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Nagata's Conjecture for a Square or Nearly-Square Number of Points

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Abstract In this paper we prove that a linear system of plane curves of degree d with n points of multiplicity m always has the expected dimension when n is a perfect square. We also prove some results on the dimension of the linear system when n is close to a perfect square.

Keywords interpolation, Nagata, linear systems, plane curves

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1 Introduction

We will denote by $\mathcal{L}_d(m^n)$ the linear system of plane curves of degree d having n general singular points of multiplicity m. The *virtual dimension* v of this system is

$$v = d(d+3)/2 - nm(m+1)/2$$

and the *expected dimension* is $e = \max\{-1, v\}$, which is a lower bound for the dimension of $\mathcal{L}_d(m^n)$. A famous conjecture by Harbourne and Hirschowitz implies that $\mathcal{L}_d(m^n)$ has the expected dimension, for all degrees d and multiplicities m, as soon as $n > 9$ (see [5], [6] and [2]). The Harbourne–Hirschowitz

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conjecture is equivalent to an earlier one by B. Segre (see [10], [8] and [3]). Another equally famous conjecture by Nagata asserts that $\mathcal{L}_d(m^n)$ is empty if $d < m\sqrt{n}$ and $n > 9$; Nagata proved this conjecture when the number of points n is a perfect square (see [7]). Note that the Segre–Harbourne–Hirschowitz conjecture implies Nagata's.

It is the purpose of this article to give a proof of Segre–Harbourne–Hirschowitz for $\mathcal{L}_d(m^n)$ for all degrees d and multiplicities m when the number of points n is a perfect square, i.e. we prove that $\mathcal{L}_d(m^n)$ has the expected dimension in this case. Our proof is based on the degeneration techniques introduced in [1] and further developed in [2]. A proof of the same result has been already given in [4] and, more recently, in [9], also using degeneration arguments. We believe however the proof in the present paper is quite straightforward, relying on a direct induction. Moreover, the technique allows us to prove results for a number of points close to a perfect square. We outline this in the final section.

2 The degeneration of the linear system

We will briefly recapitulate the degeneration of the linear system that will prove useful in the argument to follow (see [1]). Set $n = k^2$, and fix integers d and m. We note that we may assume that $k \geq 4$, since the conjecture is known for 1, 4, and 9 points.

The degeneration of the plane is obtained by taking the product $\mathbb{P}^2 \times \Delta$ and blowing up a line R in the central fiber $\mathbb{P}^2 \times \{0\}$ to obtain a threefold $\mathcal{X}$ whose central fiber is a union of two surfaces: $\mathbb{P}$, the proper transform of the original $\mathbb{P}^2$ (which is isomorphic to $\mathbb{P}^2$) and $\mathbb{F}$, the exceptional divisor for the blowup, which is a minimal ruled surface over R, isomorphic to the Hirzebruch surface $\mathbb{F}_1$. The negative section of the ruling of $\mathbb{F}$ has self-intersection -1, and is the curve R, which is still a line in the plane $\mathbb{P}$.

The natural map $\pi : \mathcal{X} \to \Delta$ induced by the second projection has as its general fiber a projective plane, but the special fiber is the union of the two surfaces $\mathbb{P}$ and $\mathbb{F}$, meeting transversally along the rational curve R. This is the degeneration of the plane.

The $n = k^2$ general points in the general fiber will degenerate to n points in the special fiber, some general on the surface $\mathbb{P}$ and some general on the surface $\mathbb{F}$. Any combination is possible; we choose a degeneration so that exactly $(k-1)^2$ points go to the plane $\mathbb{P}$, and $2k-1$ of the points go to the ruled surface $\mathbb{F}$.

