty.

Start by applying a four-point lemma with four of the $m = 6$ points. Here $s = 24$, $d = 19$, so that $s - d = 5$, and hence $t = 2$ and $e = 1$. Therefore we reduce to the system of curves of degree 16 with six general points of multiplicity six, and five other collinear points, four of multiplicity 3 and one of multiplicity 2.

We then apply another four-point lemma, to one of the $m = 3$ points and three of the $m = 6$ points. Now $s = 21$, $d = 16$, so again $s - d = 5$, $t = 2$, and $e = 1$. We therefore reduce to the system of curves of degree 13, with three general points of multiplicity six, and eight other base fat points, lying on two lines. Each of the two lines has a point with $m = 2$ and three points with $m = 3$. The intersection point between the two lines started as a point with $m = 3$, but the multiplicity was reduced by three, and therefore eliminated.

Apply another four-point lemma, to two of the $m = 6$ points, and two of the $m = 3$ points (one on each of the lines). Again since $s = 18$ and $d = 13$, we have $s - d = 5$ and $t = 2$, $e = 1$ as before. We reduce to the system of curves of degree 10, with one general $m = 6$ point, and nine other base fat points, lying on three lines; each of the lines has two $m = 3$ points and one $m = 2$ point. (Again the original two $m = 3$ points used in the four-point lemma application are removed by this process.)

Finally do one more four-point lemma, with the final $m = 6$ point, and one $m = 3$ point from each of the three lines. Again $s - d = 15 - 10 = 5$, $t = 2$, and $e = 1$. We reduce to the system of curves of degree 7, with eight general base fat points, four $m = 3$ points and four $m = 2$ points, lying on four general lines, one $m = 3$ and one $m = 2$ point on each line. (Again the three $m = 3$ points used in the four-point application are removed by this process.) At this point the eight points are in fact general points! We have reduced the problem to showing that the linear system $\mathcal{L}_7(2^4, 3^4)$ of septic curves with four general triple points and four general double points is empty.

Performing a Cremona transformation at three of the four triple points gives the system $\mathcal{L}_5(1^3, 2^4, 3)$. Performing a second Cremona transformation at the triple point and two of the double points gives the system $\mathcal{L}_3(1^4, 2^2)$. At this point one notices that the line joining the two double points must split off this system, and the residual system is the system $\mathcal{L}_2(1^6)$ of conics through six general points, which is indeed empty.

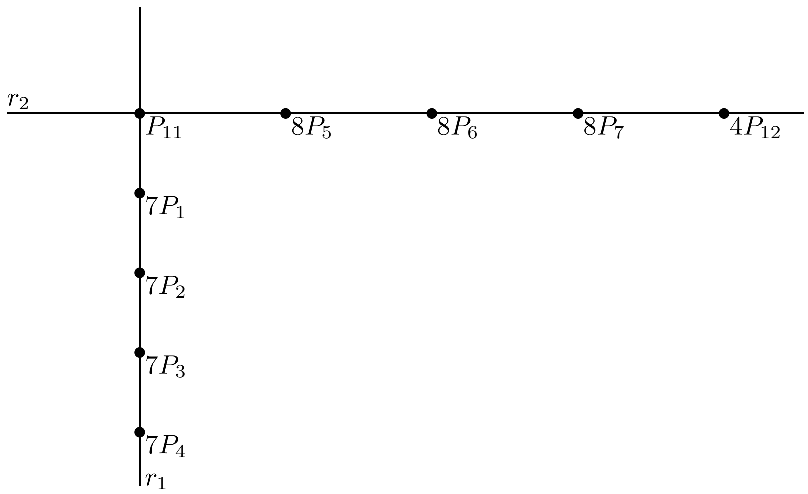
This series of relatively simple applications of the Four-Point Lemma, followed by some Cremona transformations, suffices to prove that the original system $\mathcal{L}_{19}(6^{10})$ is empty.

Example 5.6. Consider the linear system $\mathcal{L}_{38}(12^{10})$ of curves of degree 38 with ten general points of multiplicity twelve. The virtual dimension of this system is -1, and so we expect the system to be empty. This system was analyzed in Gimigliano's thesis.

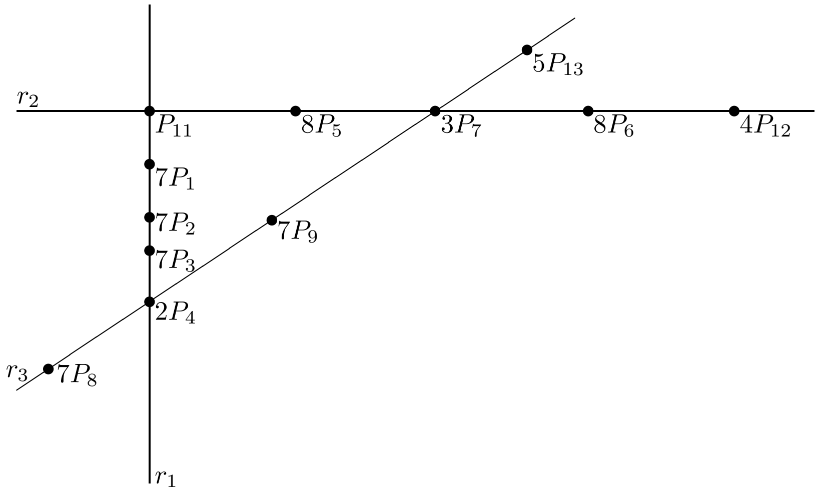
Call the points $P_1, \dots P_{10}$. Start by applying a Four-Point lemma with P_1,

P_2, P_3 and P_4. Here $s = 48$, $d = 38$, so that $s - d = 10$, and hence $t = 5$ and $e = 0$. Thus we reduce to the system of curves of degree 33 with six general points of multiplicity twelve, and five other collinear points, four of multiplicity 7 and a point P_{11} of multiplicity 5, lying on a line r_1.

