ON THE K^2 OF DEGENERATIONS OF SURFACES AND THE MULTIPLE POINT FORMULA

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Abstract. In this paper we study some properties of reducible surfaces, in particular of unions of planes. When the surface is the central fibre of an embedded flat degeneration of surfaces in a projective space, we deduce some properties of the smooth surface which is the general fibre of the degeneration from some combinatorial properties of the central fibre. In particular, we show that there are strong constraints on the invariants of a smooth surface which degenerates to configurations of planes with global normal crossings or other mild singularities.

Our interest in these problems has been raised by a series of interesting articles by Guido Zappa in the 1950's.

1. Introduction

In this paper we study in detail several properties of flat degenerations of surfaces whose general fibre is a smooth projective algebraic surface and whose central fibre is a reduced, connected surface $X \subset \mathbb{P}^r$, $r \geqslant 3$, which will be usually assumed to be a *union of planes*.

As a first application of this approach, we shall see that there are strong constraints on the invariants of a smooth projective surface which degenerates to configurations of planes with global normal crossings or other mild singularities (cf. § 8).

Our results include formulas on the basic invariants of smoothable surfaces, especially the K^2 (see e.g. Theorem 6.1).

These formulas are useful in studying a wide range of open problems, such as what happens in the curve case, where one considers *stick curves*, i.e. unions of lines with only nodes as singularities. Indeed, as stick curves are used to study moduli spaces of smooth curves and are strictly related to fundamental problems as the *Zeuthen problem* (cf. [23] and [40]), degenerations of surfaces to unions of planes naturally arise in several important instances, like toric geometry (cf. e.g. [4], [18] and [29]) and the study of the behaviour of components of moduli spaces of smooth surfaces and their compactifications. For example, see the recent paper [30], where the abelian surface case is considered, or several papers related to the $K3$ surface case (see, e.g. [9], [10] and [16]).

Using the techniques developed here, we are able to prove a Miyaoka-Yau type inequality (see Theorem 8.4 and Proposition 8.16).

In general, we expect that degenerations of surfaces to unions of planes will find many applications. These include the systematic classification of surfaces with low invariants (p_g and K^2), and especially a classification of possible boundary components to moduli spaces.

It is an open problem to understand when a family of surfaces may degenerate to a union of planes, and in some sense this is one of the most interesting questions in the subject. The techniques we develop here in some cases allow us to conclude that this is not possible. When it is possible, we obtain restrictions on the invariants which may lead to further theorems on classification, for example, the problem of bounding the irregularity of surfaces in $\mathbb{P}^4$.

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Other applications include the possibility of performing braid monodromy computations (see [11], [32], [33], [41]). We hope that future work will include an analysis of higher-dimensional analogues to the constructions and computations, leading for example to interesting degenerations of Calabi-Yau manifolds.

Our interest in degenerations to union of planes has been stimulated by a series of papers by Guido Zappa that appeared in the 1940–50's regarding in particular: (1) degenerations of scrolls to unions of planes and (2) the computation of bounds for the topological invariants of an arbitrary smooth projective surface which degenerates to a union of planes (see [44, 45, 46, 47, 48, 49, 50]).

In this paper we shall consider a reduced, connected, projective surface X which is a union of planes — or more generally a union of smooth surfaces — whose singularities are:

- in codimension one, double curves which are smooth and irreducible along which two surfaces meet transversally;
- multiple points, which are locally analytically isomorphic to the vertex of a cone over a stick curve with arithmetic genus either zero or one and which is projectively normal in the projective space it spans.

These multiple points will be called *Zappatic singularities* and X will be called a *Zappatic surface*. If moreover $X \subset \mathbb{P}^r$, for some positive r, and if all its irreducible components are planes, then X is called a *planar Zappatic surface*.

We will mainly concentrate on the so called *good Zappatic surfaces*, i.e. Zappatic surfaces having only Zappatic singularities whose associated stick curve has one of the following dual graphs (cf. Examples 2.6 and 2.7, Definition 3.5, Figures 3 and 5):

R_n: a chain of length n, with $n \geqslant 3$;
S_n: a fork with $n-1$ teeth, with $n \geqslant 4$;
E_n: a cycle of order n, with $n \geqslant 3$.

Let us call R_n-, S_n-, E_n-*point* the corresponding multiple point of the Zappatic surface X.

We first study some combinatorial properties of a Zappatic surface X (cf. § 3). We then focus on the case in which X is the central fibre of an embedded flat degeneration $\mathcal{X} \to \Delta$, where Δ is the complex unit disk and where $\mathcal{X} \subseteq \Delta \times \mathbb{P}^r$, $r \geqslant 3$, is a closed subscheme of relative dimension two. In this case, we deduce some properties of the general fibre $\mathcal{X}_t$, $t \neq 0$, of the degeneration from the aforementioned properties of the central fibre $\mathcal{X}_0 = X$ (see §'s 4, 6, 7 and 8).

A first instance of this approach can be found in [6], where we gave some partial results on the computation of $h^0(X, \omega_X)$, when X is a Zappatic surface with global normal crossings, i.e. with only E_3-points, and where ω_X is its dualizing sheaf (see Theorem 4.15 in [6]). In the particular case in which X is smoothable, namely if X is the central fibre of a flat degeneration, one can prove that $h^0(X, \omega_X)$ equals the geometric genus of the general fibre and the formula for this can be also deduced from the well-known Clemens-Schmid exact sequence (cf. e.g. [8] and [34]).

In this paper we address two main problems.

We will first compute the K^2 of a smooth surface which degenerates to a good Zappatic surface, i.e. we will compute $K^2_{\mathcal{X}_t}$, where $\mathcal{X}_t$ is the general fibre of a degeneration $\mathcal{X} \to \Delta$ such that the central fibre $\mathcal{X}_0$ is a good Zappatic surface (see § 6).

We will then prove a basic inequality, called the *Multiple Point Formula* (cf. Theorem 7.2), which can be viewed as a generalization, for good Zappatic singularities, of the well-known Triple Point Formula (see Lemma 7.7 and cf. [15]).

Both results follow from a detailed analysis of local properties of the total space $\mathcal{X}$ of the degeneration at a good Zappatic singularity of the central fibre X.

We apply the computation of K^2 and the Multiple Point Formula to prove several results concerning degenerations of surfaces. Precisely, if χ and g denote, respectively, the Euler-Poincaré characteristic and the sectional genus of the general fibre $\mathcal{X}_t$, for $t \in \Delta \setminus \{0\}$, then:

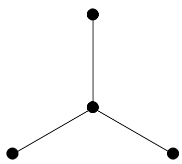
Theorem 1 (cf. Theorem 8.4). *Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration, where the central fibre $\mathcal{X}_0 = X$ has at most R_3-, E_3-, E_4- and E_5-points. Then*

$$K^2 \leqslant 8\chi + 1 - g. \tag{1.1}$$

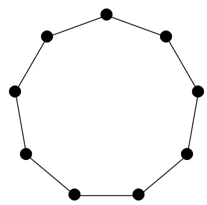
Moreover, the equality holds in (1.1) *if and only if $\mathcal{X}_t$ is either the Veronese surface in $\mathbb{P}^5$ degenerating to 4 planes with associated graph S_4 (i.e. with three R_3-points, see Figure 1.a), or an elliptic scroll of degree $n \geqslant 5$ in $\mathbb{P}^{n-1}$ degenerating to n planes with associated graph a cycle E_n (see Figure 1.b).*

Furthermore, if $\mathcal{X}_t$ is a surface of general type, then

$$K^2 < 8\chi - g.$$



(a)



(b)

FIGURE 1

In particular, we have:

Corollary (cf. Corollaries 8.10 and 8.12). *Let $\mathcal{X}$ be a good, planar Zappatic degeneration.*

(a) *Assume that $\mathcal{X}_t$, $t \in \Delta \setminus \{0\}$, is a scroll of sectional genus $g \geqslant 2$. Then $\mathcal{X}_0 = X$ has worse singularities than R_3-, E_3-, E_4- and E_5-points.*

(b) *If $\mathcal{X}_t$ is a minimal surface of general type and $\mathcal{X}_0 = X$ has at most R_3-, E_3-, E_4- and E_5-points, then*

$$g \leqslant 6\chi + 5.$$

These improve the main results of Zappa in [49].

Let us describe in more detail the contents of the paper. Section 2 contains some basic results on reducible curves and their dual graphs.

In Section 3, we give the definition of Zappatic singularities and of (planar, good) Zappatic surfaces. We associate to a good Zappatic surface X a graph G_X which encodes the configuration of the irreducible components of X as well as of its Zappatic singularities (see Definition 3.6).

In Section 4, we introduce the definition of Zappatic degeneration of surfaces and we recall some properties of smooth surfaces which degenerate to Zappatic ones.

In Section 5 we recall the notions of *minimal singularity* and *quasi-minimal singularity*, which are needed to study the singularities of the total space $\mathcal{X}$ of a degeneration of surfaces at a good Zappatic singularity of its central fibre $\mathcal{X}_0 = X$ (cf. also [26] and [27]).

Indeed, in Section 6, the local analysis of minimal and quasi-minimal singularities allows us to compute $K^2_{\mathcal{X}_t}$, for $t \in \Delta \setminus \{0\}$, when $\mathcal{X}_t$ is the general fibre of a degeneration such that the central fibre is a good Zappatic surface. More precisely, we prove the following main result (see Theorem 6.1):

Theorem 2. *Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces whose central fibre is a good Zappatic surface $X = \mathcal{X}_0 = \bigcup_{i=1}^{v} X_i$. Let $C_{ij} := X_i \cap X_j$ be a smooth (possibly reducible) curve of the double locus of X, considered as a curve on X_i, and let g_{ij} be its geometric genus, $1 \leqslant i \neq j \leqslant v$. Let v and e be the number of vertices and edges of the graph G_X associated to X. Let f_n, r_n, s_n be the number of E_n-, R_n-, S_n-points of X, respectively. If $K^2 := K^2_{\mathcal{X}_t}$, for $t \neq 0$, then:*

$$K^2 = \sum_{i=1}^{v}\left(K^2_{X_i} + \sum_{j\neq i}(4g_{ij} - C^2_{ij})\right) - 8e + \sum_{n\geqslant 3} 2nf_n + r_3 + k \tag{1.2}$$

where k depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$, and precisely:

$$\sum_{n\geqslant 4}(n-2)(r_n + s_n) \leqslant k \leqslant \sum_{n\geqslant 4}\left((2n-5)r_n + \binom{n-1}{2}s_n\right). \tag{1.3}$$

In the case that the central fibre is also planar, we have the following:

Corollary (cf. Corollary 6.4)**.** *Let $\mathcal{X} \to \Delta$ be an embedded degeneration of surfaces whose central fibre is a good, planar Zappatic surface $X = \mathcal{X}_0 = \bigcup_{i=1}^{v} \Pi_i$. Then:*

$$K^2 = 9v - 10e + \sum_{n\geqslant 3} 2nf_n + r_3 + k \tag{1.4}$$

where k is as in (1.3) *and depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$.*

The inequalities in the theorem and the corollary above reflect deep geometric properties of the degeneration. For example, if $\mathcal{X} \to \Delta$ is a degeneration with central fibre X a Zappatic surface which is the union of four planes having only a R_4-point, Theorem 2 states that $8 \leqslant K^2 \leqslant 9$. The two values of K^2 correspond to the fact that X, which is the cone over a stick curve C_{R_4} (cf. Example 2.6), can be smoothed either to the Veronese surface, which has $K^2 = 9$, or to a rational normal quartic scroll in $\mathbb{P}^5$, which has $K^2 = 8$ (cf. Remark 6.22). This in turn corresponds to different local structures of the total space of the degeneration at the R_4-point. Moreover, the local deformation space of a R_4-point is reducible.

Section 7 is devoted to the *Multiple Point Formula* (1.5) below (see Theorem 7.2):

Theorem 3. *Let X be a good Zappatic surface which is the central fibre of a good Zappatic degeneration $\mathcal{X} \to \Delta$. Let $\gamma = X_1 \cap X_2$ be the intersection of two irreducible components X_1, X_2 of X. Denote by $f_n(\gamma)$ [$r_n(\gamma)$ and $s_n(\gamma)$, respectively] the number of E_n-points [R_n-points and S_n-points, respectively] of X along γ. Denote by d_γ the number of double points of the total space $\mathcal{X}$ along γ, off the Zappatic singularities of X. Then:*

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(r_n(\gamma) + s_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{1.5}$$

In particular, if X is also planar, then:

$$2 + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(r_n(\gamma) + s_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{1.6}$$

Furthermore, if $d_{\mathcal{X}}$ denotes the total number of double points of $\mathcal{X}$, off the Zappatic singularities of X, then:

$$2e + 3f_3 - 2r_3 - \sum_{n\geqslant 4} nf_n - \sum_{n\geqslant 4}(n-1)(s_n + r_n) \geqslant d_{\mathcal{X}} \geqslant 0. \tag{1.7}$$

In § 8 we apply the above results to prove several generalizations of statements given by Zappa. For example we show that worse singularities than normal crossings are needed in order to degenerate as many surfaces as possible to unions of planes.

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2. Reducible curves and associated graphs

Let C be a projective curve and let C_i, $i = 1, \dots, n$, be its irreducible components. We will assume that:

- C is connected and reduced;
- C has at most nodes as singularities;
- the curves $C_i, i = 1, \dots, n$, are smooth.

If two components C_i, C_j, $i < j$, intersect at m_{ij} points, we will denote by P_{ij}^h, $h = 1, \dots, m_{ij}$, the corresponding nodes of C.

We can associate to this situation a simple (i.e. with no loops), weighted connected graph G_C, with vertex v_i weighted by the genus g_i of C_i:

- whose vertices $v_1, \dots, v_n$, correspond to the components $C_1, \dots, C_n$;
- whose edges η_{ij}^h, $i < j$, $h = 1, \dots, m_{ij}$, joining the vertices v_i and v_j, correspond to the nodes P_{ij}^h of C.

We will assume the graph to be *lexicographically oriented*, i.e. each edge is assumed to be oriented from the vertex with lower index to the one with higher.

We will use the following notation:

- v is the number of vertices of G_C, i.e. $v = n$;
- e is the number of edges of G_C;
- $\chi(G_C) = v - e$ is the Euler-Poincaré characteristic of G_C;
- $h^1(G_C) = 1 - \chi(G_C)$ is the first Betti number of G_C.

Notice that conversely, given any simple, connected, weighted (oriented) graph G, there is some curve C such that $G = G_C$.

One has the following basic result (cf. e.g. [2] or directly [6]) :

Theorem 2.1. *In the above situation*

$$\chi(\mathcal{O}_C) = \chi(G_C) - \sum_{i=1}^{v} g_i = v - e - \sum_{i=1}^{v} g_i. \tag{2.2}$$

We remark that formula (2.2) is equivalent to:

$$p_a(C) = h^1(G_C) + \sum_{i=1}^{v} g_i \tag{2.3}$$

(cf. Proposition 3.11.)

Notice that C is Gorenstein, i.e. the dualizing sheaf ω_C is invertible. One defines the *ω-genus* of C to be

$$p_\omega(C) := h^0(C, \omega_C). \tag{2.4}$$

Observe that, when C is smooth, the ω-genus coincides with the geometric genus of C.

In general, by the Riemann-Roch theorem, one has

$$p_\omega(C) = p_a(C) = h^1(G_C) + \sum_{i=1}^{v} g_i = e - v + 1 + \sum_{i=1}^{v} g_i. \tag{2.5}$$

If we have a flat family $\mathcal{C} \to \Delta$ over a disc Δ with general fibre $\mathcal{C}_t$ smooth and irreducible of genus g and special fibre $\mathcal{C}_0 = C$, then we can combinatorially compute g via the formula:

$$g = p_a(C) = h^1(G_C) + \sum_{i=1}^{v} g_i.$$

Often we will consider C as above embedded in a projective space $\mathbb{P}^r$. In this situation each curve C_i will have a certain degree d_i, and we will consider the graph G_C as *double weighted*, by attributing to each vertex the pair of weights (g_i, d_i). Moreover we will attribute to the graph a further marking number, i.e. r the embedding dimension of C.

The total degree of C is

$$d = \sum_{i=1}^{v} d_i$$

which is also invariant by flat degeneration.

More often we will consider the case in which each curve C_i is a line. The corresponding curve C is called a *stick curve*. In this case the double weighting is $(0,1)$ for each vertex, and it will be omitted if no confusion arises.