Finally we must degenerate the linear system, i.e., we must degenerate the sheaf $\mathcal{O}(d)$ on the general fiber of π. Before proceeding we note that we may assume that $d = km + s$ for some s with $0 \leq s \leq k-3$. Indeed, if $d = km$, then

$$v = km(km+3)/2 - k^2m(m+1)/2 = 3km/2 - k^2m/2 = (3-k)km/2$$

which is negative since $k \geq 4$. Hence if we show this system is empty, then it will be empty for all smaller degrees d. Similarly, if $d = km + k - 3$, then a simple computation shows that

$$v = km(k-3)/2 + k(k-3)/2$$

which is certainly positive since $k \geq 4$. Hence if we show that this system has the expected dimension, we will have shown that the corresponding sheaf has no H^1. Then (by using the restriction sequence to a general line in the plane) we see that all systems with higher degree also have no H^1, and therefore have they will also have the expected (positive) dimension.

We degenerate the sheaf $\mathcal{O}(d)$ on the general fiber as follows. Denote by $p_2 : \mathcal{X} \to \mathbb{P}^2$ the map induced by the first projection. Consider the sheaf $\mathcal{L} = p_2^*(\mathcal{O}(d)) \otimes \mathcal{O}_{\mathcal{X}}(m\mathbb{P})$ on the threefold $\mathcal{X}$. On the general fiber, $\mathcal{L}$ is isomorphic to $\mathcal{O}(d)$ as desired. On the central fiber, $\mathcal{L}$ restricts to the sheaf $\mathcal{O}_{\mathbb{P}}(d-m) = \mathcal{O}_{\mathbb{P}}((k-1)m+s)$ on the plane $\mathbb{P}$. On the ruled surface $\mathbb{F}$, $\mathcal{L}$ restricts to the bundle $\mathcal{O}_{\mathbb{F}}(mR+dF) = \mathcal{O}_{\mathbb{F}}(mR+(km+s)F)$, where F is the class of the fiber of the ruling of $\mathbb{F}$ and R is the negative section.

After imposing the multiple points on $\mathbb{P}$ and on $\mathbb{F}$, we obtain linear systems $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ on the two surfaces respectively. The limit system (at the sheaf level) is the fiber product system consisting of sections of each bundle which agree on the curve R.

3 The analysis of the limit system

Next we compute the dimension ℓ_0 of the limit linear system. For this we will apply Proposition 3.3 of [1], which expresses ℓ_0 in terms of the dimensions of the systems on the two surfaces $\mathbb{P}$ and $\mathbb{F}$, and the auxiliary kernel systems (of curves which vanish on the curve R).

To recall this Proposition, we introduce the following notation:

$\ell_{\mathbb{P}}$ = the dimension of the system $\mathcal{L}_{\mathbb{P}}$ on the plane $\mathbb{P}$

$\ell_{\mathbb{F}}$ = the dimension of the system $\mathcal{L}_{\mathbb{F}}$ on the ruled surface $\mathbb{F}$

$\hat{\ell}_{\mathbb{P}}$ = the dimension of the kernel system $\hat{\mathcal{L}}_{\mathbb{P}}$
consisting of curves in $\mathcal{L}_{\mathbb{P}}$ containing the curve R as a component

$\hat{\ell}_{\mathbb{F}}$ = the dimension of the kernel system $\hat{\mathcal{L}}_{\mathbb{F}}$
consisting of curves in $\mathcal{L}_{\mathbb{F}}$ containing the curve R as a component

$r_{\mathbb{P}}$ = the dimension of the system on the curve R
obtained by restricting the system $\mathcal{L}_{\mathbb{P}}$ to R

$r_{\mathbb{F}}$ = the dimension of the system on the curve R
obtained by restricting the system $\mathcal{L}_{\mathbb{F}}$ to R

We note that

$$r_{\mathbb{P}} = \ell_{\mathbb{P}} - \hat{\ell}_{\mathbb{P}} - 1 \quad \text{and} \quad r_{\mathbb{F}} = \ell_{\mathbb{F}} - \hat{\ell}_{\mathbb{F}} - 1.$$

The relevant result from [1] (Proposition 3.3 there) can now be stated:

Proposition 1 *With the above notation, the dimension ℓ_0 of the limit linear system is*

$$\ell_0 = \begin{cases} \hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}} + 1 & \text{if} \quad r_{\mathbb{P}} + r_{\mathbb{F}} \leq d - m - 1 \\ \ell_{\mathbb{P}} + \ell_{\mathbb{F}} - d + m & \text{if} \quad r_{\mathbb{P}} + r_{\mathbb{F}} \geq d - m - 1 \end{cases}$$

Remark 1 In order to prove that the original system $\mathcal{L}_d(m^n)$ has the expected dimension e it suffices to prove that $\ell_0 = e$. The assertion then follows by semi–continuity. In particular, if $\ell_0 = -1$, then $\mathcal{L}_d(m^n)$ is empty, and therefore it has the expected dimension.