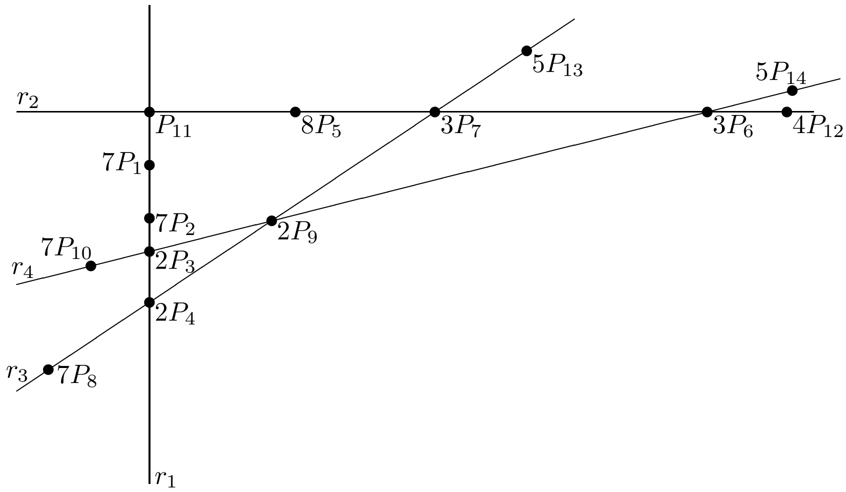
Apply a Four-Point Lemma with P_5, P_6, P_7 and P_{11}. We reduce to the system of curves of degree 29 with three points of multiplicity twelve, four points of multiplicity seven on the line r_1, three points of multiplicity eight (i.e. P_5, P_6 and P_7) and a new point of multiplicty four (i.e. P_{12}) on a line r_2. The intersection between r_1 and r_2 is the simple point P_{11}, having $m = 1$.



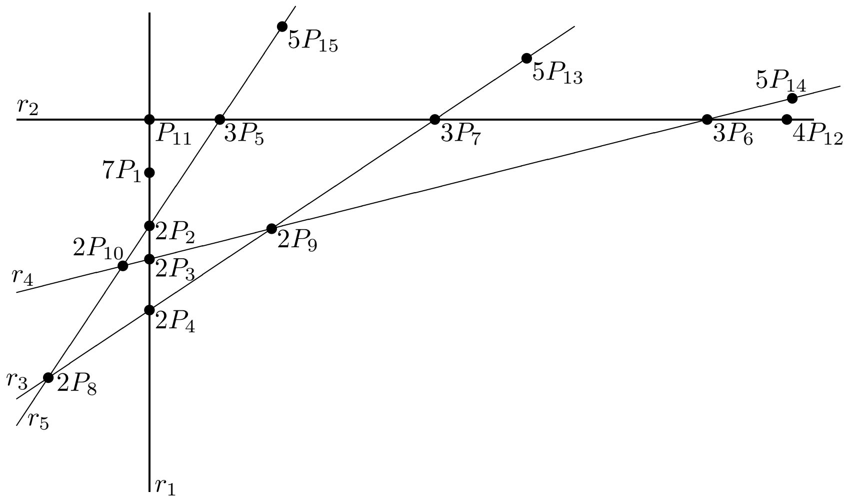
Apply a Four-Point Lemma with P_4, P_7, P_8 and P_9. Here $s = 39$, and we reduce to a system of curves of degree 24. We have a new point P_{13} of multiplicity $m_{13} = 5$. Moreover P_4 becomes a double point and P_7 a triple point, while P_8 and P_9 drop their multiplicity to $m_8 = m_9 = 7$. The points P_4, P_7, P_8, P_9 and P_{13} lie on the line r_3.



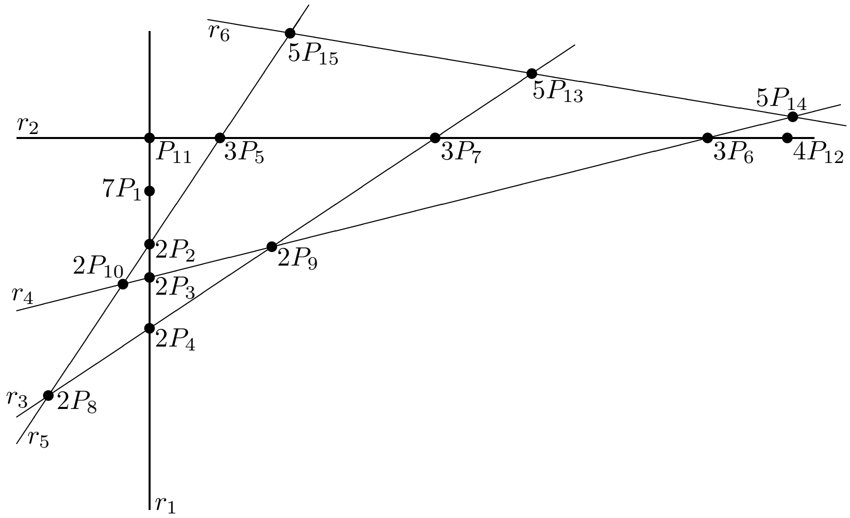
Apply a Four-Point Lemma with P_3, P_6, P_9 and P_{10}. Here $s = 34$; thus the degree of the system drops to 19. We have a new point P_{14} of multiplicity $m_{14} = 5$. The points P_3 and P_9 become double points, while P_6 drops its multiplicity to $m_6 = 3$, and P_{10} drops to $m_{10} = 7$. The points P_3, P_6, P_9 P_{10} and P_{14} lie on the line r_4.