It should be stressed that it is not true that for any simple, connected, double weighted graph G there is a curve C in a projective space such that $G_C = G$. For example there is no stick curve corresponding to the graph of Figure 2.

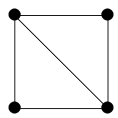
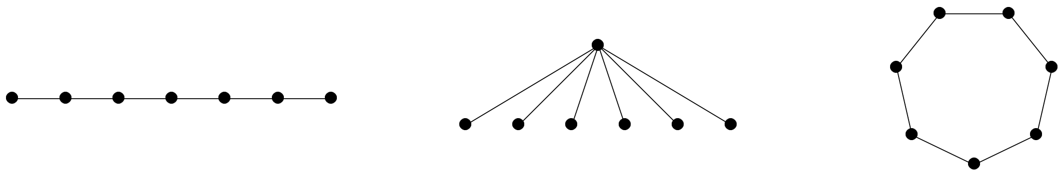


FIGURE 2. Dual graph of an "impossible" stick curve.

We now give two examples of stick curves which will be frequently used in this paper.

Example 2.6. Let T_n be any connected tree with $n \geqslant 3$ vertices. This corresponds to a non-degenerate stick curve of degree n in $\mathbb{P}^n$, which we denote by C_{T_n}. Indeed one can check that, taking a general point p_i on each component of C_{T_n}, the line bundle $\mathcal{O}_{C_{T_n}}(p_1 + \cdots + p_n)$ is very ample. Of course C_{T_n} has arithmetic genus 0 and is a flat limit of rational normal curves in $\mathbb{P}^n$.

We will often consider two particular kinds of trees T_n: a chain R_n of length n and the fork S_n with $n-1$ teeth, i.e. a tree consisting of $n-1$ vertices joining a further vertex (see Figures 3.(a) and (b)). The curve C_{R_n} is the union of n lines $l_1, l_2, \ldots, l_n$ spanning $\mathbb{P}^n$, such that $l_i \cap l_j = \emptyset$ if and only if $1 < |i-j|$. The curve C_{S_n} is the union of n lines $l_1, l_2, \ldots, l_n$ spanning $\mathbb{P}^n$, such that $l_1, \ldots, l_{n-1}$ all intersect l_n at distinct points (see Figure 4).



(a) A chain R_n (b) A fork S_n with $n-1$ teeth (c) A cycle E_n

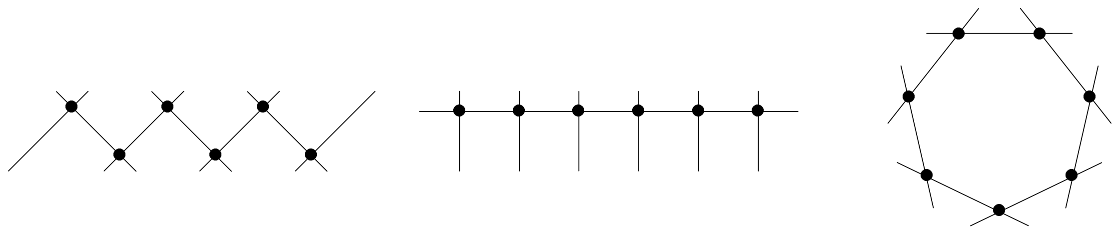
FIGURE 3. Examples of dual graphs.

Example 2.7. Let Z_n be any simple, connected graph with $n \geqslant 3$ vertices and $h^1(Z_n, \mathbb{C}) = 1$. This corresponds to an arithmetically normal stick curve of degree n in $\mathbb{P}^{n-1}$, which we denote by C_{Z_n} (as in Example 2.6). The curve C_{Z_n} has arithmetic genus 1 and it is a flat limit of elliptic normal curves in $\mathbb{P}^{n-1}$.

We will often consider the particular case of a cycle E_n of order n (see Figure 3.c). The curve C_{E_n} is the union of n lines $l_1, l_2, \ldots, l_n$ spanning $\mathbb{P}^{n-1}$, such that $l_i \cap l_j = \emptyset$ if and only if $1 < |i-j| < n-1$ (see Figure 4).

We remark that C_{E_n} is projectively Gorenstein (i.e. it is projectively Cohen-Macaulay and sub-canonical); indeed $\omega_{C_{E_n}}$ is trivial, since there is an everywhere non-zero global section of $\omega_{C_{E_n}}$, given by the meromorphic 1-form on each component with residues 1 and -1 at the nodes (in a suitable order).

All the other C_{Z_n}'s, instead, are not Gorenstein because $\omega_{C_{Z_n}}$, although of degree zero, is not trivial. Indeed a graph Z_n, different from E_n, certainly has a vertex with valence 1. This corresponds to a line l such that $\omega_{C_{Z_n}} \otimes \mathcal{O}_l$ is not trivial.



C_{R_n}: a chain of n lines, C_{S_n}: a comb with $n-1$ teeth, C_{E_n}: a cycle of n lines.

FIGURE 4. Examples of stick curves.

3. ZAPPATIC SURFACES AND ASSOCIATED GRAPHS

We will now give a parallel development, for surfaces, to the case of curves that we recalled in the previous section. Before doing this, we need to recall the singularities we will allow.

Definition 3.1 (Zappatic singularity)**.** Let X be a surface and let $x \in X$ be a point. We will say that x is a *Zappatic singularity* for X if (X, x) is locally analytically isomorphic to a pair (Y, y) where Y is the cone over either a curve C_{T_n} or a curve C_{Z_n}, $n \geqslant 3$, and y is the vertex of the cone. Accordingly we will say that x is either a T_n- or a Z_n*-point* for X.

Observe that either T_n- or Z_n-points are not classified by n, unless $n = 3$.

We will consider the following situation.

Definition 3.2 (Zappatic surface)**.** Let X be a projective surface with its irreducible components $X_1, \ldots, X_v$. We will assume that X has the following properties:

- X is reduced and connected in codimension one;
- $X_1, \ldots, X_v$ are smooth;
- the singularities in codimension one of X are at most double curves which are smooth and irreducible along which two surfaces meet transversally;
- the further singularities of X are Zappatic singularities.

A surface like X will be called a *Zappatic surface*. If moreover X is embedded in a projective space $\mathbb{P}^r$ and all of its irreducible components are planes, we will say that X is a *planar Zappatic surface*. In this case, the irreducible components of X will sometimes be denoted by Π_i instead of X_i, $1 \leqslant i \leqslant v$.

Notation 3.3. Let X be a Zappatic surface. Let us denote by:

- X_i: an irreducible component of X, $1 \leqslant i \leqslant v$;

- $C_{ij} := X_i \cap X_j$, $1 \leqslant i \neq j \leqslant v$, if X_i and X_j meet along a curve, otherwise set $C_{ij} = \emptyset$. We assume that each C_{ij} is smooth but not necessarily irreducible;
- g_{ij} : the geometric genus of C_{ij}, $1 \leqslant i \neq j \leqslant v$; i.e. g_{ij} is the sum of the geometric genera of the irreducible (equiv., connected) components of C_{ij};
- $C := \mathrm{Sing}(X) = \cup_{i<j} C_{ij}$: the union of all the double curves of X;
- $\Sigma_{ijk} := X_i \cap X_j \cap X_k$, $1 \leqslant i \neq j \neq k \leqslant v$, if $X_i \cap X_j \cap X_k \neq \emptyset$, otherwise $\Sigma_{ijk} = \emptyset$;
- m_{ijk} : the cardinality of the set Σ_{ijk};
- P^h_{ijk} : the Zappatic singular point belonging to Σ_{ijk}, for $h = 1, \ldots, m_{ijk}$.

Furthermore, if $X \subset \mathbb{P}^r$, for some r, we denote by

- $d = \deg(X)$: the degree of X;
- $d_i = \deg(X_i)$: the degree of X_i, $i \leqslant i \leqslant v$;
- $c_{ij} = \deg(C_{ij})$: the degree of C_{ij}, $1 \leqslant i \neq j \leqslant v$;
- D : a general hyperplane section of X;
- g : the arithmetic genus of D;
- D_i : the (smooth) irreducible component of D lying in X_i, which is a general hyperplane section of X_i, $1 \leqslant i \leqslant v$;
- g_i : the genus of D_i, $1 \leqslant i \leqslant v$.

Notice that if X is a planar Zappatic surface, then each C_{ij}, when not empty, is a line and each non-empty set Σ_{ijk} is a singleton.

Remark 3.4. Observe that a Zappatic surface X is Cohen-Macaulay. More precisely, X has global normal crossings except at points T_n, $n \geqslant 3$, and Z_m, $m \geqslant 4$. Thus the dualizing sheaf ω_X is well-defined. If X has only E_n-points as Zappatic singularities, then X is Gorenstein, hence ω_X is an invertible sheaf.

Definition 3.5 (Good Zappatic surface)**.** The *good Zappatic singularities* are the

- R_n-points, for $n \geqslant 3$,
- S_n-points, for $n \geqslant 4$,
- E_n-points, for $n \geqslant 3$,

which are the Zappatic singularities whose associated stick curves are respectively C_{R_n}, C_{S_n}, C_{E_n} (see Examples 2.6 and 2.7, Figures 3, 4 and 5).

A *good Zappatic surface* is a Zappatic surface with only good Zappatic singularities.

To a good Zappatic surface X we can associate an oriented complex G_X, which we will also call the *associated graph* to X.

Definition 3.6 (The associated graph to X)**.** Let X be a good Zappatic surface with Notation 3.3. The graph G_X associated to X is defined as follows (cf. Figure 6):

- each surface X_i corresponds to a vertex v_i;
- each irreducible component of the double curve $C_{ij} = C^1_{ij} \cup \ldots \cup C^{h_{ij}}_{ij}$ corresponds to an edge e^t_{ij}, $1 \leqslant t \leqslant h_{ij}$, joining v_i and v_j. The edge $e^t_{ij}, i < j$, is oriented from the vertex v_i to the one v_j. The union of all the edges e^t_{ij} joining v_i and v_j is denoted by $\tilde{e}_{ij}$, which corresponds to the (possibly reducible) double curve C_{ij};
- each E_n-point P of X is a face of the graph whose n edges correspond to the double curves concurring at P. This is called a *n-face* of the graph;
- for each R_n-point P, with $n \geqslant 3$, if $P \in X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$, where X_{i_j} meets X_{i_k} along a curve $C_{i_j i_k}$ only if $1 = |j-k|$, we add in the graph a *dashed edge* joining the vertices corresponding to X_{i_1} and X_{i_n}. The dashed edge e_{i_1,i_n}, together with the other $n-1$ edges $e_{i_j,i_{j+1}}$, $j = 1, \ldots, n-1$, bound an *open n-face* of the graph;
- for each S_n-point P, with $n \geqslant 4$, if $P \in X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$, where $X_{i_1}, \ldots, X_{i_{n-1}}$ all meet X_{i_n} along curves $C_{i_j i_n}$, $j = 1, \ldots, n-1$, concurring at P, we mark this

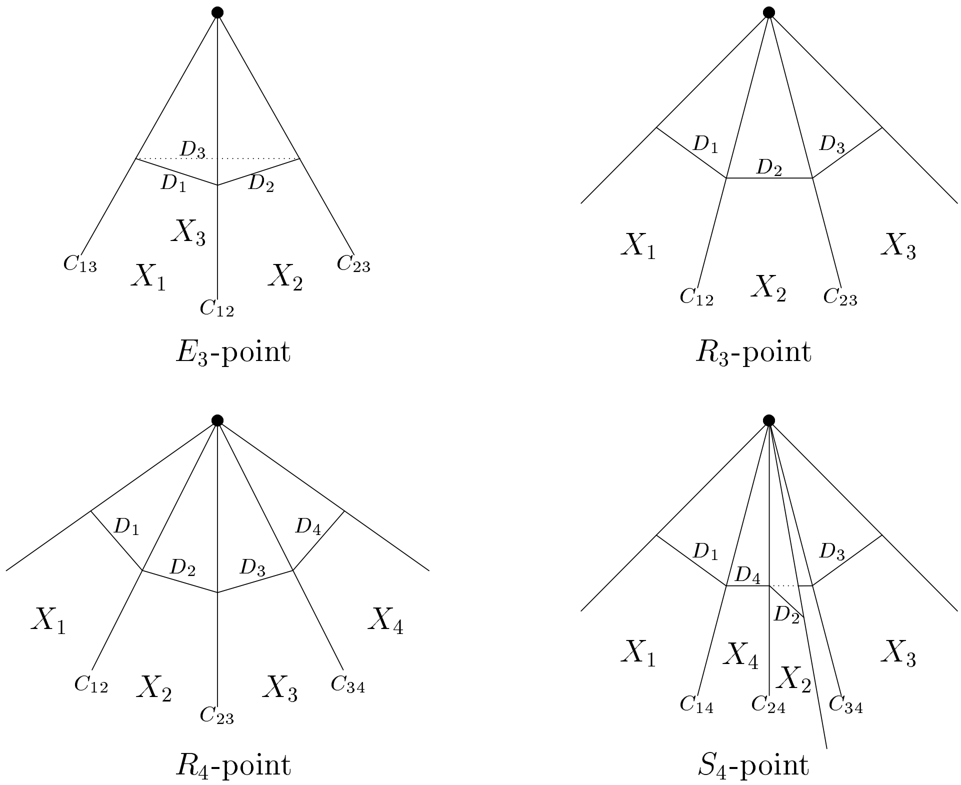


FIGURE 5. Examples of good Zappatic singularities.

in the graph by a *n-angle* spanned by the edges corresponding to the curves $C_{i_j i_n}$, $j = 1, \dots, n-1$.

In the sequel, when we speak of *faces* of G_X we always mean closed faces. Of course each vertex v_i is weighted with the relevant invariants of the corresponding surface X_i. We will usually omit these weights if X is planar, i.e. if all the X_i's are planes.

Since each R_n-, S_n-, E_n-point is an element of some set of points Σ_{ijk} (cf. Notation 3.3), we remark that there can be different faces (as well as open faces and angles) of G_X which are incident on the same set of vertices and edges. However this cannot occur if X is planar.

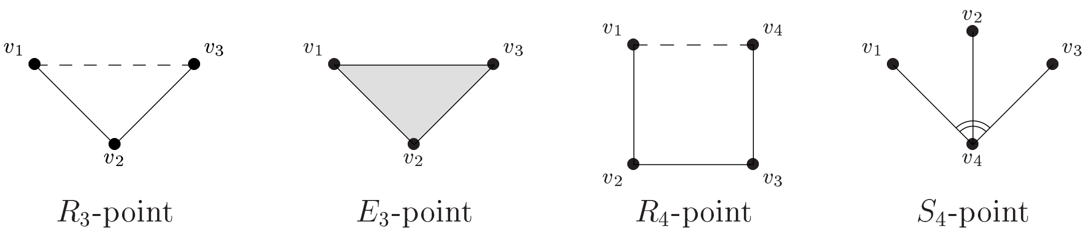


FIGURE 6. Associated graphs of R_3-, E_3-, R_4- and S_4-points (cf. Figure 5).

Consider three vertices v_i, v_j, v_k of G_X in such a way that v_i is joint with v_j and v_k. Assume for simplicity that the double curves C_{ij}, $1 \leqslant i < j \leqslant v$, are irreducible. Then, any point in $C_{ij} \cap C_{ik}$ is either a R_n-, or a S_n-, or an E_n-point, and the curves C_{ij} and C_{ik} intersect transversally, by definition of Zappatic singularities. Hence we can compute the intersection number $C_{ij} \cdot C_{ik}$ by adding the number of closed and open faces and of angles involving the edges e_{ij}, e_{ik}. In particular, if X is planar, for every pair of adjacent edges only one of the following possibilities occur: either they belong to an open face, or to a closed one, or to an angle. Therefore for good, planar Zappatic surfaces we can avoid marking open 3-faces without losing any information (see Figures 6 and 7).

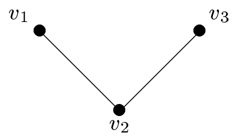


FIGURE 7. Associated graph of a R_3-point in a good, planar Zappatic surface.

As for stick curves, if G is a given graph as above, there does not necessarily exist a good planar Zappatic surface X such that its associated graph is $G = G_X$.

Example 3.7. Consider the graph G of Figure 8. If G were the associated graph of a good planar Zappatic surface X, then X should be a global normal crossing union of 4 planes with 5 double lines and two E_3 points, P_{123} and P_{134}, both lying on the double line C_{13}. Since the lines C_{23} and C_{34} (resp. C_{14} and C_{12}) both lie on the plane X_3 (resp. X_1), they should intersect. This means that the planes X_2, X_4 also should intersect along a line, therefore the edge e_{24} should appear in the graph.