Let us first analyze the system on the ruled surface $\mathbb{F}$, which, as a Hirzebruch surface $\mathbb{F}_1$, is also isomorphic to a plane blown up at one point (with exceptional curve R). If we denote by H the class of the pullback of a line in that plane, then $H \equiv R + F$ and the line bundle $\mathcal{L}_{\mathbb{F}}$ is $\mathcal{O}_{\mathbb{F}}((km + s)H - ((k-1)m + s)R)$ which corresponds to the linear system of plane curves of degree $d = km + s$ with a single point of multiplicity $d - m = (k-1)m + s$. We now impose the further general $2k - 1$ points of multiplicity m to this system, and obtain the system which on the plane would be denoted by $\mathcal{L}_{\mathbb{F}} = \mathcal{L}_d(d - m, m^{2k-1})$.

Lemma 1 *For all $k \geq 1$, the linear system on $\mathcal{L}_{\mathbb{F}}$ has the expected (positive) dimension*

$$\ell_{\mathbb{F}} = ms + m + s.$$

Proof: This goes by induction on k. For $k = 1$, we have the system $\mathcal{L}_{m+s}(s, m)$ of plane curves of degree $m+s$ with two multiple points, one of multiplicity s and one of multiplicity m. This has the expected dimension. For higher k, we can perform a Cremona transformation centered at the point of multipliciy $d - m$ and two of the points of multiplicity m. This results in a system of degree $d - m$, with one point of multiplicity $d - 2m$, and $2k - 3$ points of multiplicity m. This is the same system as before, with k reduced by one; hence by induction it has the expected dimension. Indeed, all of these systems have dimension equal to the dimension of the $k = 1$ case: after $k-1$ Cremona transformations they all Cremona reduce to $\mathcal{L}_{m+s}(s, m)$.

Q.E.D.

As noted above, the analysis of the fiber product system requires an understanding of the restriction of the systems to the curve R, and this boils down to understanding the kernel systems (formed by those elements which are identically zero on R). For $\mathcal{L}_{\mathbb{F}}$, this is the system which has one more order of vanishing on R, and hence (considered as a system on the blown-down plane) corresponds to the system $\hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}_d(d - m + 1, m^{2k-1})$.

Lemma 2 *For all $k \geq 1$, the kernel system $\hat{\mathcal{L}}_{\mathbb{F}}$ is empty, i.e., its dimension is $\hat{\ell}_{\mathbb{F}} = -1$.*

Proof: For this system, the $2k - 1$ lines joining the point of multiplicity $d - m + 1$ to the points of multiplicity m split off; the residual system is

$\mathcal{L}_{d-2k+1}(d-m+1-2k+1,(m-1)^{2k-1})$. Again we perform $k-1$ Cremona transformations as above; each one reduces the degree and the multiplicity of the first point by $m-1$, and eliminates two of the points of multiplicity $m-1$. The result then is the system $\mathcal{L}_{m+s-k}(s-k+1,m-1)$ of plane curves of degree $m+s-k$, having one point of multiplicity $s-k+1$ and another of multiplicity $m-1$. Since $s \le k-3$, this system is empty.

Q.E.D.

Therefore the linear system induced by $\mathcal{L}_{\mathbb{F}}$ on the curve R by restriction has the same dimension of $\mathcal{L}_{\mathbb{F}}$, i.e. we have

$$r_{\mathbb{F}} = ms+m+s$$

by Lemma (1).