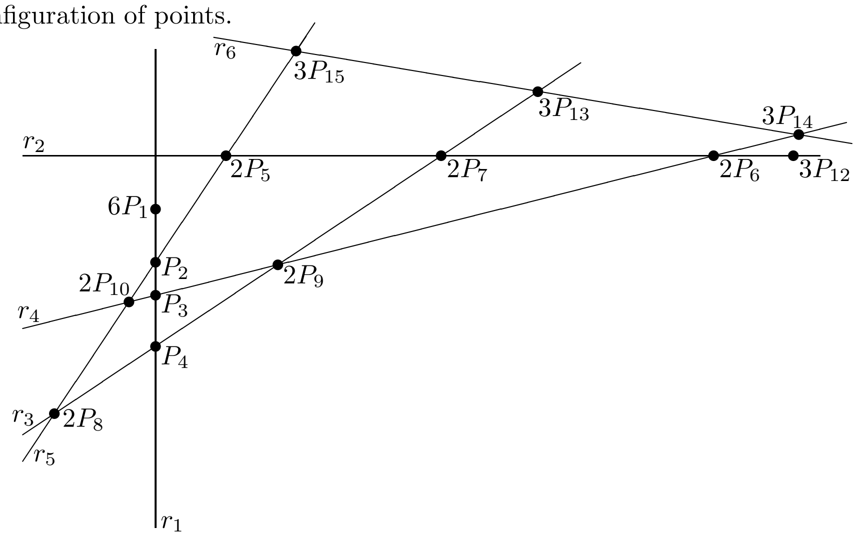
Finally apply another Four-Point Lemma with P_2, P_5, P_8 and P_{10}. In this way we reduce to a system of curves of degree 14. We have a new point P_{15} of multiplicity $m_{15} = 5$. The points P_2, P_8 and P_{10} become double points, while P_5 drops its multiplicity to $m_5 = 3$. The points P_2, P_5, P_8 P_{10} and P_{15} lie on the line r_5.



From this moment we can try to move the points in such a way that we only put three of them on a line at any one step. We start by putting P_{13}, P_{14} and P_{15} on a line r_6.



Since $m_{13} + m_{14} + m_{15} = 15 > 14$, r_6 splits off. We then reduce to a system of curves of degree 13. The points P_{13}, P_{14} and P_{15} now have multiplidity $m_{13} = m_{14} = m_{15} = 4$.
Now r_1 splits off once from the system (since $m_1 + m_2 + m_3 + m_4 + m_{11} = 7 + 2 + 2 + 2 + 1 = 14$). The point P_{11} drops its multiplicty to 0. The points P_2, P_3 and P_4 become simple points while P_1 has multiplicity $m_1 = 6$.
The degree of the system is now 12. Consider now the line r_2; we have $m_5 + m_6 + m_7 + m_{12} = 13$. Thus also r_2 splits off, leaving curves of degree 11. Moreover, the points P_5, P_6 and P_7 now have multiplicity $m_5 = m_6 = m_7 = 2$ and P_{12} has $m_{12} = 3$.
Now, since $m_{13} + m_{14} + m_{15} = 12$, the line r_6 splits once again from the system. Thus we reduce to a system of curves of degree 10 with the following configuration of points.



Now move the points such that P_1, P_8 and P_{14} lie a line r_7 and P_1, P_9

and P_{15} lie on a line r_8. Since $m_1 + m_8 + m_{14} = 6 + 2 + 3 = 11$ we can split the line r_7 from the system leaving curves of degree 9. On r_8 we now have $m_1 + m_9 + m_{15} = 5 + 2 + 3 = 10$; hence also r_8 splits. Thus the residual system has degree 8. Moreover we have P_1 with $m_1 = 4$, P_8, P_9 with $m_8 = m_9 = 1$ and P_{14}, P_{15} with $m_{14} = m_{15} = 2$.

Now the line r_2 splits ($m_5 + m_6 + m_7 + m_{12} = 9$) and we reduce to a system of curves of degree 7. The points P_5, P_6 and P_7 have multiplicity $m = 1$ while P_{12} has $m_{12} = 2$.

Now move P_1, P_5 and P_{13} such that they lie on a line r_9. Since $m_1 + m_5 + m_{13} = 8$ the line r_9 splits. We pass to a system of curves of degree 6 in which P_1 has $m_1 = 3$, P_{13} has $m_{13} = 2$ and P_5 has $m_5 = 0$. Now, the line r_4 with points P_3, P_6, P_9, P_{10} and P_{14} splits ($m_3 + m_6 + m_9 + m_{10} + m_{14} = 1 + 1 + 1 + 2 + 2 = 7$). Thus we can cancel the points P_3, P_6 and P_9 (since they have $m = 0$), while P_{10} and P_{14} drop their multiplicity to $m_{10} = m_{14} = 1$. The degree of the curves of the system drops to 5.

Move P_1, P_{10} and P_{12} in such a way they lie on a line r_{10}. Since $m_1 + m_{10} + m_{12} = 3 + 1 + 2 = 6$, the line r_{10} splits and we reduce to a system of curves of degree 4. The point P_1 has $m_1 = 2$, P_{12} has $m_{12} = 1$, while P_{10} can be canceled since $m_{10} = 0$.

Now, since $m_4 + m_7 + m_8 + m_{13} = 1 + 1 + 1 + 2 = 5 > 4$, the line r_3, with points P_4, P_7, P_8 and P_{13}, splits. Thus we can cancel the points P_4, P_7, P_8. The point P_{13} now has multiplicity $m_{13} = 1$. We reduce to a system of curves of degree 3.

Finally, we split the line r_6 with points P_{13}, P_{14} P_{15} ($m_{13} + m_{14} + m_{15} = 1 + 1 + 2 = 4$) and we reduce to the system of conics double at P_1 and passing through P_2, P_{12}, P_{15}. These give three distinct lines which must split off a system of degree two. We conclude that the system must be empty.

Example 5.7. In a similar way to the previous examples we can treat the linear system $\mathcal{L}_{158}(50^{10})$ of curves of degree 158 with ten general points of multiplicity fifty. This was posed by J. Roe during the workshop as an interesting unknown case for the Nagata problem. The virtual dimension of this system is -31, and so we expect the system to be empty. We prove that, in fact, $\mathcal{L}_{158}(50^{10})$ is empty. We just give the essential steps and we leave the details to the reader.