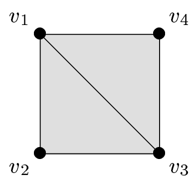


FIGURE 8. Graph associated to an impossible planar Zappatic surface.

Analogously to Example 3.7, one can easily see that, if the 1-skeleton of G is E_3 or E_4, then in order to have a planar Zappatic surface X such that $G_X = G$, the 2-skeleton of G has to consist of the face bounded by the 1-skeleton.

We can also consider an example of a good Zappatic surface with reducible double curves.

Example 3.8. Consider D_1 and D_2 two general plane curves of degree m and n, respectively. Therefore, they are smooth, irreducible and they transversally intersect each other in mn points. Consider the surfaces:

$$X_1 = D_1 \times \mathbb{P}^1 \quad \text{and} \quad X_2 = D_2 \times \mathbb{P}^1.$$

The union of these two surfaces, together with the plane $\mathbb{P}^2 = X_3$ containing the two curves, determines a good Zappatic surface X with only E_3-points as Zappatic singularities.

More precisely, by using Notation 3.3, we have:

- $C_{13} = X_1 \cap X_3 = D_1$, $C_{23} = X_2 \cap X_3 = D_2$, $C_{12} = X_1 \cap X_2 = \sum_{k=1}^{mn} F_k$, where each F_k is a fibre isomorphic to $\mathbb{P}^1$;
- $\Sigma_{123} = X_1 \cap X_2 \cap X_3$ consists of the mn points of the intersection of D_1 and D_2 in X_3.

Observe that C_{12} is smooth but not irreducible. Therefore, the graph G_X consists of 3 vertices, $mn + 2$ edges and mn triangles incident on them.

In order to combinatorially compute some of the invariants of a good Zappatic surface, we need some notation.

Notation 3.9. Let X be a good Zappatic surface (with invariants as in Notation 3.3) and let $G = G_X$ be its associated graph. We denote by

- V : the (indexed) set of vertices of G;
- v : the cardinality of V, the number of irreducible components of X;
- E : the set of edges of G; this is indexed by the ordered triples $(i, j, t) \in V \times V \times \mathbb{N}$, where $i < j$ and $1 \leqslant t \leqslant h_{ij}$, such that the corresponding surfaces X_i, X_j meet along the curve $C_{ij} = C_{ji} = C_{ij}^1 \cup \ldots \cup C_{ij}^{h_{ij}}$;
- e : the cardinality of E, i.e. the number of irreducible components of double curves in X;
- $\tilde{E}$: the set of double curves C_{ij} of X; this is indexed by the ordered pairs $(i, j) \in V \times V$, where $i < j$, such that the corresponding surfaces X_i, X_j meet along the curve $C_{ij} = C_{ji}$;
- $\tilde{e}$: the cardinality of $\tilde{E}$, i.e. the pairs of vertices of G_X which are joint by at least one edge;
- f_n : the number of n-faces of G, i.e. the number of E_n-points of X, for $n \geqslant 3$;
- $f := \sum_{n \geqslant 3} f_n$, the number of faces of G, i.e. the total number of E_n-points of X, for all $n \geqslant 3$;
- r_n : the number of open n-faces of G, i.e. the number of R_n-points of X, for $n \geqslant 3$;
- $r := \sum_{n \geqslant 3} r_n$, the total number of R_n-points of X, for all $n \geqslant 3$;
- s_n : the number of n-angles of G, i.e. the number of S_n-points of X, for $n \geqslant 4$;
- $s: = \sum_{n \geqslant 4} s_n$: the total number of S_n-points of X, for all $n \geqslant 4$;
- w_i: the valence of the i^{th} vertex v_i of G, i.e. the number of irreducible double curves lying on X_i;
- $\chi(G) := v - e + f$, i.e. the Euler-Poincaré characteristic of G;
- $G^{(1)}$: the 1-*skeleton* of G, i.e. the graph obtained from G by forgetting all the faces, dashed edges and angles;
- $\chi(G^{(1)}) = v - e$, i.e. the Euler-Poincaré characteristic of $G^{(1)}$.

Remark 3.10. Observe that, when X is a good, planar Zappatic surface, $E = \tilde{E}$ and the 1-skeleton $G_X^{(1)}$ of G_X coincides with the dual graph G_D of the general hyperplane section D of X.

As a straightforward generalization of what proved in [6], one can compute the following invariants:

Proposition 3.11. *Let $X = \bigcup_{i=1}^{v} X_i \subset \mathbb{P}^r$ be a good Zappatic surface. Let $G = G_X$ be its associated graph, whose number of faces is f. Let C be the double locus of X, i.e. the union of the double curves of X, $C_{ij} = C_{ji} = X_i \cap X_j$ and let $c_{ij} = \deg(C_{ij})$. Let D_i be a general hyperplane section of X_i, and denote by g_i its genus. Then:*
(i) the arithmetic genus of a general hyperplane section D of X is:

$$g = \sum_{i=1}^{v} g_i + \sum_{1 \leqslant i < j \leqslant v} c_{ij} - v + 1. \tag{3.12}$$

In particular, when X is a good, planar Zappatic surface, then

$$g = e - v + 1 = 1 - \chi(G^{(1)}); \tag{3.13}$$

(ii) the Euler-Poincaré characteristic of X is:

$$\chi(\mathcal{O}_X) = \sum_{i=1}^{v} \chi(\mathcal{O}_{X_i}) - \sum_{1 \leqslant i < j \leqslant v} \chi(\mathcal{O}_{C_{ij}}) + f. \tag{3.14}$$

In particular, when X is a good, planar Zappatic surface, then

$$\chi(\mathcal{O}_X) = \chi(G_X) = v - e + f. \tag{3.15}$$

Proof. For complete details when C_{ij} are irreducible, the reader is referred to [6], Propositions 3.12 and 3.15. □

Not all of the invariants of X can be directly computed by the graph G_X. For example, if ω_X denotes the dualizing sheaf of X, the computation of the $h^0(X, \omega_X)$, which plays a fundamental role in degeneration theory, is actually much more involved, even if X has mild Zappatic singularities (cf. [6] or [7]).

To conclude this section, we observe that in the particular case of good, planar Zappatic surface one can determine a simple relation among the numbers of Zappatic singularities, as the next lemma shows.

Lemma 3.16. *Let G be the associated graph to a good, planar Zappatic surface $X = \bigcup_{i=1}^{v} X_i$. Then, with Notation* (3.9)*, we have*

$$\sum_{i=1}^{v} \frac{w_i(w_i-1)}{2} = \sum_{n\geqslant 3} (nf_n + (n-2)r_n) + \sum_{n\geqslant 4} \binom{n-1}{2} s_n. \tag{3.17}$$

Proof. The dual graph of three planes which form a R_3-point consists of two adjacent edges (cf. Figure 7). The total number of two adjacent edges in G is the left hand side member of (3.17) by definition of valence w_i. On the other hand, a n-face (resp. an open n-face, resp. a n-angle) clearly contains exactly n (resp. $n-2$, resp. $\binom{n-1}{2}$) pairs of adjacent edges. □

4. Degenerations to Zappatic surfaces

In this section we will focus on flat degenerations of smooth surfaces to Zappatic ones.

Definition 4.1. Let Δ be the spectrum of a DVR (equiv. the complex unit disk). A *degeneration* of relative dimension n is a proper and flat morphism

$$\begin{array}{c} \mathfrak{X} \\ \downarrow \pi \\ \Delta \end{array}$$

such that $\mathfrak{X}_t = \pi^{-1}(t)$ is a smooth, irreducible, n-dimensional, projective variety, for $t \neq 0$.

If Y is a smooth, projective variety, the degeneration

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "\pi"'] & \subseteq & \Delta \times Y \arrow[dll, "\mathrm{pr}_1"] \\ \Delta & & \end{tikzcd}$$

is said to be an *embedded degeneration* in Y of relative dimension n. When it is clear from the context, we will omit the term *embedded*.

A degeneration is said to be *semistable* (see, e.g., [34]) if the total space $\mathfrak{X}$ is smooth and if the central fibre $\mathfrak{X}_0$ is a divisor in $\mathfrak{X}$ with global normal crossings, i.e. $\mathfrak{X}_0 = \sum X_i$ is a sum of smooth, irreducible components X_i's which meet transversally so that locally analytically the morphism π is defined by

$$(x_1, \ldots, x_{n+1}) \to x_1 x_2 \cdots x_k = t \in \Delta, \quad k \leqslant n+1.$$

Given an arbitrary degeneration $\pi : \mathcal{X} \to \Delta$, the well-known Semistable Reduction Theorem (see [25]) states that there exists a base change $\beta : \Delta \to \Delta$ (defined by $\beta(t) = t^m$, for some m), a semistable degeneration $\psi : \mathcal{Z} \to \Delta$ and a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, dashed, "f"] \arrow[dr, "\psi"'] & \mathcal{X}_\beta \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \Delta \arrow[r, "\beta"] & \Delta \end{tikzcd}$$

such that f is a birational map obtained by blowing-up and blowing-down subvarieties of the central fibre.

From now on, we will be concerned with degenerations of relative dimension two, namely degenerations of smooth, projective surfaces.

Definition 4.2. Let $\mathcal{X} \to \Delta$ be a degeneration (equiv. an embedded degeneration) of surfaces. Denote by $\mathcal{X}_t$ the general fibre, which is by definition a smooth, irreducible and projective surface; let $X = \mathcal{X}_0$ denote the central fibre. We will say that the degeneration is *Zappatic* if X is a Zappatic surface, the total space $\mathcal{X}$ is smooth except for:

- ordinary double points at points of the double locus of X, which are not the Zappatic singularities of X;
- further singular points at the Zappatic singularities of X of type T_n, for $n \geqslant 3$, and Z_n, for $n \geqslant 4$,

and there exists a birational morphism $\mathcal{X}' \to \mathcal{X}$, which is the composition of blow-ups at points of the central fibre, such that $\mathcal{X}'$ is smooth.

A Zappatic degeneration will be called *good* if the central fibre is moreover a good Zappatic surface. Similarly, an embedded degeneration will be called a *planar Zappatic degeneration* if its central fibre is a planar Zappatic surface.

Notice that we require the total space $\mathcal{X}$ to be smooth at E_3-points of X.

The singularities of the total space $\mathcal{X}$ of an arbitrary degeneration with Zappatic central fibre will be described in Section 5.

Notation 4.3. Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces and let $\mathcal{X}_t$ be the general fibre, which is by definition a smooth, irreducible and projective surface. Then, we consider the following intrinsic invariants of $\mathcal{X}_t$:

- $\chi := \chi(\mathcal{O}_{\mathcal{X}_t})$;
- $K^2 := K^2_{\mathcal{X}_t}$;

If the degeneration is assumed to be embedded in $\mathbb{P}^r$, for some r, then we also have:

- $d := \deg(\mathcal{X}_t)$;
- $g := (K + H)H/2 + 1$, the sectional genus of $\mathcal{X}_t$.

We will be mainly interested in computing these invariants in terms of the central fibre X. For some of them, this is quite simple. For instance, when $\mathcal{X} \to \Delta$ is an embedded degeneration in $\mathbb{P}^r$, for some r, and if the central fibre $\mathcal{X}_0 = X = \bigcup_{i=1}^{v} X_i$, where the X_i's are smooth, irreducible surfaces of degree d_i, $1 \leqslant i \leqslant v$, then by the flatness of the family we have

$$d = \sum_{i=1}^{v} d_i.$$

When $X = \mathcal{X}_0$ is a good Zappatic surface (in particular a good, planar Zappatic surface), we can easily compute some of the above invariants by using our results of § 3. Indeed, by Proposition 3.11 and by the flatness of the family, we get:

Proposition 4.4. *Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces and suppose that the central fibre $\mathcal{X}_0 = X = \bigcup_{i=1}^{v} X_i$ is a good Zappatic surface. Let $G = G_X$ be its associated graph (cf. Notation 3.9). Let C be the double locus of X, i.e. the union of the double curves of X, $C_{ij} = C_{ji} = X_i \cap X_j$ and let $c_{ij} = \deg(C_{ij})$.*
(i) If f denotes the number of (closed) faces of G, then

$$\chi = \sum_{i=1}^{v} \chi(\mathcal{O}_{X_i}) - \sum_{1 \leqslant i < j \leqslant v} \chi(\mathcal{O}_{C_{ij}}) + f. \tag{4.5}$$

Moreover, if $X = \mathcal{X}_0$ is a good, planar Zappatic surface, then

$$\chi = \chi(G) = v - e + f, \tag{4.6}$$

where e denotes the number of edges of G.
(ii) Assume further that $\mathcal{X} \to \Delta$ is embedded in $\mathbb{P}^r$. Let D be a general hyperplane section of X; let D_i be the i^{th}-smooth, irreducible component of D, which is a general hyperplane section of X_i, and let g_i be its genus. Then

$$g = \sum_{i=1}^{v} g_i + \sum_{1 \leqslant i < j \leqslant v} c_{ij} - v + 1. \tag{4.7}$$

When X is a good, planar Zappatic surface, if $G^{(1)}$ denotes the 1-skeleton of G, then:

$$g = 1 - \chi(G^{(1)}) = e - v + 1. \tag{4.8}$$

In the particular case that $\mathcal{X} \to \Delta$ is a semistable Zappatic degeneration, i.e. if X has only E_3-points as Zappatic singularities and the total space $\mathcal{X}$ is smooth, then χ can be computed also in a different way by topological methods (cf. e.g. [34]).

Proposition 4.4 is indeed more general: X is allowed to have any good Zappatic singularity, namely R_n-, S_n- and E_n-points, for any $n \geqslant 3$, the total space $\mathcal{X}$ is possibly singular, even in dimension one, and, moreover, our computations do not depend on the fact that X is smoothable, i.e. that X is the central fibre of a degeneration.

5. Minimal and quasi-minimal singularities

In this section we shall describe the singularities that the total space of a degeneration of surfaces has at the Zappatic singularities of its central fibre. We need to recall a few general facts about reduced Cohen-Macaulay singularities and two fundamental concepts introduced and studied by Kollár in [26] and [27].

Recall that $V = V_1 \cup \cdots \cup V_r \subseteq \mathbb{P}^n$, a reduced, equidimensional and non-degenerate scheme is said to be *connected in codimension one* if it is possible to arrange its irreducible components $V_1, \ldots, V_r$ in such a way that

$$\operatorname{codim}_{V_j} V_j \cap (V_1 \cup \cdots \cup V_{j-1}) = 1, \text{ for } 2 \leqslant j \leqslant r.$$

Remark 5.1. Let X be a surface in $\mathbb{P}^r$ and C be a hyperplane section of X. If C is a projectively Cohen-Macaulay curve, then X is connected in codimension one. This immediately follows from the fact that X is projectively Cohen-Macaulay.

Given Y an arbitrary algebraic variety, if $y \in Y$ is a reduced, Cohen-Macaulay singularity then

$$\operatorname{emdim}_y(Y) \leqslant \operatorname{mult}_y(Y) + \dim_y(Y) - 1, \tag{5.2}$$

where $\operatorname{emdim}_y(Y) = \dim(\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2)$ is the *embedding dimension* of Y at the point y, where $\mathfrak{m}_{Y,y} \subset \mathcal{O}_{Y,y}$ denotes the maximal ideal of y in Y (see, e.g., [26]).

For any singularity $y \in Y$ of an algebraic variety Y, let us set

$$\delta_y(Y) = \operatorname{mult}_y(Y) + \dim_y(Y) - \operatorname{emdim}_y(Y) - 1. \tag{5.3}$$

If $y \in Y$ is reduced and Cohen-Macaulay, then formula (5.2) states that $\delta_y(Y) \geqslant 0$.

Let H be any effective Cartier divisor of Y containing y. Of course one has

$$\operatorname{mult}_y(H) \geqslant \operatorname{mult}_y(Y).$$

Lemma 5.4. *In the above setting, if* $\operatorname{emdim}_y(Y) = \operatorname{emdim}_y(H)$, *then* $\operatorname{mult}_y(H) > \operatorname{mult}_y(Y)$.