We may now turn our attention to the system $\mathcal{L}_{\mathbb{P}} = \mathcal{L}_{d-m}(m^{(k-1)^2})$ on the plane $\mathbb{P}$, which is the system of plane curves of degree $d-m = (k-1)m+s$ having $(k-1)^2$ general points of multiplicity m. The kernel system $\hat{\mathcal{L}}_{\mathbb{P}}$, corresponding to sections of $\mathcal{L}_{\mathbb{P}}$ which vanish identically on the curve R (which the reader will recall is a line in the plane $\mathbb{P}$), is therefore $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}_{d-m-1}(m^{(k-1)^2})$, the system of plane curves as in $\mathcal{L}_{\mathbb{P}}$ having degree one less.

We now invoke the induction and conclude:

Lemma 3 *Both $\mathcal{L}_{\mathbb{P}}$ and $\hat{\mathcal{L}}_{\mathbb{P}}$ have the expected dimension.*

If we denote these dimensions by $\ell_{\mathbb{P}}$ and $\hat{\ell}_{\mathbb{P}}$ respectively, we see that

$$\ell_{\mathbb{P}} = \max\{-1,(d-m)(d-m+3)/2-(k-1)^2m(m+1)/2\}$$

and

$$\hat{\ell}_{\mathbb{P}} = \max\{-1,(d-m-1)(d-m+2)/2-(k-1)^2m(m+1)/2.\}$$

To finish the argument, we find it convenient to analyze separately several cases.

Case 1: $\mathcal{L}_{\mathbb{P}}$ is empty.
In this case the fiber product analysis says that the sections of $\mathcal{L}_{\mathbb{P}}$ vanish identically, so that a matching section of $\mathcal{L}_{\mathbb{F}}$ must vanish on the curve R, and therefore must actually lie in the kernel system $\hat{\mathcal{L}}_{\mathbb{F}}$. By Lemma 2, this system is also empty, and so the limit system is empty. Therefore by semicontinuity the general system is also empty, hence it has the expected dimension (see Remark 1).

In what follows we will assume that $\mathcal{L}_{\mathbb{P}}$ is not empty, and therefore by Lemma (3),

$$\ell_{\mathbb{P}} = (d-m)(d-m+3)/2-(k-1)^2m(m+1)/2.$$

In this case the reader can readily check that

$$\begin{aligned}\ell_{\mathbb{P}}+\ell_{\mathbb{F}}-d+m &= [(d-m)(d-m+3)/2-(k-1)^2m(m+1)/2] \\ &\quad +[ms+m+s]-d+m \\ &= d(d+3)/2-k^2m(m+1)/2 = v, \qquad (3.1)\end{aligned}$$

which is the virtual dimension of the original linear system. This remark is useful below, in applying the second case of Proposition (1).

Case 2: $\mathcal{L}_{\mathbb{P}}$ is not empty, but $\hat{\mathcal{L}_{\mathbb{P}}}$ is empty.
In this case $r_{\mathbb{P}} = \ell_{\mathbb{P}}$, and therefore

$$\begin{aligned} r_{\mathbb{P}} + r_{\mathbb{F}} &= [(d-m)(d-m+3)/2 - (k-1)^2 m(m+1)/2] + [ms+m+s] \\ &= v + d - m. \end{aligned}$$

If $v \leq -1$, then $r_{\mathbb{P}} + r_{\mathbb{F}} \leq d-m-1$. Then Proposition (1) implies that $\ell_0 = -1$ which, as above, which finishes the proof. If $v \geq -1$, then $r_{\mathbb{P}} + r_{\mathbb{F}} \geq d-m-1$, and Proposition (1) implies that $\ell_0 = v = e$, by (3.1). This finishes the proof in this case, by Remark 1.

Case 3: $\mathcal{L}_{\mathbb{P}}$ is not empty, and $\hat{\mathcal{L}_{\mathbb{P}}}$ is not empty.
In this case we have

$$\begin{aligned} r_{\mathbb{P}} &= \ell_{\mathbb{P}} - \hat{\ell_{\mathbb{P}}} - 1 \\ &= [(d-m)(d-m+3)/2 - (k-1)^2 m(m+1)/2] \\ &\quad - [(d-m-1)(d-m+2)/2 - (k-1)^2 m(m+1)/2] - 1 \\ &= d - m \end{aligned}$$

so that the second case of Proposition (1) again applies, and we have $\ell_0 = \ell_{\mathbb{P}} + \ell_{\mathbb{F}} - d + m = v$ by (3.1) as above.