We represent the system $\mathcal{L}_{158}(50^{10})$ by the following table

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
158	50	50	50	50	50	50	50	50	50	50

We start applying a four-point lemma to P_1, P_2, P_3 and P_4. Since $s = 200$ and $s - d = 200 - 158 = 42$, we have $t = 21$ and $e = 0$. Thus we reduce to the system with the following data

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
137	29	29	29	29	50	50	50	50	50	50	21
1	1	1	1	1	0	0	0	0	0	0	1

Here we have indicated that the points are no longer all general in the third row, by indicating the curve (in this case a line, of degree one) which passes through P_1, P_2, P_3, P_4, and P_{11}. Apply now a Cremona transformation centered at P_5, P_6 and P_7. Apply a second Cremona centered at P_8, P_9, and P_{10}. Apply a third Cremona centered at P_5, P_6 and P_7, and finally a fourth Cremona centered at P_1, P_2, and P_3. We obtain

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
83	27	27	27	29	24	24	24	24	24	24	21
7	3	3	3	1	2	2	2	2	2	2	1

so that the resulting system has degree 83 with the indicated multiplicities, and the points lie on a curve of degree 7 with the indicated multiplicities. (The curve of degree 7 is the image of the line under the four Cremona transformations.)

Now execute a Four Point Lemma with points P_4, P_5, P_6, and P_7, which results in:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
74	27	27	27	20	15	15	15	24	24	24	21	9
7	3	3	3	1	2	2	2	2	2	2	1	0
1	0	0	0	1	1	1	1	0	0	0	0	1

Apply two more Cremona transformations. The first one is centered at P_1, P_2 and P_3 and the second one is centered at P_8, P_9 and P_{10}. The result is:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
62	20	20	20	20	15	15	15	19	19	19	21	9
4	1	1	1	1	2	2	2	1	1	1	1	0
4	1	1	1	1	1	1	1	2	2	2	0	1

The two quartics indicated above are the Cremona images of the septic and the line in the previous table.

At this point, we can use the four-point lemma on P_1, P_2, P_3 and P_{11}. We reduce to the system represented by

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
52	10	10	10	20	15	15	15	19	19	19	11	9	9
4	1	1	1	1	2	2	2	1	1	1	1	0	0
4	1	1	1	1	1	1	1	2	2	2	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1	0	1

At this point one notices that the second quartic splits off the system, three times; and then the line splits off once. This results in the system of degree 39 indicated by:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
39	6	6	6	17	12	12	12	13	13	13	10	6	8
4	1	1	1	1	2	2	2	1	1	1	1	0	0
4	1	1	1	1	1	1	1	2	2	2	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1	0	1

Now apply three Cremona transformations respectively centered at (P_4, P_8, P_9), (P_4, P_5, P_{10}) and (P_4, P_6, P_7). This reduces the system to

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
30	6	6	6	8	9	10	10	9	9	10	10	6	8
4	1	1	1	1	2	1	1	2	2	1	1	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1

where now the constraints on the points are that they lie on the indicated three quartic curves.

Now execute a Four Point Lemma with P_{10}, P_{11}, P_{12}, and P_{13}, to obtain the system

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
28	6	6	6	8	9	10	10	9	9	8	8	4	6	2
4	1	1	1	1	2	1	1	2	2	1	1	0	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1

At this point the first quartic splits off the system, then the second quartic splits off, then the line splits off; at this point the first quartic splits off again, then the third quartic splits, then finally the line splits again. This leaves us with the system of degree ten:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
10	2	2	2	2	3	5	5	3	3	1	3	1	3	0
4	1	1	1	1	2	1	1	2	2	1	1	0	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1

Now three Cremona transformations, centered respectively at (P_6, P_7, P_{11}), (P_5, P_8, P_9) and (P_3, P_6, P_{13}), give a system of cubics which are double at P_1, P_2, P_4, and P_7, and passing simply through P_5, P_8, P_9, P_{10}, P_{12}, and P_{13}. This system is clearly empty (the six lines passing pairwise through the four double points must split off; but the degree of the system is only three).

Since this system is empty, the claim follows for $\mathcal{L}_{158}(50^{10})$.

6 Lecture Six: Special Effect Varieties

This section is devoted to the definition and the study of two different kinds of varieties called *special effect varieties.* The $\alpha-$*special effect variety* is defined by requiring some numerical conditions, while the definition of h^1-*special effect variety* concerns cohomology groups. We will start with the case of special effect curves in $\mathbb{P}^2$. As we will see, the existence of these curves is related to the speciality of a given linear system. This suggests two new conjectures for special systems in the planar case. Whenever not otherwise specified, we work over the field $\mathbb{C}$.

6.1 Basic Definitions

6.1.1 $\alpha-$Special effect curves

We start with some preliminary definitions.

Definition 6.1. *Let* $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ *and* $\mathcal{L}' := \mathcal{L}_{2,d'}(-\sum_{i=1}^{s} c_{j_i} P_{j_i})$ *be two linear systems in* $\mathbb{P}^2$*. We will write* $\mathcal{L}' <_{LS} \mathcal{L}$ *if*

1) $d' \leq d$*;*

2) $\{P_{j_1}, \ldots P_{j_s}\} \subseteq \{P_1, \ldots P_h\}$*;*

3) $c_{j_k} \leq m_{j_k}$ *for all* $k = 1, \ldots, s$*.*

Let $Y \subset \mathbb{P}^2$ *be a curve. Then we write* $Y <_{LS} \mathcal{L}$ *if the degree of* Y *is less than or equal to* d *and* $\operatorname{mult}_{P_i}(Y) \leq m_i$ *for each* i*.*

Definition 6.2. *Let* $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ *be a linear system of curves of degree* d *passing through the points* P_i *with multiplicity at least* m_i*. Let* Y *be an irreducible curve passing through the points* $P_{j_1}, \ldots, P_{j_s}$ *with multiplicity at least* $c_{j_1}, \ldots, c_{j_s}$*, such that* $Y <_{LS} \mathcal{L}$*. Then* Y *has the* **weak special effect property for** $\mathcal{L}$ *if*

(iP) $\nu(|Y|) \geq 0$*,*

(iiW) $\nu(\mathcal{L} - Y) \geq \nu(\mathcal{L})$*.*

Moreover, we will say that Y *has the* **special effect property for** $\mathcal{L}$ *if the inequality in* (iiW) *is strict, i.e.*

(iiP) $\nu(\mathcal{L} - Y) > \nu(\mathcal{L})$*.*

Example 6.3. Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ and consider a $(-1)-$curve E such that $\mathcal{L} \cdot E = -N < 0$. Thus $\mathcal{L} = NE + \mathcal{M}$, where $E \cdot \mathcal{M} = 0$. Using Riemann-Roch it is easy to prove

$$\nu(\mathcal{M}) = \nu(\mathcal{L}) + \binom{N}{2}.$$

Hence E has the special effect property if $N \geq 2$ and the weak special effect property if $N = 1$.