Proof. Let $f \in \mathcal{O}_{Y,y}$ be a local equation defining H around y. If $f \in \mathfrak{m}_{Y,y} \setminus \mathfrak{m}_{Y,y}^2$ (non-zero), then f determines a non-trivial linear functional on the Zariski tangent space $T_y(Y) \cong (\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2)^\vee$. By the definition of $\operatorname{emdim}_y(H)$ and the fact that $f \in \mathfrak{m}_{Y,y} \setminus \mathfrak{m}_{Y,y}^2$, it follows that $\operatorname{emdim}_y(H) = \operatorname{emdim}_y(Y) - 1$. Thus, if $\operatorname{emdim}_y(Y) = \operatorname{emdim}_y(H)$, then $f \in \mathfrak{m}_{Y,y}^h$, for some $h \geqslant 2$. Therefore, $\operatorname{mult}_y(H) \geqslant h \operatorname{mult}_y(Y) > \operatorname{mult}_y(Y)$. □

We let

$$\nu := \nu_y(H) = \min\{n \in \mathbb{N} \mid f \in m_{Y,y}^n\}. \tag{5.5}$$

Notice that:

$$\operatorname{mult}_y(H) \geqslant \nu \operatorname{mult}_y(Y), \qquad \operatorname{emdim}_y(H) = \begin{cases} \operatorname{emdim}_y(Y) & \text{if } \nu > 1, \\ \operatorname{emdim}_y(Y) - 1 & \text{if } \nu = 1. \end{cases} \tag{5.6}$$

Lemma 5.7. *One has*

$$\delta_y(H) \geqslant \delta_y(Y).$$

Furthermore:

- (i) *if the equality holds, then either*
  - (1) $\operatorname{mult}_y(H) = \operatorname{mult}_y(Y)$, $\operatorname{emdim}_y(H) = \operatorname{emdim}_y(Y) - 1$ *and* $\nu_y(H) = 1$, *or*
  - (2) $\operatorname{mult}_y(H) = \operatorname{mult}_y(Y) + 1$, $\operatorname{emdim}_y(H) = \operatorname{emdim}_y(Y)$, *in which case* $\nu_y(H) = 2$ *and* $\operatorname{mult}_y(Y) = 1$;
- (ii) *if* $\delta_y(H) = \delta_y(Y) + 1$, *then either*
  - (1) $\operatorname{mult}_y(H) = \operatorname{mult}_y(Y) + 1$, $\operatorname{emdim}_y(H) = \operatorname{emdim}_y(Y) - 1$, *in which case* $\nu_y(H) = 1$, *or*
  - (2) $\operatorname{mult}_y(H) = \operatorname{mult}_y(Y) + 2$ *and* $\operatorname{emdim}_y(H) = \operatorname{emdim}_y(Y)$, *in which case either*
    (a) $2 \leqslant \nu_y(H) \leqslant 3$ *and* $\operatorname{mult}_y(Y) = 1$, *or*
    (b) $\nu_y(H) = \operatorname{mult}_y(Y) = 2$.

Proof. It is a straightforward consequence of (5.3), of Lemma 5.4 and of (5.6). □

We will say that H has *general behaviour* at y if

$$\operatorname{mult}_y(H) = \operatorname{mult}_y(Y). \tag{5.8}$$

We will say that H has *good behaviour* at y if

$$\delta_y(H) = \delta_y(Y). \tag{5.9}$$

Notice that if H is a general hyperplane section through y, than H has both general and good behaviour.

We want to discuss in more details the relations between the two notions. We note the following facts:

Lemma 5.10. *In the above setting:*

- (i) *if H has general behaviour at y, then it has also good behaviour at y;*
- (ii) *if H has good behaviour at y, then either*
  - (1) *H has also general behaviour and* $\operatorname{emdim}_y(Y) = \operatorname{emdim}_y(H) + 1$, *or*

(2) $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H)$, *in which case* $\mathrm{mult}_y(Y) = 1$ *and* $\nu_y(H) = \mathrm{mult}_y(H) = 2$.

Proof. The first assertion is a trivial consequence of Lemma 5.4.

If H has good behaviour and $\mathrm{mult}_y(Y) = \mathrm{mult}_y(H)$, then it is clear that $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H) + 1$. Otherwise, if $\mathrm{mult}_y(Y) \neq \mathrm{mult}_y(H)$, then $\mathrm{mult}_y(H) = \mathrm{mult}_y(Y) + 1$ and $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H)$. By Lemma 5.7, (i), we have the second assertion. □

As mentioned above, we can now give two fundamental definitions (cf. [26] and [27]):

Definition 5.11. Let Y be an algebraic variety. A reduced, Cohen-Macaulay singularity $y \in Y$ is called *minimal* if the tangent cone of Y at y is geometrically reduced and $\delta_y(Y) = 0$.

Remark 5.12. Notice that if y is a smooth point for Y, then $\delta_y(Y) = 0$ and we are in the minimal case.

Definition 5.13. Let Y be an algebraic variety. A reduced, Cohen-Macaulay singularity $y \in Y$ is called *quasi-minimal* if the tangent cone of Y at y is geometrically reduced and $\delta_y(Y) = 1$.

It is important to notice the following fact:

Proposition 5.14. *Let Y be a projective threefold and $y \in Y$ be a point. Let H be an effective Cartier divisor of Y passing through y.*

(i) *If H has a minimal singularity at y, then Y has also a minimal singularity at y. Furthermore H has general behaviour at y, unless Y is smooth at y and $\nu_y(H) = \mathrm{mult}_y(H) = 2$.*

(ii) *If H has a quasi-minimal, Gorenstein singularity at y then Y has also a quasi-minimal singularity at y, unless either*

  (1) $\mathrm{mult}_y(H) = 3$ *and* $1 \leqslant \mathrm{mult}_y(Y) \leqslant 2$, *or*

  (2) $\mathrm{emdim}_y(Y) = 4$, $\mathrm{mult}_y(Y) = 2$ *and* $\mathrm{emdim}_y(H) = \mathrm{mult}_y(H) = 4$.

Proof. Since $y \in H$ is a minimal (resp. quasi-minimal) singularity, hence reduced and Cohen-Macaulay, the singularity $y \in Y$ is reduced and Cohen-Macaulay too.

Assume that $y \in H$ is a minimal singularity, i.e. $\delta_y(H) = 0$. By Lemma 5.7, (i), and by the fact that $\delta_y(Y) \geqslant 0$, one has $\delta_y(Y) = 0$. In particular, H has good behaviour at y. By Lemma 5.10, (ii), either Y is smooth at y and $\nu_y(H) = 2$, or H has general behaviour at y. In the latter case, the tangent cone of Y at y is geometrically reduced, as is the tangent cone of H at y. Therefore, in both cases Y has a minimal singularity at y, which proves (i).

Assume that $y \in H$ is a quasi-minimal singularity, namely $\delta_y(H) = 1$. By Lemma 5.7, then either $\delta_y(Y) = 1$ or $\delta_y(Y) = 0$.

If $\delta_y(Y) = 1$, then the case (i.2) in Lemma 5.7 cannot occur, otherwise we would have $\delta_y(H) = 0$, against the assumption. Thus H has general behaviour and, as above, the tangent cone of Y at y is geometrically reduced, as the tangent cone of H at y is. Therefore Y has a quasi-minimal singularity at y.

If $\delta_y(Y) = 0$, we have the possibilities listed in Lemma 5.7, (ii). If (1) holds, we have $\mathrm{mult}_y(H) = 3$, i.e. we are in case (ii.1) of the statement. Indeed, Y is Gorenstein at y as H is, and therefore $\delta_y(Y) = 0$ implies that $\mathrm{mult}_y(Y) \leqslant 2$ by Corollary 3.2 in [39], thus $\mathrm{mult}_y(H) \leqslant 3$, and in fact $\mathrm{mult}_y(H) = 3$ because $\delta_y(H) = 1$. Also the possibilities listed in Lemma 5.7, (ii.2) lead to cases listed in the statement. □

Remark 5.15. From an analytic viewpoint, case (1) in Proposition 5.14 (ii), when Y is smooth at y, can be thought of as $Y = \mathbb{P}^3$ and H a cubic surface with a triple point at y.

On the other hand, case (2) can be thought of as Y being a quadric cone in $\mathbb{P}^4$ with vertex at y and as H being cut out by another quadric cone with vertex at y. The resulting singularity is therefore the cone over a quartic curve Γ in $\mathbb{P}^3$ with arithmetic genus 1, which is the complete intersection of two quadrics.

Now we describe the relation between minimal and quasi-minimal singularities and Zappatic singularities. First we need the following straightforward remark:

Lemma 5.16. *Any T_n-point (resp. Z_n-point) is a minimal (resp. quasi-minimal) surface singularity.*

The following direct consequence of Proposition 5.14 will be important for us:

Proposition 5.17. *Let X be a surface with a Zappatic singularity at a point $x \in X$ and let $\mathcal{X}$ be a threefold containing X as a Cartier divisor.*

- *If x is a T_n-point for X, then x is a minimal singularity for $\mathcal{X}$ and X has general behaviour at x;*
- *If x is an E_n-point for X, then $\mathcal{X}$ has a quasi-minimal singularity at x and X has general behaviour at x, unless either:*
  (i) $\mathrm{mult}_x(X) = 3$ *and* $1 \leqslant \mathrm{mult}_x(\mathcal{X}) \leqslant 2$, *or*
  (ii) $\mathrm{emdim}_x(\mathcal{X}) = 4$, $\mathrm{mult}_x(\mathcal{X}) = 2$ *and* $\mathrm{emdim}_x(X) = \mathrm{mult}_x(X) = 4$.

In the sequel, we will need a description of a surface having as a hyperplane section a stick curve of type C_{S_n}, C_{R_n}, and C_{E_n} (cf. Examples 2.6 and 2.7).

First of all, we recall well-known results about *minimal degree surfaces* (cf. [20], page 525).

Theorem 5.18 (del Pezzo). *Let X be an irreducible, non-degenerate surface of minimal degree in $\mathbb{P}^r$, $r \geqslant 3$. Then X has degree $r-1$ and is one of the following:*

(i) *a rational normal scroll;*
(ii) *the Veronese surface, if $r = 5$.*

Next we recall the result of Xambó concerning reducible minimal degree surfaces (see [42]).

Theorem 5.19 (Xambó). *Let X be a non-degenerate surface which is connected in codimension one and of minimal degree in $\mathbb{P}^r$, $r \geqslant 3$. Then, X has degree $r-1$, any irreducible component of X is a minimal degree surface in a suitable projective space and any two components intersect along a line.*

Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projectively Cohen-Macaulay surface with canonical singularities, i.e. with Du Val singularities. We recall that X is called a *del Pezzo surface* if $\mathcal{O}_X(-1) \simeq \omega_X$. We note that a del Pezzo surface is projectively Gorenstein (for connections between Commutative Algebra and Projective Geometry, we refer the reader to e.g. [13], [19] and [28]).

Theorem 5.20 (del Pezzo, [12]). *Let X be an irreducible, non-degenerate, linearly normal surface of degree r in $\mathbb{P}^r$. Then one of the following occurs:*

(i) *one has $3 \leqslant r \leqslant 9$ and X is either*
  a. *the image of the blow-up of $\mathbb{P}^2$ at $9-r$ suitable points, mapped to $\mathbb{P}^r$ via the linear system of cubics through the $9-r$ points, or*
  b. *the 2-Veronese image in $\mathbb{P}^8$ of a quadric in $\mathbb{P}^3$.*

  In each case, X is a del Pezzo surface.
(ii) *X is a cone over a smooth elliptic normal curve of degree r in $\mathbb{P}^{r-1}$.*

Proof. This is a classical result. For a complete proof in modern language, see e.g. [7]. □

Since cones as in (ii) above are projectively Gorenstein surfaces, the surfaces listed in Theorem 5.20 will be called *minimal Gorenstein surfaces*.

We shall make use of the following easy consequence of the Riemann-Roch theorem.

Lemma 5.21. *Let $D \subset \mathbb{P}^r$ be a reduced (possibly reducible), non-degenerate and linearly normal curve of degree $r+d$ in $\mathbb{P}^r$, with $0 \leqslant d < r$. Then $p_a(D) = d$.*

Theorem 5.22. *Let X be a non-degenerate, projectively Cohen-Macaulay surface of degree r in $\mathbb{P}^r$, $r \geqslant 3$, which is connected in codimension one. Then, any irreducible component of X is either*

(i) *a minimal Gorenstein surface, and there is at most one such component, or*
(ii) *a minimal degree surface.*

If there is a component of type (i), then the intersection in codimension one of any two distinct components can be only a line.

If there is no component of type (i), then the intersection in codimension one of any two distinct components is either a line or a (possibly reducible) conic. Moreover, if two components meet along a conic, all the other intersections are lines.

Furthermore, X is projectively Gorenstein if and only if either

(a) *X is irreducible of type (i), or*
(b) *X consists of only two components of type (ii) meeting along a conic, or*
(c) *X consists of ν, $3 \leqslant \nu \leqslant r$, components of type (ii) meeting along lines and the dual graph G_D of a general hyperplane section D of X is a cycle E_ν.*

Proof. Consider D a general hyperplane section of X. Since X is projectively Cohen-Macaulay, it is arithmetically Cohen-Macaulay. This implies that D is an arithmetically Cohen-Macaulay (equiv. arithmetically normal) curve. By Lemma 5.21, $p_a(D) = 1$. Therefore, for each connected subcurve D' of D, one has $0 \leqslant p_a(D') \leqslant 1$ and there is at most one irreducible component D'' with $p_a(D'') = 1$. In particular two connected subcurves of D can meet at most in two points. This implies that two irreducible components of X meet either along a line or along a conic. The linear normality of X immediately implies that each irreducible component is linearly normal too. As a consequence of Theorem 5.20 and of Lemma 5.21, all this proves the statement about the components of X and their intersection in codimension one.

It remains to prove the final part of the statement.

If X is irreducible, the assertion is trivial, so assume X reducible.

Suppose that all the intersections in codimension one of the distinct components of X are lines. If either the dual graph G_D of a general hyperplane section D of X is not a cycle or there is an irreducible component of D which is not rational, then D is not Gorenstein (see the discussion at the end of Example 2.7), contradicting the assumption that X is Gorenstein.

Conversely, if G_D is a cycle E_ν and each component of D is rational, then D is projectively Gorenstein. In particular, if all the components of D are lines, then D isomorphic to C_{E_ν} (cf. again Example 2.7). Therefore X is projectively Gorenstein too.

Suppose that X consists of two irreducible components meeting along a conic. Then D consists of two rational normal curves meeting at two points; thus the dualizing sheaf ω_D is trivial, i.e. D is projectively Gorenstein and Gorenstein, therefore so is X.

Conversely, let us suppose that X is projectively Gorenstein and there are two irreducible components X_1 and X_2 meeting along a conic. If there are other components, then there is a component X' meeting all the rest along a line. Thus, the hyperplane section contains a rational curve meeting all the rest at a point. Therefore the dualizing sheaf of D is not trivial, hence D is not Gorenstein, thus X is not Gorenstein. □

By using Theorems 5.18, 5.19 and 5.20, we can prove the following result:

Proposition 5.23. *Let X be a non-degenerate surface in $\mathbb{P}^r$, for some r, and let $n \geqslant 3$ be an integer.*

(i) *If $r = n+1$ and if a hyperplane section of X is C_{R_n}, then either:*
   a. *X is a smooth rational cubic scroll, possible only if $n = 3$, or*
   b. *X is a Zappatic surface, with ν irreducible components of X which are either planes or smooth quadrics, meeting along lines, and the Zappatic singularities of X are $h \geqslant 1$ points of type R_{m_i}, $i = 1, \dots, h$, such that*

$$\sum_{i=1}^{h}(m_i - 2) = \nu - 2. \tag{5.24}$$

In particular X has global normal crossings if and only if $\nu = 2$, i.e. if and only if either $n = 3$ and X consists of a plane and a quadric meeting along a line, or $n = 4$ and X consists of two quadrics meeting along a line.