This completes the proof of Segre–Harbourne–Hirschowitz conjecture for $\mathcal{L}_d(m^n)$, when n is a perfect square:

Theorem 1 *For any d, m, and k, the linear system of plane curves of degree d having k^2 general multiple points of multiplicity m has the expected dimension.*

4 Results for a nearly square number of points

The techniques used above can be applied in case the number of points is close to a perfect square. Let us illustrate the required variation by considering the linear system $\mathcal{L}_d(m^{k^2-h})$ of curves of degree d with k^2-h points of multiplicity m, with h a positive integer. We consider the same type of degeneration, putting $(k-1)^2-h$ points on the plane $\mathbb{P}$, and $2k-1$ points on the ruled surface $\mathbb{F}$; pulling back $\mathcal{O}(d)$ and twisting by $\mathcal{O}(m\mathbb{P})$, the relevant systems are:

$$\begin{aligned} \hat{\mathcal{L}_{\mathbb{F}}} &= \mathcal{L}_d(d-m+1, m^{2k-1}) \\ \mathcal{L}_{\mathbb{F}} &= \mathcal{L}_d(d-m, m^{2k-1}) \\ \mathcal{L}_{\mathbb{P}} &= \mathcal{L}_{d-m}(m^{(k-1)^2-h}) \\ \hat{\mathcal{L}_{\mathbb{P}}} &= \mathcal{L}_{d-m-1}(m^{(k-1)^2-h}) \end{aligned}$$

Again we write $s = d - km$, so that $d = km + s$.

We note that the systems on $\mathbb{F}$ are the same as those considered in the previous section, and therefore the same analysis gives the following:

Lemma 4 *If $s \leq k-3$, then $\hat{\ell_{\mathbb{F}}} = -1$; if $s \geq -1$, then $\ell_{\mathbb{F}} = ms + m + s$; if $s \leq -1$, then $\ell_{\mathbb{F}} = -1$. Therefore if $-1 \leq s \leq k-3$, we have $r_{\mathbb{F}} = ms+m+s$; If $s \leq -1$, then $\ell_{\mathbb{F}} = r_{\mathbb{F}} = -1$.*

We will prove the following result.

Theorem 2 *Suppose that the number of points is $n = k^2 - h$, with $k \geq 3$, and $1 \leq h \leq 9$. Then the system $\mathcal{L}_d(m^n)$ is empty if $d < km$ and moreover*

$$d < f_h(k, m)$$

where:

- $f_1(k,m) = (k - \frac{3}{17})m + k - 3$;
- $f_2(k,m) = (k - \frac{3}{8})m + k - 3$;
- $f_3(k,m) = (k - \frac{3}{5})m + k - 3$;
- $f_h(k,m) = (k - 1)m + k - 3$ *if* $h = 4, 5$;
- $f_6(k,m) = (k - \frac{3}{2})m + k - 3$;
- $f_h(k,m) = (k - 2)m + k - 3$ *if* $h = 7, 8$;
- $f_9(k,m) = (k - 3)(m + 1)$.

Proof: We go by induction on k. For $k = 3$ we have $0 \leq n \leq 8$ and $\mathcal{L}_d(m^n)$ is well known to be empty under the hypothesis $d < f_h(3, m)$ (see [2], Theorem 2.4).

Suppose the theorem is true for $k - 1$; let us show it for k. Since $d < km$, in the degeneration described above, the system $\mathcal{L}_{\mathbb{F}}$ is empty. Therefore to prove the theorem, it is sufficient to show that the kernel system $\hat{\mathcal{L}}_{\mathbb{P}}$ on $\mathbb{P}$ is empty. This is the linear system of plane curves of degree $d - m - 1$, with $(k-1)^2 - h$ general base points of multiplicity m. By induction, this system will be empty if

$$d - m - 1 < f_h(k-1, m), \quad \text{and} \quad d - m - 1 < (k-1)m.$$

For the second inequality, since $d < km$, we have $d - m - 1 < km - m - 1 < (k-1)m$, as required. Similarly, the first inequality follows from the hypothesis $d < f_h(k, m)$, since $f_h(k, m) - m - 1 = f_h(k-1, m)$.