Example 6.4. Let $\mathcal{L}$ be the system $\mathcal{L}_{2,6}(-\sum_{i=1}^{9} 2P_i)$. The only element in $\mathcal{L}$ is the double cubic through the nine points $C = 3H - \sum_{i=1}^{9} P_i$. Since

$$\nu(\mathcal{L}) = \frac{6 \cdot (3+6)}{2} - 9 \cdot 3 = 0$$

and

$$\nu(\mathcal{L} - C) = \nu(C) = \frac{3 \cdot (3+3)}{2} - 9 = 0$$

we conclude that the cubic C has the weak special effect property for $\mathcal{L}$.

Example 6.5. Let $\mathcal{L}$ be the system $\mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$. The only element in $\mathcal{L}$ is $3Y$, where Y is the union of the lines passing through two of the three points, i.e. $Y = L_{12} + L_{13} + L_{23}$, where L_{ij} is the line through P_i and P_j. We claim that each of the lines L_{ij} has the special effect property. We prove this for L_{12}. Obviously one has $\nu(|L_{12}|) \geq 0$; indeed, it is a (-1)-curve. Moreover $\mathcal{L} - L_{12}$ is the system $\mathcal{L}' := \mathcal{L}_{2,8}(-5P_1 - 5P_2 - 6P_3)$ and its virtual dimension is

$$\nu(\mathcal{L}') := \frac{8 \cdot 11}{2} - 2\frac{5 \cdot 6}{2} - \frac{6 \cdot 7}{2} = 44 - 30 - 21 = -7$$

while $\nu(\mathcal{L}) = -9$. So the claim follows.

It is clear, now, in which way we proceed. If Y has one of the special effect properties, we substitute the system $\mathcal{L}$ with $\mathcal{L} - Y$ and we investigate this new system.

Definition 6.6. *Let $\mathcal{L}$ be a system as above. Fix a sequence of (not necessarily distinct) irreducible curves $Y_1, \dots Y_\alpha$, such that any two distinct members are disjoint. Suppose further that*

(1) Y_j has the weak special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} Y_i$, for $j = 1, \dots, \alpha$,

(2) there exists at least one index j such that Y_j has the special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} Y_i$,

(3) $\nu(\mathcal{L} - \sum_{i=1}^{\alpha} Y_i) \geq 0$.

Then $X := \sum_{i=1}^{\alpha} Y_i$ is called a **special effect configuration for** $\mathcal{L}$. *In particular we write $X := \sum_{i=1}^{r} \alpha_i Y_i$ if r is the number of distinct curves and Y_i occurs α_i times in the list. We call both X and $\{Y_1, \dots, Y_r\}$ an $(\alpha_1, \dots, \alpha_r)-$***special effect configuration**. *Finally, when $Y_1 = Y_2 = \dots = Y_\alpha = Y$ we write $X = \alpha Y$ and we call both X and Y an $\alpha-$***special effect curve**.

Let us analyze the three requirements. Since $\mathcal{L} - \sum_{i=1}^{\alpha} Y_i$ is nothing else than the (residual) system $\mathcal{L}' := |(d - \sum_{i=1}^{\alpha} \deg(Y_i))H - \sum_{i=1}^{h}(m_i - \sum_{k=1}^{s_i} c_{j_{i_k}})P_i|$, condition (3) says that $\mathcal{L}'$ is not empty. Conditions (1) and (2) are surely the most interesting. As a matter of fact they tell us that the number of conditions imposed on the system of curves of degree d by imposing the curves $Y_1, \dots, Y_\alpha$

and the points P_i with multiplicity $m_i - \sum_{k=1}^{s_i} c_{j_{i_k}}$ (such that the final multiplicity at the point P_i is at least m_i, $i = 1, \ldots, n$) is less than the number of conditions imposed to the same system $|dH|$ only imposing each P_i with multiplicity at least m_i, $i = 1, \ldots, n$. This sounds like a crazy requirement because, in general, we expect that a positive dimensional variety imposes more conditions than a zero-dimensional variety. It is important to notice the similarity with the "strange" requirement in the case of $(-1)-$curves (see for example [21]). We asked there for a curve C whose double is not expected to exist.

We recall that the existence of a $(-1)-$curve C such that $\mathcal{L} := NC + \mathcal{M}$ leads us to the inequality

$$\dim(\mathcal{L}) = \dim(\mathcal{M}) \geq \nu(\mathcal{M}) = \nu(\mathcal{L}) + \binom{N}{2}$$

which, under the assumption $\nu(\mathcal{M}) \geq 0$ and $N \geq 2$, implies that $\mathcal{L}$ is special. Also the existence of $\alpha-$special effect variety or special effect configuration X for a system $\mathcal{L}$ forces the system itself to be special. In fact we have the following chain of inequalities:

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L} - X) \geq \nu(\mathcal{L} - X) > \nu(\mathcal{L})$$

and, together with condition (3), one has $\dim(\mathcal{L}) > \epsilon(\mathcal{L})$.

Example 6.7. Let $\mathcal{L} := \mathcal{L}_{2,2}(-2P_1 - 2P_2)$ be the linear system of conics with two double points. Let Y be a line through P_1 and P_2, i.e $Y = H - P_1 - P_2$. Since

$$\nu(\mathcal{L}) = -1$$

and

$$\nu(\mathcal{L} - Y) = \nu(Y) = 0$$

we conclude that Y has the special effect property for $\mathcal{L}$ and it has the weak special effect property for $\mathcal{L} - Y$. From $\nu(\mathcal{L} - 2Y) = 0$ it follows that the line through P_1 and P_2 is a $2-$special effect curve for $\mathcal{L}$ and so $\mathcal{L}$ is special.