(ii) *If $r = n+1$ and if a hyperplane section of X is C_{S_n}, then either:*
   a. *X is the union of a smooth rational normal scroll $X_1 = S(1, d-1)$ of degree d, $2 \leqslant d \leqslant n$, and of $n-d$ disjoint planes each meeting X_1 along different lines of the same ruling, in which case X has global normal crossings; or*
   b. *X is planar Zappatic surface with $h \geqslant 1$ points of type S_{m_i}, $i = 1, \dots, h$, such that*

$$\sum_{i=1}^{h}\binom{m_i - 1}{2} = \binom{n-1}{2}. \tag{5.25}$$

(iii) *If $r = n$ and if a hyperplane section of X is C_{E_n} then either:*
   a. *X is an irreducible del Pezzo surface of degree n in $\mathbb{P}^n$, possible only if $n \leqslant 6$; in particular X is smooth if $n = 6$; or*
   b. *X has two irreducible components X_1 and X_2, meeting along a (possibly reducible) conic; X_i, $i = 1, 2$, is either a smooth rational cubic scroll, or a quadric, or a plane; in particular X has global normal crossings if $X_1 \cap X_2$ is a smooth conic and neither X_1 nor X_2 is a quadric cone;*
   c. *X is a Zappatic surface whose irreducible components $X_1, \dots, X_\nu$ of X are either planes or smooth quadrics. Moreover X has a unique E_ν-point, and no other Zappatic singularity, the singularities in codimension one being double lines.*

Proof. (i) According to Remark 5.1 and Theorem 5.19, X is connected in codimension one and is a union of minimal degree surfaces meeting along lines. Since a hyperplane section is a C_{R_n}, then each irreducible component Y of X has to contain some line and therefore it is a rational normal scroll, or a plane. Furthermore Y has a hyperplane section which is a connected subcurve of C_{R_n}. It is then clear that Y is either a plane, or a quadric or a smooth rational normal cubic scroll.

We claim that Y cannot be a quadric cone. In fact, in this case, the hyperplane sections of Y consisting of lines pass through the vertex $y \in Y$. Since $Y \cap \overline{(X \setminus Y)}$ also consists of lines passing through y, we see that no hyperplane section of X is a C_{R_n}.

Reasoning similarly, one sees that if a component Y of X is a smooth rational cubic scroll, then Y is the only component of X, i.e. $Y = X$, which proves statement a.

Suppose now that X is reducible, so its components are either planes or smooth quadrics. The dual graph G_D of a general hyperplane section D of X is a chain of length ν and any connecting edge corresponds to a double line of X. Let $x \in X$ be a singular point and let $Y_1, \dots, Y_m$ be the irreducible components of X containing x. Let G' be the subgraph of G_D corresponding to $Y_1 \cup \dots \cup Y_m$. Since X is projectively Cohen-Macaulay, then clearly G' is connected, hence it is a chain. This shows that x is a Zappatic singularity of type R_m.

Finally we prove formula (5.24). Suppose that the Zappatic singularities of X are h points $x_1, \dots, x_h$ of type $R_{m_1}, \dots, R_{m_h}$, respectively. Notice that the hypothesis that a hyperplane section of X is a C_{R_n} implies that two double lines of X lying on the same irreducible component have to meet at a point, because they are either lines in a plane or fibres of different rulings on a quadric.

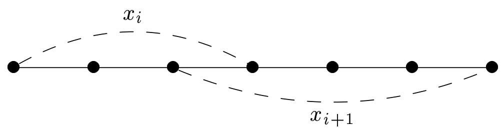


FIGURE 9. The points x_i and x_{i+1} share a common edge in the associated graph G_X.

So the graph G_X consists of h open faces corresponding to the points x_i, $1 \leqslant i \leqslant h$, and two contiguous open faces must share a common edge, as shown in Figure 9. Thus, both formula (5.24) and the last part of statement b. immediately follow.

(ii) Arguing as in the proof of (i), one sees that any irreducible component Y of X is either a plane, or a smooth quadric or a smooth rational normal scroll with a line as a directrix.

If Y is a rational normal scroll $S(1, d-1)$ of degree $d \geqslant 2$, the subgraph of S_n corresponding to the hyperplane section of Y is S_d. Then a. follows in this case, namely all the other components of X are planes meeting Y along lines of the ruling. Note that, since X spans a $\mathbb{P}^{n+1}$, these planes are pairwise skew and therefore X has global normal crossings.

Suppose now that X is a union of planes. Then X consists of a plane Π and of $n-1$ more planes meeting Π along distinct lines. Arguing as in part (i), one sees that the planes different from Π pairwise meet only at a point in Π. Hence X is smooth off Π. On the other hand, it is clear that the singularities x_i in Π are Zappatic of type S_{m_i}, $i = 1, \ldots, h$. This corresponds to the fact that $m_i - 1$ planes different from Π pass through the same point $x_i \in \Pi$. Formula (5.25) follows by suitably counting the number of pairs of double lines in the configuration.

(iii) If X is irreducible, then a. holds by elementary properties of lines on a del Pezzo surface.

Suppose that X is reducible. Every irreducible component Y of X has a hyperplane section which is a stick curve strictly contained in C_{E_n}. By an argument we already used in part (i), then Y is either a plane, or a quadric or a smooth rational normal cubic scroll.

Suppose that an irreducible component Y meets $\overline{X \setminus Y}$ along a conic. Since C_{E_n} is projectively Gorenstein, then also X is projectively Gorenstein; so, by Theorem 5.22, X consists of only two irreducible components and b. follows.

Again by Theorem 5.22 and reasoning as in part (i), one sees that all the irreducible components of X are either planes or smooth quadrics and the dual graph G_D of a general hyperplane section D of X is a cycle E_ν of length ν.

As we saw in part (i), two double lines of X lying on the same irreducible component Y of X meet at a point of Y. Hence X has some singularity besides the general points on the double lines. Again, as we saw in part (i), such singularity can be either of type R_m or of type E_m, where R_m or E_n are subgraphs of the dual graph G_D of a general hyperplane section D of X. Since X is projectively Gorenstein, it has only Gorenstein singularities, in particular R_m-points are excluded. Thus, the only singularity compatible with the above graph is a E_ν-point. □

Remark 5.26. At the end of the proof of part *(iii)*, instead of using the Gorenstein property, one can prove by a direct computation that a surface X of degree n, which is a union of planes and smooth quadrics and such that the dual graph G_D of a general hyperplane section D of X is a cycle of length ν, must have an E_ν-point and no other Zappatic singularity in order to span a $\mathbb{P}^n$.

Corollary 5.27. *Let $\mathfrak{X} \to \Delta$ be a degeneration of surfaces whose central fibre X is Zappatic. Let $x \in X$ be a T_n-point. Let $\mathfrak{X}'$ be the blow-up of $\mathfrak{X}$ at x. Let E be the exceptional divisor, let X' be the proper transform of X, $\Gamma = C_{T_n}$ be the intersection curve of E and X'. Then E is a minimal degree surface of degree n in $\mathbb{P}^{n+1} = \mathbb{P}(T_{\mathfrak{X},x})$, and Γ is one of its hyperplane sections.*

In particular, if x is either a R_n- or a S_n-point, then E is as described in Proposition 5.23.

Proof. The first part of the statement directly follows from Lemma 5.16, Proposition 5.17 and Theorem 5.19. □

We close this section by stating a result which will be useful in the sequel:

Corollary 5.28. *Let y be a point of a projective threefold Y. Let H be an effective Cartier divisor on Y passing through y. If H has an E_n-point at y, then Y is Gorenstein at y.*

Proof. Recall that H is Gorenstein at y (cf. Remark 3.4) and H locally behaves as a hyperplane section of Y at y (cf. the proof of Proposition 5.14), therefore Y is Gorenstein at y. □

Let $\mathfrak{X} \to \Delta$ be a degeneration of surfaces whose central fibre X is good Zappatic. From Definition 3.2 and Corollary 5.28, it follows that $\mathfrak{X}$ is Gorenstein at all the points of X, except at its R_n- and S_n-points.

6. Combinatorial computation of K^2

The results contained in § 5 will be used in this section to prove combinatorial formulas for $K^2 = K^2_{\mathfrak{X}_t}$, where $\mathfrak{X}_t$ is a smooth surface which degenerates to a good Zappatic surface $\mathfrak{X}_0 = X = \bigcup_{i=1}^v X_i$, i.e. $\mathfrak{X}_t$ is the general fibre of a degeneration of surfaces whose central fibre is good Zappatic (cf. Notation 4.3).

Indeed, by using the combinatorial data associated to X and G_X (cf. Definition 3.6 and Notation 3.9), we shall prove the following main result:

Theorem 6.1. *Let $\mathfrak{X} \to \Delta$ be a degeneration of surfaces whose central fibre is a good Zappatic surface $X = \mathfrak{X}_0 = \bigcup_{i=1}^v X_i$. Let $C_{ij} = X_i \cap X_j$ be a double curve of X, which is considered as a curve on X_i, for $1 \leqslant i \neq j \leqslant v$.*

If $K^2 := K^2_{\mathfrak{X}_t}$, for $t \neq 0$, then (cf. Notation 3.9):

$$K^2 = \sum_{i=1}^{v} \left(K^2_{X_i} + \sum_{j \neq i} (4g_{ij} - C^2_{ij}) \right) - 8e + \sum_{n \geqslant 3} 2nf_n + r_3 + k, \tag{6.2}$$

where k depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$, and precisely:

$$\sum_{n \geqslant 4} (n-2)(r_n + s_n) \leqslant k \leqslant \sum_{n \geqslant 4} \left((2n-5) r_n + \binom{n-1}{2} s_n \right). \tag{6.3}$$

In case $\mathfrak{X}$ is an embedded degeneration and X is also planar, we have the following:

Corollary 6.4. *Let $\mathfrak{X} \to \Delta$ be an embedded degeneration of surfaces whose central fibre is a good planar Zappatic surface $X = \mathfrak{X}_0 = \bigcup_{i=1}^v \Pi_i$. Then:*

$$K^2 = 9v - 10e + \sum_{n \geqslant 3} 2nf_n + r_3 + k \tag{6.5}$$

where k is as in (6.3) *and depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$.*

Proof. Clearly $g_{ij} = 0$, for each $1 \leqslant i \neq j \leqslant v$, whereas $C^2_{ij} = 1$, for each pair (i,j) s.t. $e_{ij} \in E$, otherwise $C^2_{ij} = 0$. □

The proof of Theorem 6.1 will be done in several steps. The first one is to compute K^2 when X has only E_n-points. In this case, and only in this case, K_X is a Cartier divisor.

Theorem 6.6. *Under the assumptions of Theorem 6.1, if $X = \bigcup_{i=1}^v X_i$ has only E_n-points, for $n \geqslant 3$, then:*

$$K^2 = \sum_{i=1}^{v} \left(K_{X_i}^2 + \sum_{j \neq i} (4g_{ij} - C_{ij}^2) \right) - 8e + \sum_{n \geqslant 3} 2nf_n. \tag{6.7}$$

Proof. Note that the total space $\mathfrak{X}$ is Gorenstein: this is clear for the double points, for the E_n-points of X, see Corollary 5.28. Thus, $K_{\mathfrak{X}}$ is a Cartier divisor on $\mathfrak{X}$. Therefore K_X is also Cartier and it makes sense to consider K_X^2 and the adjunction formula states $K_X = (K_{\mathfrak{X}} + X)_{|X}$.

We claim that

$$K_{X|X_i} = (K_{\mathfrak{X}} + X)_{|X_i} = K_{X_i} + C_i, \tag{6.8}$$

where $C_i = \sum_{j \neq i} C_{ij}$ is the union of the double curves of X lying on the irreducible component X_i, for each $1 \leqslant i \leqslant v$. Since $\mathcal{O}_X(K_X)$ is invertible, it suffices to prove (6.8) off the E_n-points. In other words, we can consider the surfaces X_i as if they were Cartier divisors on $\mathfrak{X}$. Then, we have:

$$K_{X|X_i} = (K_{\mathfrak{X}} + X)_{|X_i} = \big(K_{\mathfrak{X}} + X_i + \sum_{j \neq i} X_j\big)_{|X_i} = K_{X_i} + C_i, \tag{6.9}$$

as we had to show. Furthermore:

$$K^2 = (K_{\mathfrak{X}} + \mathfrak{X}_t)^2 \cdot \mathfrak{X}_t = (K_{\mathfrak{X}} + X)^2 \cdot X = (K_{\mathfrak{X}} + X)^2 \cdot \sum_{i=1}^{v} X_i = \sum_{i=1}^{v} \big((K_{\mathfrak{X}} + X)_{|X_i}\big)^2 =$$

$$= \sum_{i=1}^{v} (K_{X_i}^2 + 2C_i K_{X_i} + C_i^2) = \sum_{i=1}^{v} K_{X_i}^2 + \sum_{i=1}^{v} C_i K_{X_i} + \sum_{i=1}^{v} C_i(C_i + K_{X_i}) =$$

$$= \sum_{i=1}^{v} K_{X_i}^2 + \sum_{i=1}^{v} (\sum_{j \neq i} C_{ij}) K_{X_i} + \sum_{i=1}^{v} 2(p_a(C_i) - 1). \tag{6.10}$$

As in Notation 3.9, $C_{ij} = \sum_{t=1}^{h_{ij}} C_{ij}^t$ is the sum of its disjoint, smooth, irreducible components, where h_{ij} is the number of these components. Thus,

$$C_{ij} K_{X_i} = \sum_{t=1}^{h_{ij}} (C_{ij}^t K_{X_i}),$$

for each $1 \leqslant i \neq j \leqslant v$. If we denote by g_{ij}^t the geometric genus of the smooth, irreducible curve C_{ij}^t, by the adjunction formula on each C_{ij}^t, we have the following intersection number on the surface X_i:

$$C_{ij} K_{X_i} = \sum_{t=1}^{h_{ij}} (2g_{ij}^t - 2 - (C_{ij}^t)^2) = 2g_{ij} - 2h_{ij} - C_{ij}^2,$$

where the last equality follows from the definition of geometric genus of C_{ij} and the fact that $C_{ij}^s C_{ij}^t = 0$, for any $1 \leqslant t \neq s \leqslant h_{ij}$.

Therefore, by the distributivity of the intersection form and by (6.10), we get:

$$K^2 = \sum_{i=1}^{v} K_{X_i}^2 + \sum_{i=1}^{v} (\sum_{j \neq i} (2g_{ij} - 2h_{ij}) - C_{ij}^2) + \sum_{i=1}^{v} 2(p_a(C_i) - 1). \tag{6.11}$$

For each index i, consider now the normalization $\nu_i : \tilde{C}_i \to C_i$ of the curve C_i lying on X_i; this determines the short exact sequence:

$$0 \to \mathcal{O}_{C_i} \to (\nu_i)_*(\mathcal{O}_{\tilde{C}_i}) \to \underline{t}_i \to 0, \tag{6.12}$$

where $\underline{t}_i$ is a sky-scraper sheaf supported on $\mathrm{Sing}(C_i)$, as a curve in X_i. By using Notation 3.9, the long exact sequence in cohomology induced by (6.12) gives that:

$$\chi(\mathcal{O}_{C_i}) + h^0(\underline{t}_i) = \sum_{j\neq i}\sum_{t=1}^{h_{ij}} \chi(\mathcal{O}_{C_{ij}^t}) = \sum_{j\neq i}(h_{ij} - g_{ij}).$$

Since $\chi(\mathcal{O}_{C_i}) = 1 - p_a(C_i)$, we get

$$p_a(C_i) - 1 = \sum_{j\neq i}(g_{ij} - h_{ij}) + h^0(\underline{t}_i),\ 1 \leqslant i \leqslant v. \tag{6.13}$$

By plugging formula (6.13) in (6.11), we get:

$$K^2 = \sum_{i=1}^{v}\left(K_{X_i}^2 + \sum_{j\neq i}(4g_{ij} - C_{ij}^2)\right) - 8e + 2\sum_{i=1}^{v} h^0(\underline{t}_i). \tag{6.14}$$

To complete the proof, we need to compute $h^0(\underline{t}_i)$. By definition of $\underline{t}_i$, this computation is a local problem. Suppose that p is an E_n-point of X lying on X_i, for some i. By the very definition of E_n-point (cf. Definition 3.1 and Example 2.7), p is a node for the curve $C_i \subset X_i$; therefore $h^0(\underline{t}_{i|p}) = 1$. The same holds on each of the other $n-1$ curves $C_j \subset X_j$, $1 \leqslant j \neq i \leqslant n$, concurring at the E_n-point p. Therefore, by (6.14), we get (6.7). $\square$

Proof of Theorem 6.1. The previous argument proves that, in this more general case, one has:

$$K^2 = \sum_{i=1}^{v}\left(K_{X_i}^2 + \sum_{j\neq i}(4g_{ij} - C_{ij}^2)\right) - 8e + 2\sum_{i=1}^{v} h^0(\underline{t}_i) - c \tag{6.15}$$

where c is a positive correction term which depends only on the points where $\mathcal{X}$ is not Gorenstein, i.e. at the R_n- and S_n-points of its central fibre X.