Q.E.D.

Remark 2 Starting the induction with $k = 4, 5$, one can prove a similar result for $k^2 - h$ points, with $10 \leq h \leq 25$. The reader can easily figure out the statement, and therefore we will not dwell on this here.

It would be good to have a result similar to Theorem 2, to the effect that if d is large enough with respect to m and k, then the system $\mathcal{L}_d(m^n)$ is not empty, with the expected dimension, if $n = k^2 - h$, with $k \geq 3$. Unfortunately, the estimate on d one can get with the above techniques is $d \geq k(m+1)$, which is not really satisfactory. In fact, according to Theorem 1, if $d \geq k(m+1)$ then $\mathcal{L}_d(m^{k^2})$ is not empty, with the expected dimension, so of course the same is true for $\mathcal{L}_d(m^n)$.

The case of $n = k^2 + h$, with h a positive integer, can be analysed in a similar way. One considers the same type of degeneration, putting $(k-1)^2+h$ points on the plane $\mathbb{P}$, and $2k-1$ points on the ruled surface $\mathbb{F}$; pulling back $\mathcal{O}(d)$ and twisting by $\mathcal{O}(m\mathbb{P})$, the relevant systems are:

$$\begin{aligned}
\hat{\mathcal{L}}_{\mathbb{F}} &= \mathcal{L}_d(d-m+1, m^{2k-1}) \\
\mathcal{L}_{\mathbb{F}} &= \mathcal{L}_d(d-m, m^{2k-1}) \\
\mathcal{L}_{\mathbb{P}} &= \mathcal{L}_{d-m}(m^{(k-1)^2+h}) \\
\hat{\mathcal{L}}_{\mathbb{P}} &= \mathcal{L}_{d-m-1}(m^{(k-1)^2+h})
\end{aligned}$$

The result is now:

Theorem 3 *Suppose that the number of points is $n = k^2+h$, with $k \geq 1$, and $1 \leq h \leq 8$. Then the system $\mathcal{L}_d(m^n)$ is not empty of the expected dimension if*

$$d > f_h(k, m)$$

where:

- $f_h(k, m) = (k+1)m + k - 3$ *if* $h = 1, 2$;
- $f_3(k, m) = (k+1)m + k - 2$;
- $f_h(k, m) = (k + \frac{3}{2})m + k - 2$ *if* $h = 4, 5$;
- $f_6(k, m) = (k + \frac{5}{3})m + k - \frac{5}{3}$;
- $f_7(k, m) = (k + \frac{11}{6})m + k - \frac{8}{6}$;
- $f_8(k, m) = (k+2)m + k - 2$.

Proof: We go by induction on k. For $k = 1$, the result follows by [2], Theorem 2.4.

Suppose the theorem is true for $k-1$; let us prove it for k. Since $d > km$, the system $\mathcal{L}_{\mathbb{F}}$ is non empty, of the expected dimension, by Lemma 1. In order to prove the theorem, it suffices to prove that the kernel system $\hat{\mathcal{L}}_{\mathbb{P}}$ is not empty, of the expected dimension (see [1], Corollary 3.4, (b)). This is the linear system of plane curves of degree $d-m-1$, with $(k-1)^2+h$ general base points of multiplicity m. By induction, this system is non-empty, and has the expected dimension, since $d > f_h(k, m)$ implies $d-m-1 > f_h(k, m)-m-1 = f_h(k-1, m)$.

Q.E.D.

Remark 3 It would be nice in this case of $n > k^2$ to have a result such as Theorem 3, saying that if d is small enough with respect to m and k, then the system $\mathcal{L}_d(m^n)$ is empty. The estimate on d one can get with the above techniques is $d < km$, which is not satisfactory. Indeed, by Theorem 1, if $d < km$ then $\mathcal{L}_d(m^{k^2})$ is empty so the same happens for $\mathcal{L}_d(m^n)$.

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