Example 6.8. Consider again the system $\mathcal{L} := \mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$. We prove that $X = 3L_{12} + 3L_{13} + 3L_{23}$ is a $(3, 3, 3)-$special effect configuration. Recall that $\nu(\mathcal{L}) = -9$. In Example 6.5 we proved that L_{12} has the special effect property for $\mathcal{L}$ because

$$\nu(\mathcal{L} - L_{12}) = \nu(|8H - 5P_1 - 5P_2 - 6P_3|) = -7.$$

Define now $\mathcal{L}' := \mathcal{L} - L_{12}$. We have

$$\nu(\mathcal{L}' - L_{12}) = \nu(\mathcal{L} - 2L_{12}) = \nu(|7H - 4P_1 - 4P_2 - 6P_3|) = -6$$

so that L_{12} has the special effect property for $\mathcal{L}' = \mathcal{L} - L_{12}$. Define $\mathcal{L}'' := \mathcal{L}' - L_{12} = \mathcal{L} - 2L_{12}$. If we compute the virtual dimension of $\mathcal{L}'' - L_{12}$ we discover that it is again -6. Thus L_{12} has the weak special effect property for

$\mathcal{L}''$. We can go ahead and apply the previous procedure with L_{13} and L_{23}. We obtain:

$$\begin{array}{l}
\nu(\mathcal{L} - 3L_{12} - L_{13}) = \nu(|5H - 2P_1 - 3P_2 - 5P_3|) = -4 \\
\nu(\mathcal{L} - 3L_{12} - 2L_{13}) = \nu(|4H - P_1 - 3P_2 - 4P_3|) = -3 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13}) = \nu(|3H - 3P_2 - 3P_3|) = -3 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - L_{23}) = \nu(|2H - 2P_2 - 2P_3|) = -1 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - 2L_{23}) = \nu(|H - P_2 - P_3|) = 0. \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - 3L_{23}) = 0.
\end{array}$$

Thus $X = 3L_{12} + 3L_{13} + 3L_{23}$ is a $(3,3,3)-$special effect configuration for $\mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$.

Definition 6.9. *A special system arising from the existence of an $\alpha-$special effect curve (or an $(\alpha_1, \ldots \alpha_r)-$special effect configuration) is called* **Numerically Special**.

Finally, we can state the following

Conjecture 6.10 ((NSEC) "Numerical Special Effect" Conjecture). *A linear system of plane curves $\mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is numerically special.*

6.1.2 h^1-Special effect curves

The second class of curves we introduce are defined via some particular conditions on certain cohomology groups. The original idea for these curves comes from a detailed analysis of the base locus in the special systems listed in Theorem 1.29, that is, linear systems with imposed double points in $\mathbb{P}^n$, $n \geq 2$.

Definition 6.11. *Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of plane curves with general multiple base points. An irreducible curve $Y \subset \mathbb{P}^2$, with $\mathcal{O}_{\mathbb{P}^2}(Y) \ncong \mathcal{L}$, is an h^1-***special effect curve** *for the system $\mathcal{L}$ if the following conditions are satisfied:*

(a) $h^0(\mathcal{L}_{|Y}) = 0$;

(b) $h^0(\mathcal{L} - Y) \neq 0$;

(c) $h^1(\mathcal{L}_{|Y}) > 0$.

Example 6.12. Let $\mathcal{L} := \mathcal{L}_{2,2}(-2P_1 - 2P_2)$ be the linear system of conics with two double points. Let Y be a line through P_1 and P_2, i.e $Y = H - P_1 - P_2$. Since $\mathcal{L} \cdot Y = -2$ the restricted system $\mathcal{L}_{|Y}$ has no effective divisors and $h^0(\mathcal{L}_{|Y})$ is empty. By Riemann–Roch we easily compute $h^1(\mathcal{L}_{|Y}) = g_Y - 1 - \deg(\mathcal{L}_{|Y}) = 1 > 0$. Finally $\mathcal{L} - Y$ is $|H - P_1 - P_2|$, so that $h^0(\mathcal{L} - Y) \neq 0$. Hence the line Y through P_1 and P_2 is an h^1-special effect curve for $\mathcal{L}$.

Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ and consider, on the blow-up of $\mathbb{P}^2$ at the points P_i $i = 1, \dots, n$, the exact sequence

$$0 \to \mathcal{L} - Y \to \mathcal{L} \to \mathcal{L}_{|Y} \to 0$$

which gives the following long exact sequence in cohomology:

$$0 \to H^0(\mathcal{L}-Y) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|Y}) \to H^1(\mathcal{L}-Y) \to H^1(\mathcal{L}) \to H^1(\mathcal{L}_{|Y}) \to 0.$$

Conditions (a) and (b) assure us that $H^0(\mathcal{L}) \neq 0$, while condition (c) implies $H^1(\mathcal{L}) \neq 0$. Thus the existence of such Y forces the system $\mathcal{L}$ to have $h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) \neq 0$ so that, by (1.12), $\mathcal{L}$ is special. Again, we can give a particular name to this kind of system:

Definition 6.13. *A special system arising from the existence of an h^1-special effect curve is called* **Cohomologically Special**.

And again we can state a conjecture:

Conjecture 6.14 ((CSEC) "Cohomological Special Effect" Conjecture). *A linear system of plane curves $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is cohomologically special.*

6.1.3 The four conjectures

In the previous sections we introduced two new conjectures for the characterization of special linear systems in the planar case. At this point it is natural to ask if these new conjectures are equivalent to the Segre and Harbourne–Hirschowitz conjectures. The answer is given in the following

Theorem 6.15. *Conjectures (SC) [1.13], (HHC) [1.15], (NSEC) [6.10] and (CSEC) [6.14] are equivalent.*

The proof of the previous theorem can be found in [10], Chapter 3. Here one can find additional interesting evidence relating these ideas with other conjectures for special systems on surfaces, in particular on Hirzebruch and K3 surfaces.

6.2 Results in Higher Dimensions

As already mentioned in Section 1, very little is known for special linear systems on a variety X with $\dim(X) > 2$, even when $X = \mathbb{P}^n$. In this last case the most important result is the classification of the homogeneous special systems for double points given by Alexander and Hirschowitz in Theorem 1.29.