To prove the statement, we have to compute:

(i) the contribution of $h^0(\underline{t}_i)$ given by the R_n- and the S_n-points of X, for each $1 \leqslant i \leqslant v$;
(ii) the correction term c.

For (i), suppose first that p is a R_n-point of X and let C_i be one of the curves passing through p. By definition (cf. Example 2.6), the point p is either a smooth point or a node for $C_i \subset X_i$. In the first case we have $h^0(\underline{t}_{i|p}) = 0$ whereas, in the latter, $h^0(\underline{t}_{i|p}) = 1$. More precisely, among the n indexes involved in the R_n-point there are exactly two indexes, say i_1 and i_n, such that C_{i_j} is smooth at p, for $j = 1$ and $j = n$, and $n-2$ indexes such that C_{i_j} has a node at p, for $2 \leqslant j \leqslant n-1$. On the other hand, if we assume that p is a S_n-point, then p is an ordinary $(n-1)$-tuple point for only one of the curves concurring at p, say $C_i \subset X_i$, and a simple point for all the other curves $C_j \subset X_j$, $1 \leqslant j \neq i \leqslant n$. Recall that an ordinary $(n-1)$-tuple point contributes $\binom{n-1}{2}$ to $h^0(\underline{t}_i)$.

Therefore, from (6.15), we have:

$$K^2 = \sum_{i=1}^{v}\left(K_{X_i}^2 + \sum_{j\neq i}(4g_{ij} - C_{ij}^2)\right) - 8e + \sum_{n\geqslant 3} 2nf_n + \sum_{n\geqslant 3} 2(n-2)r_n + \sum_{n\geqslant 4}(n-1)(n-2)s_n - c.$$

In order to compute the correction term c, we have to perform a partial resolution of $\mathcal{X}$ at the R_n- and S_n-points of X, which makes the total space Gorenstein. This will give us (6.2), i.e.

$$K^2 = \sum_{i=1}^{v} \left(K_{X_i}^2 + \sum_{j \neq i} (4g_{ij} - C_{ij}^2) \right) - 8e + \sum_{n \geqslant 3} 2nf_n + r_3 + k,$$

where

$$k := \sum_{n \geqslant 3} 2(n-2)r_n \ - r_3 \ + \sum_{n \geqslant 4} (n-1)(n-2)s_n \ - c.$$

It is clear that the contribution to c of each such point is purely local. In other words,

$$c = \sum_x c_x$$

where x varies in the set of R_n- and S_n-points of X and where c_x is the contribution at x to the computation of K^2 as above.

In the next Proposition 6.16, we shall compute such local contributions. This result, together with Theorem 6.6, will conclude the proof. ☐

Proposition 6.16. *In the hypothesis of Theorem 6.1, if $x \in X$ is a R_n-point then:*

$$n - 2 \geqslant c_x \geqslant 1,$$

whereas if $x \in X$ is a S_n-point then:

$$(n-2)^2 \geqslant c_x \geqslant \binom{n-1}{2}.$$

Proof. Since the problem is local, we may (and will) assume that $\mathcal{X}$ is Gorenstein, except at a point x, and that each irreducible component X_i of X passing through x is a plane, denoted by Π_i.

First we will deal with the case $n = 3$.

Claim 6.17. *If x is a R_3-point, then*

$$c_x = 1.$$

Proof of the claim. Let us blow-up the point $x \in \mathcal{X}$ as in Corollary 5.27.

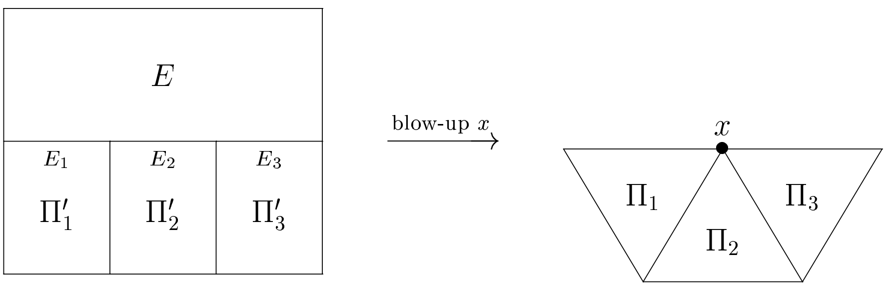


FIGURE 10. Blowing-up a R_3-point x.

We get a new total space $\mathcal{X}'$. We denote by E the exceptional divisor, by Π'_i the proper transform of Π_i and by $X' = \cup \Pi'_i$ the proper transform of X, as in Figure 10. We remark that the three planes Π_i, $i = 1, 2, 3$, concurring at x, are blown-up in this process, whereas the remaining planes stay untouched. We call E_i the exceptional divisor on the blown-up plane Π_i. Let $\Gamma = E_1 + E_2 + E_3$ be the intersection curve of E and X'. By Corollary 5.27, E is a non-degenerate surface of degree 3 in $\mathbb{P}^4$, with Γ as a hyperplane section.

Suppose first that E is irreducible. Then $\mathcal{X}'$ is Gorenstein and by adjunction:

$$K^2 = (K_{X'} + \Gamma)^2 + (K_E + \Gamma)^2. \tag{6.18}$$

Since E is a rational normal cubic scroll in $\mathbb{P}^4$, then:

$$(K_E + \Gamma)^2 = 1, \tag{6.19}$$

whereas the other term is:

$$(K_{X'} + \Gamma)^2 = \sum_i (K_{X'|\Pi'_i} + \Gamma_{\Pi'_i})^2 = \sum_{i=1}^{3} (K_{X'|\Pi'_i} + E_i)^2 + \sum_{j\geqslant 4} K^2_{X'|\Pi'_j}.$$

Reasoning as in the proof of Theorem 6.6, one sees that

$$\sum_{j\geqslant 4} K^2_{X'|\Pi'_j} = \sum_{j\geqslant 4} (w_j - 3)^2.$$

On the other hand, one has

$$(K_{X'|\Pi'_i} + E_i)^2 = (w_i - 3)^2 - 1, \quad i = 1, 3, \qquad (K_{X'|\Pi'_2} + E_2)^2 = (w_2 - 3)^2.$$

Putting all together, it follows that $c_x = 1$.

Suppose now that E is reducible and X' is still Gorenstein. In this case E is as described in Proposition 5.23 (ii), b, and in Corollary 5.27 and the proof proceeds as above, once one remarks that (6.19) holds. This can be left to the reader to verify (see Figure 11).

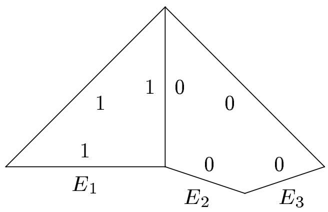


FIGURE 11. E splits in a plane and a quadric.

Suppose that E is reducible and X' is not Gorenstein. This means that E consists of a cone over a C_{R_3} with vertex x', hence x' is again a R_3-point. Therefore we have to repeat the process by blowing-up x'. After finitely many steps this procedure stops (cf. e.g. Proposition 3.4.13 in [26]). In order to conclude the proof in this case, one has simply to remark that no contribution to K^2 comes from the surfaces created in the intermediate steps.

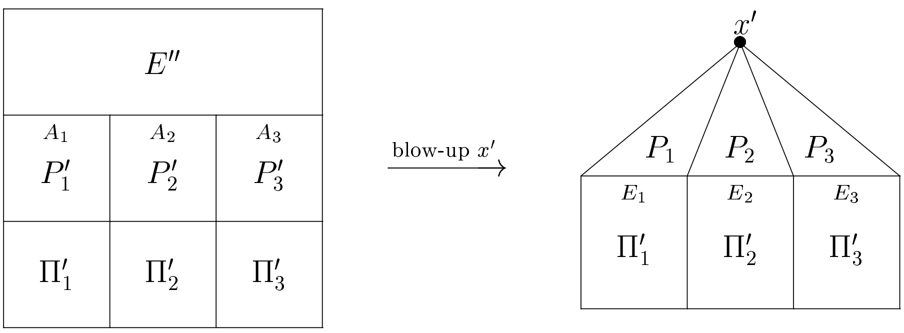


FIGURE 12. blowing-up a R_3-point x' infinitely near to the R_3-point x

To see this, it suffices to make this computation when only two blow-ups are needed. This is the situation showed in Figure 12 where:

- $\mathcal{X}'' \to \mathcal{X}'$ is the blow-up at x',

- $X' = \sum \Pi_i'$ the proper transform of X' on $\mathcal{X}''$,
- $E' = P_1' + P_2' + P_3'$ is the strict transform of $E = P_1 + P_2 + P_3$ on $\mathcal{X}''$,
- E'' is the exceptional divisor of the blow-up.

We remark that P_i', $i = 1, 2, 3$, is the blow-up of the plane P_i. We denote by Λ_i the pullback to P_i' of a line, and by A_i the exceptional divisor of P_i'. Then their contributions to the computation of K^2 are:

$$(K_{P_i'} + \Lambda_i + (\Lambda_i - A_i) + A_i)^2 = (-\Lambda_i + A_i)^2 = 0, \quad i = 1, 3,$$
$$(K_{P_2'} + \Lambda_2 + 2(\Lambda_2 - A_2) + A_2)^2 = 0.$$

This concludes the proof of Claim 6.17. □

Consider now the case that $n = 4$ and x is a R_4-point.

Claim 6.20. *If x is a R_4-point, then*

$$2 \geqslant c_x \geqslant 1.$$

Proof of the claim. As before, we blow-up the point $x \in \mathcal{X}$; let $\mathcal{X}'$ be the new total space and let E be the exceptional divisor. By Corollary 5.27, E is a non-degenerate surface of minimal degree in $\mathbb{P}^5$ with $\Gamma = E_1 + E_2 + E_3 + E_4$ as a hyperplane section. By Proposition 5.23, E is reducible and the following cases may occur:

(i) E has global normal crossings, in which case E consists of two quadrics Q_1, Q_2 meeting along a line (see Figure 13);

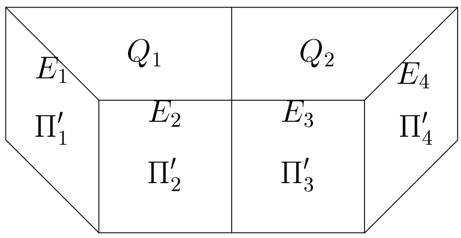
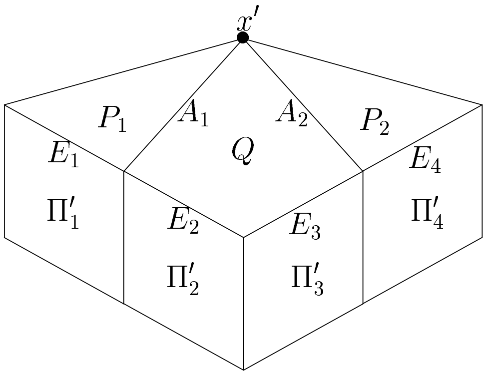
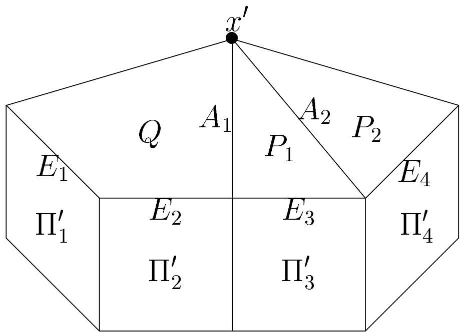


FIGURE 13. The exceptional divisor E has global normal crossings.

(ii) E has one R_3-point x', in which case E consists of a quadric Q and two planes P_1, P_2 (see Figure 14);



a) The quadric in the middle



b) The quadric on one side

FIGURE 14. E consists of a quadric and two planes and has a R_3-point x'.

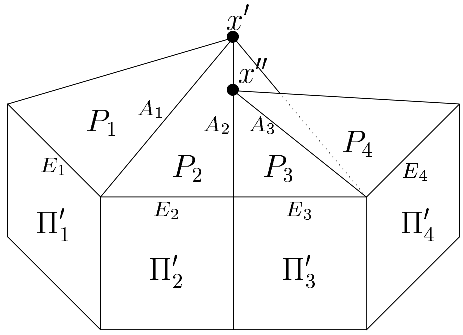


FIGURE 15. E consists of four planes and has two R_3-points x', x''.

(iii) E has two R_3-points x', x'', in which case E consists of four planes $P_1, \dots, P_4$, i.e. a planar Zappatic surface whose associated graph is the tree R_4 (see Figure 15);

(iv) E has one R_4-point x', in which case E consists of four planes, i.e. a planar Zappatic surface whose associated graph is an open 4-face (cf. Figures 5, 6 and 16).

In case (i), $\mathfrak{X}'$ is Gorenstein and we can compute K^2 as we did in the proof of Claim 6.17. Formula (6.18) still holds and one has $(K_E + \Gamma)^2 = 0$, whereas:

$$(6.21)\quad (K_{X'}+\Gamma)^2 = \sum_i (K_{X'|\Pi'_i} + \Gamma_{\Pi'_i})^2 = \sum_{i=1}^{4} (K_{X'|\Pi'_i} + E_i)^2 + \sum_{j\geqslant 4} K^2_{X'|\Pi'_j} = \sum_{j\geqslant 1} (w_j - 3)^2 - 2,$$

because the computations on the blown-up planes $\Pi'_1, \dots, \Pi'_4$ give:

$$(K_{X'|\Pi'_i} + E_i)^2 = (w_i - 3)^2 - 1, \quad i = 1, 4, \qquad (K_{X'|\Pi'_i} + E_i)^2 = (w_i - 3)^2, \quad i = 2, 3.$$

This proves that $c_x = 2$ in this case.

In case (ii), there are two possibilities corresponding to cases (a) and (b) of Figure 14. Let us first consider the former possibility. By Claim 6.17, in order to compute K^2 we have to add up three quantities:

- the contribution of $(K_{X'} + \Gamma)^2$, which is computed in (6.21);
- the contribution to K^2 of E, as if E had only global normal crossings, i.e.:
$$(K_{P_1} + A_1 + E_1)^2 + (K_{P_2} + A_2 + E_4)^2 + (K_Q + A_1 + A_2 + E_2 + E_3)^2 = 2$$
- the contribution of the R_3-point x', which is $c_{x'} = 1$ by Claim 6.17.

Putting all this together, it follows that $c_x = 1$ in this case. Consider now the latter possibility, i.e. suppose that the quadric meets only one plane. We can compute the three contributions to K^2 as above: the contribution of $(K_{X'} + \Gamma)^2$ and of the R_3-point x' do not change, whereas the contribution to K^2 of E, as if E had only global normal crossings, is:

$$(K_Q + A_1 + E_1 + E_2)^2 + (K_{P_1} + A_1 + A_2 + E_3)^2 + (K_{P_4} + A_3 + E_4)^2 = 1,$$

therefore we find that $c_x = 2$, which concludes the proof for case (ii).

In case (iii), we use the same strategy as in case (ii), namely we add up $(K_{X'} + \Gamma)^2$, the contribution to K^2 of E, as if E had only global normal crossings, which turns out to be 2, and then subtract 2, because of the contribution of the two R_3-points x', x''. Summing up, one finds $c_x = 2$ in this case.

In case (iv), we have to repeat the process by blowing-up x', see Figure 16. After finitely many steps (cf. e.g. Proposition 3.4.13 in [26]), this procedure stops in the sense that the exceptional divisor will be as in case (i), (ii) or (iii).

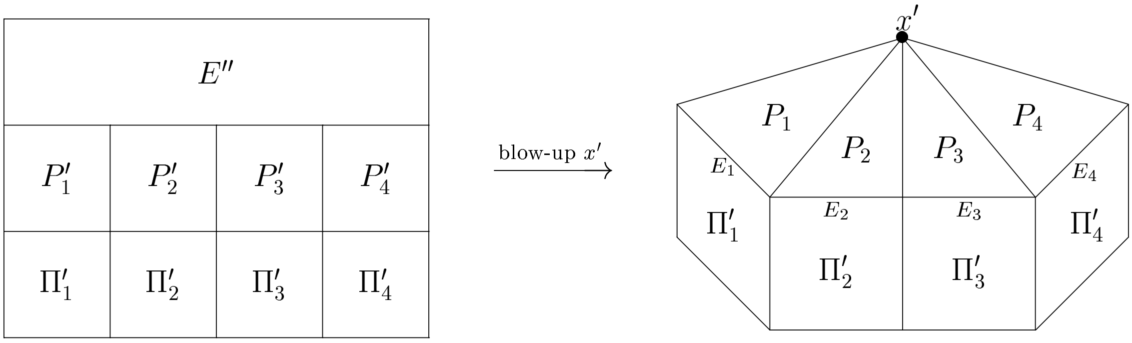


FIGURE 16. Blowing-up a R_4-point x' infinitely near to x.