Continuing with $\mathbb{P}^n$, $n \geq 3$ we can notice that there is not a precise conjecture. Although the Segre Conjecture can be generalized in every ambient variety using the statement concerning $H^1 \neq 0$, there is nothing that characterizes the special systems from a geometric point of view as, for example, in the case of $(-1)-$curves in $\mathbb{P}^2$.

A worthy goal would be to find a conjecture (C) in $\mathbb{P}^n$, [or in a generic variety X] such that, when we read (C) in $\mathbb{P}^2$, (C) is equivalent to the Segre (1.13) and Harbourne–Hirschowitz (1.15) Conjectures.

This goal is one of the main topics in [10]. Here we can see how both *Numerical Special Effect Conjecture* and *Cohomological Special Effect Conjecture* are potential candidates for the above-mentioned goal. Unluckily, in both cases it could be difficult to work with a generic special effect variety Y of codimension $c > 1$; we do not have, for example, a precise definition of virtual dimension of $\mathcal{L} - Y$ or it could be hard to compute $h^2(\mathcal{L} - Y)$.

Thus it is not so easy to define the special effect varieties in $\mathbb{P}^n$ with $n \geq 3$. Obviously, when the $\alpha-$special effect variety Y is a divisor on $\mathbb{P}^n$ (but in general, on every variety) we can generalize the definitions given in section 6.1 most easily. In [10] there are some different approaches to avoid the previous problem and, although there is not yet a general theory for the higher dimension case, several examples of $\alpha-$special effect varieties are shown. In particular we have the following theorem

Theorem 6.16 ([10], Theorem 4.1.17). *There exists a $2-$special effect variety Y for each of the special systems listed in Theorem 1.29.*

Example 6.17. Consider the system $\mathcal{L} := \mathcal{L}_{3,4}(2^9)$. One has $\nu(\mathcal{L}) = -2$. Let Q the quadric in $\mathbb{P}^3$ through the nine points. Since $\nu(\mathcal{L} - 2Q) = 0 > \nu(\mathcal{L})$, Q is a $2-$special effect variety for $\mathcal{L}$.

We turn now to analyzing h^1-special effect varieties in higher dimension. Let $\mathcal{L} := \mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of hypersurfaces with general multiple base points and let X be the blow-up of $\mathbb{P}^n$ at the points $\{P_i\}$. Let $\tilde{\mathcal{L}}$ be the strict transform of $\mathcal{L}$. In general, if confusion cannot arise, we will denote both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ by $\mathcal{L}$. If we denote by $\tilde{Y}$ the strict transform of a variety $Y \subset \mathbb{P}^n$, then we define $\mathcal{L} - Y := \mathcal{L} \otimes \mathcal{I}_{\tilde{Y}}$. The definition of the h^1-special effect variety is slightly modified by respect to the planar case.

Definition 6.18. *Let $\mathcal{L}$ and Y be as above with Y irreducible. Moreover, if $codim(Y, \mathbb{P}^n) = 1$ then we require $\mathcal{O}_{\mathbb{P}^n}(Y) \ncong \mathcal{L}$. Then $Y \subset \mathbb{P}^n$ is an h^1-**special effect variety** for the system $\mathcal{L}$ if the following conditions are satisfied:*

(a) $h^0(\mathcal{L}_{|Y}) = 0$;

(b) $h^0(\mathcal{L} - Y) \neq 0$;

(c) $h^1(\mathcal{L}_{|Y}) > h^2(\mathcal{L} - Y)$.

The h^1-special effect varieties seem easier to treat than the $\alpha-$special effect varieties. In fact, we do not need to define the virtual dimension, but we just work with elements in cohomology. However, in several situations, it is very difficult to compute some cohomology groups, in particular $h^2(\mathcal{L} - Y)$.

As in the case of $\alpha-$special effect varieties, we do not have problems when Y is a divisor since $h^2(\mathcal{L} - Y) = 0$ if $\mathcal{L} - Y$ is effective. To see this, write $\mathcal{L}$ as $\mathcal{L} := |dH - \sum_{i=1}^{h} m_i P_i|$ and $Y := eH - \sum_{i=1}^{h} c_i P_i$; then $\mathcal{L} - Y = |(d-e)H -$

$\sum_{i=1}^{h}(m_i - c_i)P_i|$, with $d \geq e$ and $m_i \geq c_i$. Hence the system $\mathcal{L} - Y$ has the form $\mathcal{L} - Y = |aH - \sum_{i=1}^{h} s_i P_i|$, with $a \geq 0$. Define Z as the union of the fat points $s_i P_i$; then we have the following exact sequence

$$0 \to \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(a) \to \mathcal{O}_{\mathbb{P}^n}(a) \to \mathcal{O}_Z \to 0.$$

where $\mathcal{L} - Y$ is exactly $\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(a)$. When we consider the cohomology groups, we have $h^i(\mathcal{O}_Z) = 0$ for $i \geq 1$, since Z is a zero-dimensional scheme. Moreover, $h^i(\mathcal{O}_{\mathbb{P}^2}(a)) = 0$ for $i \geq 1$. Thus $h^i(\mathcal{L} - Y) = 0$ for $i \geq 2$ (this motivates also conditions (c) in the planar case).

Unluckily, when Y is a divisor, it can be difficult to study the behaviour of $\mathcal{L}_{|Y}$. Instead, when $\mathrm{codim}(Y, \mathbb{P}^n) \geq 2$, the groups $h^i(\mathcal{L} - Y)$, $i = 1, 2$ can be computed on the blow-up of $\mathbb{P}^n$ along Y, but we need a deep understanding of the geometry and cohomology of Y.

In any case there is interesting evidence of the relationship between special systems and h^1-special effect varieties; in particular, we have the following:

Theorem 6.19 ([10], Ch 4, Theorem 4.2.2). *There exists an h^1-special effect variety Y for each of the special systems listed in Theorem 1.29.*

We conclude this section with some examples of $\alpha-$special effect varieties on $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$ with $t \geq 2$ and $n_i \geq 1$ for $i = 1, \ldots, t$. In [10] the case mainly explored concerns $m = \alpha = 2$ and Y is a divisor. Surely this does not exhaust all possible special effect varieties on X, but we can observe how our results fit with the ones by Catalisano, Geramita and Gimigliano on secant varieties of products of projective spaces ([15], [16], [17], [18]). We suppose that the reader knows the relationship between special systems and defective varieties; we suggest [20] as reference.