In order to conclude the proof of Claim 6.20, one has to remark that no contribution to K^2 comes from the surfaces created in the intermediate steps (the blown-up planes P'_i in Figure 16). This can be done exactly in the same way as we did in the proof of Claim 6.17. $\square$

Remark 6.22. The proof of Claim 6.20 is purely combinatorial. However there is a nice geometric motivation for the two cases $c_x = 2$ and $c_x = 1$, when x is a R_4-point, which resides in the fact that the local deformation space of a R_4-point is reducible. This corresponds to the fact that the cone over C_{R_4} can be smoothed in both a Veronese surface and a rational normal quartic scroll, which have $K^2 = 9$ and $K^2 = 8$, respectively.

Consider now the case that x is a R_n-point.

Claim 6.23. *If x is a R_n-point, then*

$$n - 2 \geqslant c_x \geqslant 1. \tag{6.24}$$

Proof of the claim. The claim for $n = 3, 4$ has already been proved, so we assume $n \geqslant 5$ and proceed by induction on n. As usual, we blow-up the point $x \in \mathcal{X}$.

By Corollary 5.27, the exceptional divisor E is a non-degenerate surface of minimal degree in $\mathbb{P}^{n+1}$ with $\Gamma = E_1 + \ldots + E_n$ as a hyperplane section. By Proposition 5.23, E is reducible and the following cases may occur:

(i) E consists of $\nu \geqslant 3$ irreducible components $P_1, \ldots, P_\nu$, which are either planes or smooth quadrics, and E has h Zappatic singular points $x_1, \ldots, x_h$ of type $R_{m_1}, \ldots, R_{m_h}$ such that $m_i < n$, $i = 1, \ldots, h$;

(ii) E has one R_n-point x', in which case E consists of n planes, i.e. a planar Zappatic surface whose associated graph is an open n-face.

In case (ii), one has to repeat the process by blowing-up x'. After finitely many steps (cf. e.g. Proposition 3.4.13 in [26]), the exceptional divisor will necessarily be as in case (i). We remark that no contribution to K^2 comes from the surfaces created in the intermediate steps, as one can prove exactly in the same way as we did in the proof of Claim 6.17.

Thus, it suffices to prove the statement for the case (i). Notice that $\mathcal{X}'$ is not Gorenstein, nonetheless we can compute K^2 since we know (the upper and lower bounds of) the contribution of x_i by induction. We can indeed proceed as in case (ii) of the proof of Claim 6.20, namely, we have to add up three quantities:

- the contribution of $(K_{X'} + \Gamma)^2$;
- the contribution to K^2 of E, as if E had only global normal crossings;
- the contributions of the points x_i which is known by induction.

Let us compute these contributions. As for the first one, one has:

$$(K_{X'}+\Gamma)^2=\sum_{i=1}^{n}(K_{X'|\Pi'_i}+E_i)^2+\sum_{j\geqslant n}K^2_{X'|\Pi'_j}=\sum_{j\geqslant 1}(w_j-3)^2-2,$$

since the computations on the blown-up planes $\Pi'_1,\dots,\Pi'_n$ give:

$$(K_{X'|\Pi'_i}+E_i)^2=(w_i-3)^2-1,\quad i=1,n,$$
$$(K_{X'|\Pi'_i}+E_i)^2=(w_i-3)^2,\quad 2\leqslant i\leqslant n-1.$$

In order to compute the second contribution, one has to introduce some notation, precisely we let:

- $P_1,\dots,P_\nu$ be the irreducible components of E, which are either planes or smooth quadrics, ordered in such a way that the intersections in codimension one are as follows: P_i meets P_{i+1}, $i=1,\dots,\nu-1$, along a line;
- A_i be the line which is the intersection of P_i and P_{i+1};
- $\varepsilon_i=\deg(P_i)-1$, which is 0 if P_i is a plane and 1 if P_i is a quadric;
- $j(i)=i+\sum_{k=1}^{i-1}\varepsilon_j$. With this notation, if P_i is a plane, it meets the blown-up plane $\Pi'_{j(i)}$ along $E_{j(i)}$, whereas if P_i is a quadric, it meets the blown-up planes $\Pi'_{j(i)}$ and $\Pi'_{j(i)+1}$ along $E_{j(i)}$ and $E_{j(i)+1}$, respectively.

Then the contribution to K^2 of E, as if E had only global normal crossings, is:

$$(K_{P_1}+A_1+E_1+\varepsilon_1E_2)^2+(K_{P_\nu}+A_{\nu-1}+\varepsilon_\nu E_{n-1}+E_n)^2+$$
$$+\sum_{i=2}^{\nu-1}(K_{P_i}+A_{i-1}+A_i+E_{j(i)}+\varepsilon_iE_{j(i)+1})^2=2-\varepsilon_1-\varepsilon_\nu.$$

Finally, by induction, the contribution $\sum_{i=1}^{h}c_{x_i}$ of the points x_i is such that:

$$\nu-2=\sum_{i=1}^{h}(m_i-2)\geqslant\sum_{i=1}^{h}c_{x_i}\geqslant\sum_{i=1}^{h}1=h,$$

where the first equality is just (5.24).

Putting all this together, it follows that:

$$c_x=\varepsilon_1+\varepsilon_\nu+\sum_{i=1}^{h}c_{x_i},$$

hence an upper bound for c_x is

$$c_x\leqslant\varepsilon_1+\varepsilon_\nu+\nu-2\leqslant n-2,$$

because $n=\nu+\sum_{i=1}^{\nu}\varepsilon_i$, whereas a lower bound is

$$c_x\geqslant\varepsilon_1+\varepsilon_\nu+h\geqslant h\geqslant 1,\tag{6.25}$$

which concludes the proof of Claim 6.23. □

Remark 6.26. If $c_x=1$, then in (6.25) all inequalities must be equalities, thus $h=1$ and $\varepsilon_1=\varepsilon_\nu=0$. This means that there is only one point x_1 infinitely near to x, of type R_ν, and that the external irreducible components of E, i.e. P_1 and P_ν, are planes. There is no combinatorial obstruction to this situation.

For example, let x be a R_n-point such that the exceptional divisor E consists of $\nu=n-1$ irreducible components, namely $n-2$ planes and a quadric adjacent to two planes, forming a R_{n-1}-point x'. By the proof of Claim 6.20 (case (ii), former possibility), it follows that

$c_x = c_{x'}$. Since, as we saw, the contribution of an R_4-point can be 1, by induction we may have that also a R_n-point contributes by 1.

From the proof of Claim 6.23, it follows that the upper bound $c_x = n-2$ is attained when for example the exceptional divisor E consists of n planes forming $n-2$ points of type R_3.

More generally, one can see that there is no combinatorial obstruction for c_x to attain any possible value between the upper and lower bounds in (6.24).

Finally, consider the case that x is of type S_n.

Claim 6.27. *If x is a S_n-point, then*

$$(n-2)^2 \geqslant c_x \geqslant \binom{n-1}{2}. \tag{6.28}$$

Proof. We remark that we do not need to take care of 1-dimensional singularities of the total space of the degeneration, as we have already noted in Claim 6.23.

Notice that $S_3 = R_3$ and, for $n = 3$, formula (6.28) trivially follows from Claim 6.17. So we assume $n \geqslant 4$. Blow-up x, as usual; let $\mathfrak{X}'$ be the new total space and E the exceptional divisor. By Proposition 5.23, three cases may occur: either

(i) E has global normal crossings, i.e. E is the union of a smooth rational normal scroll $X_1 = S(1, d-1)$ of degree d, $2 \leqslant d \leqslant n$, and of $n-d$ disjoint planes $P_1, \ldots, P_{n-d}$, each meeting X_1 along different lines of the same ruling; or

(ii) E is a union of n planes $P_1, \ldots, P_n$ with h Zappatic singular points $x_1, \ldots, x_h$ of type $S_{m_1}, \ldots, S_{m_h}$ such that $3 \leqslant m_i < n$, $i = 1, \ldots, h$, and (5.25) holds; or

(iii) E is a union of n planes with one S_n-point x'.

In case (iii), one has to repeat the process by blowing-up x'. After finitely many steps (cf. e.g. Proposition 3.4.13 in [26]), the exceptional divisor will necessarily be as in cases either (i) or (ii). We remark that no contribution to K^2 comes from the surfaces created in the intermediate steps. Indeed, by using the same notation of the R_n-case in Claim 6.17, if x is a S_n point and if Π_1 is the plane corresponding to the vertex of valence $n-1$ in the associated graph, we have (cf. Figure 17):

$$(K_{P'_1} + \Lambda_1 + A_1 + (n-1)(\Lambda_1 - A_1))^2 = (n-3)^2 - (n-3)^2 = 0,$$
$$(K_{P'_i} + \Lambda_i + A_i + (\Lambda_i - A_i))^2 = 1 - 1 = 0, \ \ 2 \leqslant i \leqslant n.$$

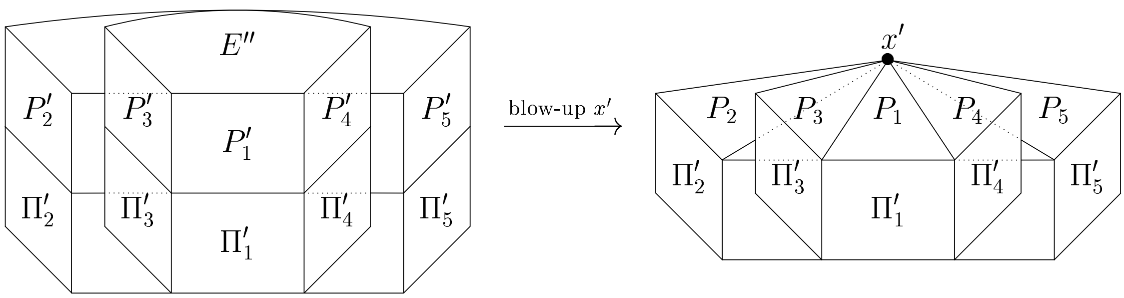


FIGURE 17. Blowing-up a S_5-point x' infinitely near to a S_5-point x

Thus, it suffices to prove the statement for the first two cases (i) and (ii).

Consider the case (i), namely E has global normal crossings. Then $\mathfrak{X}'$ is Gorenstein and we may compute K^2 as in (6.18). The contribution of the blown-up planes $\Pi'_1, \ldots, \Pi'_n$ (choosing again the indexes in such a way that Π'_1 meets $\Pi'_2, \dots, \Pi'_n$ in a line) is:

$$
\begin{aligned}
(K_{X'|\Pi'_i} + E_i)^2 &= (w_i - 3)^2 - 1, \qquad i = 2, \dots, n,\\
(K_{X'|\Pi'_1} + E_1)^2 &= (w_1 - 3)^2 - (n-3)^2,
\end{aligned}
\tag{6.29}
$$

whereas the contribution of E turns out to be:

$$(K_E + \Gamma)^2 = 4 - n. \tag{6.30}$$

Indeed, one finds that:

$$
\begin{aligned}
\big((K_E + \Gamma)_{|X_1}\big)^2 &= (-A + (n-d-1)F)^2 = d + 4 - 2n,\\
\big((K_E + \Gamma)_{|P_i}\big)^2 &= 1, \quad i = 1, \dots, n-d,
\end{aligned}
$$

where A is the linear directrix of X_1 and F is its fibre, therefore (6.30) holds. Summing up, it follows that

$$c_x = n - 4 + (n-1) + (n-3)^2 = (n-2)^2, \tag{6.31}$$

which proves (6.28) in this case (i).

In case (ii), E is not Gorenstein, nonetheless we can compute K^2 since we know (the upper and lower bounds of) the contribution of x_i by induction. We can indeed proceed as in case (ii) of the proof of Claim 6.20, namely, we have to add up three quantities:

- the contribution of $(K_{X'} + \Gamma)^2$, which has been computed in (6.29);
- the contribution to K^2 of E, as if E had only global normal crossings, which is:
$$\Big(K_{P_1} + E_1 + \sum_{i=2}^{n} A_i\Big)^2 + \sum_{i=2}^{n} (K_{P_i} + E_i + A_i)^2 = (n-3)^2 + n - 1,$$
where Π'_1 is the blown-up plane meeting all the other blown-up planes in a line, E_i is the exceptional curve on Π'_i and A_i is the double line intersection of P_1 with P_i;
- the contribution $\sum_{i=1}^{h} c_{x_i}$ of the points x_i, which by induction, is such that:
$$\sum_{i=1}^{h} (m_i - 2)^2 \geqslant \sum_{i=1}^{h} c_{x_i} \geqslant \sum_{i=1}^{h} \binom{m_i - 1}{2} = \binom{n-1}{2}, \tag{6.32}$$
where the last equality is just (5.25).

Putting all together, one sees that

$$c_x = \sum_{i=1}^{h} c_{x_i},$$

hence (6.32) gives the claimed lower bound, as for the upper bound:

$$
\begin{aligned}
c_x &\leqslant \sum_{i=1}^{h} (m_i - 2)^2 = \sum_{i=1}^{h} (m_i - 1)(m_i - 2) - \sum_{i=1}^{h} (m_i - 2) \overset{(*)}{=}\\
&\overset{(*)}{=} (n-1)(n-2) - \sum_{i=1}^{h} (m_i - 2) \leqslant (n-1)(n-2) - (n-2) = (n-2)^2,
\end{aligned}
$$

where the equality $(*)$ follows from (5.25). This completes the proof of Claim 6.27. $\square$

The above Claims 6.23 and 6.27 prove Proposition 6.16 and, so, Theorem 6.1. $\square$

Remark 6.33. Notice that the upper bound $c_x = (n-2)^2$ is attained when for example the exceptional divisor E has global normal crossings (cf. case (i) in Claim 6.27). The lower bound $c_x = \binom{n-1}{2}$ can be attained if the exceptional divisor E consists of n planes forming $\binom{n-1}{2}$ points of type $S_3 = R_3$.

Contrary to what happens for the R_n-points, not all the values between the upper and the lower bound are realised by c_x, for a S_n-point x. Indeed they are not even combinatorially possible. For example, there are combinatorial obstructions for a S_6-point x to have $c_x = 15$.

7. The Multiple Point Formula

The aim of this section is to prove a fundamental inequality, which involves the Zappatic singularities of a given good Zappatic surface X (see Theorem 7.2), under the hypothesis that X is the central fibre of a good Zappatic degeneration as in Definition 4.2. This inequality can be viewed as an extension of the well-known Triple Point Formula (see Lemma 7.7 and cf. [15]), which holds only for semistable degenerations. As corollaries, we will obtain, among other things, the main result contained in Zappa's paper [49] (cf. Section 8).

Let us introduce some notation.

Notation 7.1. Let X be a good Zappatic surface. We denote by:

- $\gamma = X_1 \cap X_2$ the intersection of two irreducible components X_1, X_2 of X;
- F_γ the divisor on γ consisting of the E_3-points of X along γ;
- $f_n(\gamma)$ the number of E_n-points of X along γ; in particular, $f_3(\gamma) = \deg(F_\gamma)$;
- $r_n(\gamma)$ the number of R_n-points of X along γ;
- $s_n(\gamma)$ the number of S_n-points of X along γ;
- $\rho_n(\gamma) := r_n(\gamma) + s_n(\gamma)$, for $n \geqslant 4$, and $\rho_3(\gamma) = r_3(\gamma)$.

If X is the central fibre of a good Zappatic degeneration $\mathfrak{X} \to \Delta$, we denote by:

- D_γ the divisor of γ consisting of the double points of $\mathfrak{X}$ along γ off the Zappatic singularities of X;
- $d_\gamma = \deg(D_\gamma)$;
- $d_{\mathfrak{X}}$ the total number of double points of $\mathfrak{X}$ off the Zappatic singularities of X.