Also in the case of $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$ we have some interesting evidence. First of all we recall a result by Catalisano, Geramita and Gimigliano.

Theorem 6.20 ([18], Theorem 2.1). *Let $\mathcal{L} := \mathcal{L}_{a_1,a_2}(2^h)$ be the linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ of divisors of bidegree (d_1, d_2) with h imposed double points. Then $\mathcal{L}$ is non-special unless*

$$a_1 = 2d, a_2 = 2, d \geq 1, \text{ and } h = 2d + 1.$$

From the study of $\alpha-$special effect varieties on $\mathbb{P}^a \times \mathbb{P}^b$ we are able to prove the following

Theorem 6.21. *There exists a $2-$special effect curve for each of the special systems listed in Theorem 6.20.*

The second result we mention is related to the study of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 6.22 ([18], Theorem 2.5). *Let $a_1 \geq a_2 \geq a_3 \geq 1$, $\alpha \in \mathbb{N}$ and $V = V_{\mathbf{a}}$ be a Segre–Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $Sec_k(V)$ has the expected dimension, except for:*

$$(a_1, a_2, a_3) = (2, 2, 2) \text{ and } k = 6;$$

$$(a_1, a_2, a_3) = (2\alpha, 1, 1) \text{ and } k = 2\alpha.$$

In these cases $Sec_k(V)$ is defective, and its defectivity is 2 in the first case and 1 in the second.

Once again we can try to check if there are special effect varieties for the special systems corresponding to the defective varieties listed before. It is easy to observe that, by numerical reasons, the second case cannot be treated with a $2-$special effect variety. However, using special effect configurations we can state a result as Theorem 6.21.

Theorem 6.23. *There exists a $2-$special effect variety or a $(1,1)-$special effect configuration for each of the special systems listed in Theorem 6.22.*

The Theorems 6.21 and 6.23 follow from a deep studying of the combinatorial behaviour of $\alpha-$special effect varieties on $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$. In particular we can prove the following results.

Proposition 6.24. *Let $Y \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ be a divisor of bidegree (e_1, e_2), with $e_i \neq 0$ for at least one i; then Y is a $2-$special effect variety for $\mathcal{L}_{(d_1,d_2)}(2^h)$, with $d_1 \cdot d_2 \neq 0$, in the following cases*

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$	(d_1, d_2)	(e_1, e_2)	h
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2e_2)$	$(1, e_2)$	$2e_2 + 1$
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2e_1, 2)$	$(e_1, 1)$	$2e_1 + 1$
$\mathbb{P}^1 \times \mathbb{P}^2$	$(2e_1, 2)$	$(e_1, 1)$	$3e_1 + 2$
$\mathbb{P}^1 \times \mathbb{P}^{n_2}$	$(2, 2)$	$(1, 1)$	$m_1(n_2) < h \leq M_1(n_2)$
$\mathbb{P}^2 \times \mathbb{P}^{n_2}$	$(2, 2)$	$(1, 1)$	$m_2(n_2) < h \leq M_2(n_2)$
$\mathbb{P}^3 \times \mathbb{P}^3$	$(2, 2)$	$(1, 1)$	15
$\mathbb{P}^3 \times \mathbb{P}^4$	$(2, 2)$	$(1, 1)$	19

where

$$m_1(n_2) := \frac{3n_2^2 + 9n_2 + 4}{2(n_2 + 2)} \qquad M_1(n_2) := 2n_2 + 1$$

$$m_2(n_2) := \frac{3n_2^2 + 9n_2 + 5}{n_2 + 3} \qquad M_2(n_2) := 3n_2 + 2.$$

Proposition 6.25. *Let $t \geq 3$. Let $\mathcal{L} := \mathcal{L}_{(d_1,\dots d_t)}(2^h)$ be a linear system of multidegree $(d_1, \dots, d_t)$, with $d_i \neq 0$ for $i = 1, \dots, t$, on $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ passing through h double points in general position and let Y be a divisor of multidegree $(e_1, \dots, e_t)$ on X with $e_i \neq 0$ for $i = 1, \dots, t$. Then Y is a $2-$special effect variety on X for $\mathcal{L}$ only if $t = 3$ and for the following values:*

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$	(d_1, d_2, d_3)	(e_1, e_2, e_3)	h
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2, 2)$	$(1, 1, 1)$	7
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	$(2, 2, 2)$	$(1, 1, 1)$	11
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	$(2, 2, 2)$	$(1, 1, 1)$	15

We conclude with a short list of interesting special effect varieties.

- A curve of type $(n,1)$ [resp. of type $(1,n)$] on a quadric $Q \subset \mathbb{P}^3$ is a $2-$special effect variety on Q for the system $\mathcal{L}(2n,2)(2^{2n+1})$ [resp. for $\mathcal{L}(2,2n)(2^{2n+1})$];
- the line in $\mathbb{P}^3$ is a $2-$special effect curve for $\mathcal{L}_{3,2}(2^2)$;
- the conic in $\mathbb{P}^3$ is a $2-$special effect curve for $\mathcal{L}_{3,2}(2^3)$;
- the union of the $\binom{n+1}{2}$ lines passing through the coordinate points in $\mathbb{P}^n$, $n \geq 3$, is an $(n-1)-$special effect variety for $\mathcal{L}_{n,n+1}(n^{n+1})$;
- the quadric $Q \subset \mathbb{P}^3$ is both a $1-$special effect variety and an h^1-special effect variety for the Laface–Ugaglia example (see [39]).

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C. Bocci:
Department of Mathematics
University of Milan
Via Saldini, 50
I-20133 Milano, Italy.
E-mail: cristiano.bocci@unimi.it

R. Miranda:
Department of Mathematics
Colorado State University
Fort Collins, CO 80523 USA.
E-mail: Rick.Miranda@Colostate.Edu