The main result of this section is the following:

Theorem 7.2 (Multiple Point Formula)**.** *Let X be a surface which is the central fibre of a good Zappatic degeneration $\mathfrak{X} \to \Delta$. Let $\gamma = X_1 \cap X_2$ be the intersection of two irreducible components X_1, X_2 of X. Then*

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(\rho_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{7.3}$$

In the planar case, one has:

Corollary 7.4. *Let X be a surface which is the central fibre of a good, planar Zappatic degeneration $\mathfrak{X} \to \Delta$. Let γ be a double line of X. Then*

$$2 + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(\rho_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{7.5}$$

Therefore:

$$2e + 3f_3 - 2r_3 - \sum_{n\geqslant 4} n f_n - \sum_{n\geqslant 4}(n-1)\rho_n \geqslant d_{\mathfrak{X}} \geqslant 0. \tag{7.6}$$

As for Theorem 6.1, the proof of Theorem 7.2 will be done in several steps, the first of which is the classical:

Lemma 7.7 (Triple Point Formula)**.** *Let X be a good Zappatic surface with global normal crossings, which is the central fibre of a good Zappatic degeneration with smooth total space $\mathcal{X}$. Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X. Then:*

$$\mathcal{N}_{\gamma|X_1} \otimes \mathcal{N}_{\gamma|X_2} \otimes \mathcal{O}_\gamma(F_\gamma) \cong \mathcal{O}_\gamma. \tag{7.8}$$

In particular,

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = 0. \tag{7.9}$$

Proof. By Definition 4.2, since the total space $\mathcal{X}$ is assumed to be smooth, the Zappatic degeneration $\mathcal{X} \to \Delta$ is semistable. Let $X = \bigcup_{i=1}^v X_i$. Since X is a Cartier divisor in $\mathcal{X}$ which is a fibre of the morphism $\mathcal{X} \to \Delta$, then $\mathcal{O}_X(X) \cong \mathcal{O}_X$. Tensoring by $\mathcal{O}_\gamma$ gives $\mathcal{O}_\gamma(X) \cong \mathcal{O}_\gamma$. Thus,

$$\mathcal{O}_\gamma \cong \mathcal{O}_\gamma(X_1) \otimes \mathcal{O}_\gamma(X_2) \otimes \mathcal{O}_\gamma(Y), \tag{7.10}$$

where $Y = \cup_{i=3}^v X_i$. One concludes by observing that in (7.10) one has $\mathcal{O}_\gamma(X_i) \cong \mathcal{N}_{\gamma|X_{3-i}}$, $1 \leqslant i \leqslant 2$, and $\mathcal{O}_\gamma(Y) \cong \mathcal{O}_\gamma(F_\gamma)$. □

It is useful to consider the following slightly more general situation. Let X be a union of surfaces such that X_{red} is a good Zappatic surface with global normal crossings. Then $X_{\mathrm{red}} = \cup_{i=1}^v X_i$ and let m_i be the multiplicity of X_i in X, $i = 1, \ldots, v$. Let $\gamma = X_1 \cap X_2$ be the intersection of two irreducible components of X. For every point p of γ, we define the weight $w(p)$ of p as the multiplicity m_i of the component X_i such that $p \in \gamma \cap X_i$.

Of course $w(p) \neq 0$ only for E_3-points of X_{red} on γ. Then we define the divisor F_γ on γ as

$$F_\gamma := \sum_p w(p)p.$$

The same proof of Lemma 7.7 shows the following:

Lemma 7.11 (Generalized Triple Point Formula)**.** *Let X be a surface such that $X_{\mathrm{red}} = \cup_i X_i$ is a good Zappatic surface with global normal crossings. Let m_i be the multiplicity of X_i in X. Assume that X is the central fibre of a degeneration $\mathcal{X} \to \Delta$ with smooth total space $\mathcal{X}$. Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X_{red}. Then:*

$$\mathcal{N}_{\gamma|X_1}^{\otimes m_2} \otimes \mathcal{N}_{\gamma|X_2}^{\otimes m_1} \otimes \mathcal{O}_\gamma(F_\gamma) \cong \mathcal{O}_\gamma. \tag{7.12}$$

In particular,

$$m_2 \deg(\mathcal{N}_{\gamma|X_1}) + m_1 \deg(\mathcal{N}_{\gamma|X_2}) + \deg(F_\gamma) = 0. \tag{7.13}$$

The second step is given by the following result:

Proposition 7.14. *Let X be a good Zappatic surface with global normal crossings, which is the central fibre of a good Zappatic degeneration $\mathcal{X} \to \Delta$. Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X. Then:*

$$\mathcal{N}_{\gamma|X_1} \otimes \mathcal{N}_{\gamma|X_2} \otimes \mathcal{O}_\gamma(F_\gamma) \cong \mathcal{O}_\gamma(D_\gamma). \tag{7.15}$$

In particular,

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = d_\gamma. \tag{7.16}$$

Proof. By the very definition of good Zappatic degeneration, the total space $\mathcal{X}$ is smooth except for ordinary double points along the double locus of X, which are not the E_3-points of X. We can modify the total space $\mathcal{X}$ and make it smooth by blowing-up its double points.

Since the computations are of local nature, we can focus on the case of $\mathcal{X}$ having only one double point p on γ. We blow-up the point p in $\mathcal{X}$ to get a new total space $\mathcal{X}'$, which is smooth.

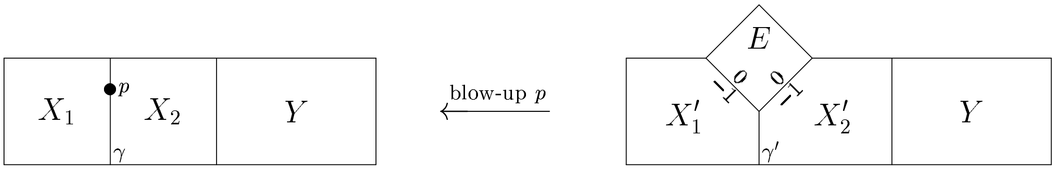


FIGURE 18. Blowing-up an ordinary double point of $\mathcal{X}$

Notice that, according to our hypotheses, the exceptional divisor $E := E_{\mathcal{X},p} = \mathbb{P}(T_{\mathcal{X},p})$ is isomorphic to a smooth quadric in $\mathbb{P}^3$ (see Figure 18).

The proper trasform of X is:

$$X' = X_1' + X_2' + Y$$

where X_1', X_2' are the proper transforms of X_1, X_2, respectively. Let γ' be the intersection of X_1' and X_2', which is clearly isomorphic to γ. Let p_1 be the intersection of γ' with E.

Since $\mathcal{X}'$ is smooth, we can apply Lemma 7.11 to γ'. Therefore, by (7.12), we get

$$\mathcal{O}_{\gamma'} \cong \mathcal{N}_{\gamma'|X_1'} \otimes \mathcal{N}_{\gamma'|X_2'} \otimes \mathcal{O}_{\gamma'}(F_{\gamma'}).$$

In the isomorphism between γ' and γ, one has:

$$\mathcal{O}_{\gamma'}(F_{\gamma'} - p_1) \cong \mathcal{O}_{\gamma}(F_{\gamma}), \quad \mathcal{N}_{\gamma'|X_i'} \cong \mathcal{N}_{\gamma|X_i} \otimes \mathcal{O}_{\gamma}(-p), \quad 1 \leqslant i \leqslant 2.$$

Putting all this together, one has the result. $\square$

Taking into account Lemma 7.11, the same proof of Proposition 7.14 gives the following result:

Corollary 7.17. *Let X be a surface such that $X_{\mathrm{red}} = \cup_i X_i$ is a good Zappatic surface with global normal crossings. Let m_i be the multiplicity of X_i in X. Assume that X is the central fibre of a degeneration $\mathcal{X} \to \Delta$ with total space $\mathcal{X}$ having at most ordinary double points outside the Zappatic singularities of X_{red}.*

Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X_{red}. Then:

$$\mathcal{N}_{\gamma|X_1}^{\otimes m_2} \otimes \mathcal{N}_{\gamma|X_2}^{\otimes m_1} \otimes \mathcal{O}_{\gamma}(F_{\gamma}) \cong \mathcal{O}_{\gamma}(D_{\gamma})^{\otimes(m_1+m_2)}. \tag{7.18}$$

In particular,

$$m_2 \deg(\mathcal{N}_{\gamma|X_1}) + m_1 \deg(\mathcal{N}_{\gamma|X_2}) + \deg(F_{\gamma}) = (m_1 + m_2) d_{\gamma}. \tag{7.19}$$

Now we can come to the:

Proof of Theorem 7.2. Recall that, by Definition 4.2 of Zappatic degenerations, the total space $\mathcal{X}$ has only isolated singularities. We want to apply Corollary 7.17 after having resolved the singularities of the total space $\mathcal{X}$ at the Zappatic singularities of the central fibre X, i.e. at the R_n-points of X, for $n \geqslant 3$, and at the E_n- and S_n-points of X, for $n \geqslant 4$.

Now we briefly describe the resolution process, which will become even clearer in the second part of the proof, when we will enter into the details of the proof of formula (7.3).

Following the blowing-up process at the R_n- and S_n-points of the central fibre X, as described in Section 6, one gets a degeneration such that the total space is Gorenstein, with isolated singularities, and the central fibre is a Zappatic surface with only E_n-points.

The degeneration will not be Zappatic, if the double points of the total space occurring along the double curves, off the Zappatic singularities, are not ordinary. According to our hypotheses, this cannot happen along the proper transform of the double curves of the original central fibre. All these non-ordinary double points can be resolved with finitely many subsequent blow-ups and they will play no role in the computation of formula (7.3).

Recall that the total space $\mathcal{X}$ is smooth at the E_3-points of the central fibre, whereas $\mathcal{X}$ has multiplicity either 2 or 4 at an E_4-point of X. Thus, we can consider only E_n-points $p \in X$, for $n \geqslant 4$.

By Proposition 5.17, p is a quasi-minimal singularity for $\mathcal{X}$, unless $n = 4$ and $\mathrm{mult}_p(\mathcal{X}) = 2$. In the latter case, this singularity is resolved by a sequence of blowing-ups at isolated double points.

Assume now that p is a quasi-minimal singularity for $\mathcal{X}$. Let us blow-up $\mathcal{X}$ at p and let E' be the exceptional divisor. Since a hyperplane section of E' is C_{E_n}, the possible configurations of E' are described in Proposition 5.23, (iii).

If E' is irreducible, that is case (iii.a) of Proposition 5.23, then E' has at most isolated rational double points, where the new total space is either smooth or it has a double point. This can be resolved by finitely many blowing-ups at analogous double points.

Suppose we are in case (iii.b) of Proposition 5.23. If E' has global normal crossings, then the desingularization process proceeds exactly as before.

If E' does not have global normal crossings then, either E' has a component which is a quadric cone or the two components of E' meet along a singular conic. In the former case, the new total space has a double point at the vertex of the cone. In the latter case, the total space is either smooth or it has an isolated double point at the singular point of the conic. In either cases, one resolves the singularities by a sequence of blowing-ups as before.

Suppose finally we are in case (iii.c) of Proposition 5.23, i.e. the new central fibre is a Zappatic surface with one point p' of type E_m, with $m \leqslant n$. Then we can proceed by induction on n. Note that if an exceptional divisor has an E_3-point p'', then p'' is either a smooth, or a double, or a triple point for the total space. In the latter two cases, we go on by blowing-up p''. After finitely many blow-ups (by Definition 4.2, cf. Proposition 3.4.13 in [26]), we get a central fibre which might be non-reduced, but its support has only global normal crossings, and the total space has at most ordinary double points off the E_3-points of the reduced part of the central fibre.

Now we are in position to apply Corollary 7.17. In order to do this, we have to understand the relations between the invariants of a double curve of the original Zappatic surface X and the invariants appearing in formula (7.19) for the double curve of the strict transform of X.

Since all the computations are of local nature, we may assume that X has a single Zappatic singularity p, which is not an E_3-point. We will prove the theorem in this case. The general formula will follow by iterating these considerations for each Zappatic singularity of $\mathcal{X}$.

Let X_1, X_2 be irreducible components of X containing p and let γ be their intersection. As we saw in the above resolution process, we blow-up $\mathcal{X}$ at p. We obtain a new total space $\mathcal{X}'$, with the exceptional divisor $E' := E_{\mathcal{X},p} = \mathbb{P}(T_{\mathcal{X},p})$ and the proper transform X_1', X_2' of X_1, X_2. Let γ' be the intersection of X_1', X_2'. We remark that $\gamma' \cong \gamma$ (see Figure 19).

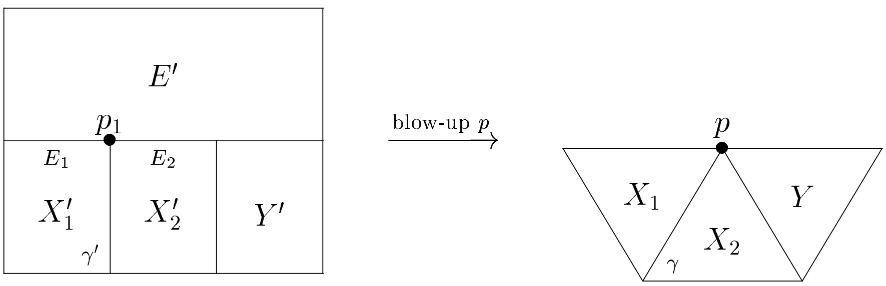


FIGURE 19. Blowing-up $\mathcal{X}$ at p

Notice that $\mathcal{X}'$ might have Zappatic singularities off γ'. These will not affect our considerations. Therefore, we can assume that there are no singularities of $\mathcal{X}'$ of this sort. Thus, the only point of $\mathcal{X}'$ we have to take care of is $p_1 := E' \cap \gamma'$.

If p_1 is smooth for E', then it must be smooth also for $\mathcal{X}'$. Moreover, if p_1 is singular for E', then p_1 is a double point of E' as it follows from the above resolution process and from Proposition 5.23. Therefore, p_1 is at most double also for $\mathcal{X}'$; since p_1 is a quasi-minimal, Gorenstein singularity of multiplicity 4 for the central fibre of $\mathcal{X}'$, then p_1 is a double point of $\mathcal{X}'$ by Proposition 5.17.

Thus there are two cases to be considered: either

(i) p_1 is smooth for both E' and $\mathcal{X}'$, or
(ii) p_1 is a double point for both E' and $\mathcal{X}'$.

In case (i), the central fibre of $\mathcal{X}'$ is $\mathcal{X}'_0 = X'_1 \cup X'_2 \cup Y' \cup E'$ and we are in position to use the enumerative information (7.16) from Proposition 7.14 which reads:

$$\deg(\mathcal{N}_{\gamma'|X'_1}) + \deg(\mathcal{N}_{\gamma'|X'_2}) + f_3(\gamma') = d_{\gamma'}.$$

Observe that $f_3(\gamma')$ is the number of E_3-points of the central fibre $\mathcal{X}'_0$ of $\mathcal{X}'$ along γ', therefore

$$f_3(\gamma') = f_3(\gamma) + 1.$$

On the other hand:

$$\deg(\mathcal{N}_{\gamma'|X'_i}) = \deg(\mathcal{N}_{\gamma|X_i}) - 1, \ \ 1 \leqslant i \leqslant 2.$$

Finally,

$$d_\gamma = d_{\gamma'}$$

and therefore we have

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - 1 = d_\gamma \tag{7.20}$$

which proves the theorem in this case (i).

Consider now case (ii), i.e. p_1 is a double point for both E' and $\mathcal{X}'$.

If p_1 is an ordinary double point for $\mathcal{X}'$, we blow-up $\mathcal{X}'$ at p_1 and we get a new total space $\mathcal{X}''$. Let X''_1, X''_2 be the proper transforms of X'_1, X'_2, respectively, and let γ'' be the intersection of X''_1 and X''_2, which is isomorphic to γ. Notice that $\mathcal{X}''$ is smooth and the exceptional divisor E'' is a smooth quadric (see Figure 20).

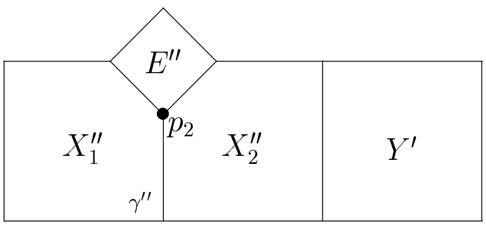


FIGURE 20. Blowing-up $\mathcal{X}'$ at p_1 when p_1 is ordinary for both $\mathcal{X}'$ and E'

We remark that the central fibre of $\mathcal{X}''$ is now non-reduced, since it contains E'' with multiplicity 2. Thus we apply Corollary 7.17 and we get

$$\mathcal{O}_{\gamma''} \cong \mathcal{N}_{\gamma''|X''_1} \otimes \mathcal{N}_{\gamma''|X''_2} \otimes \mathcal{O}_{\gamma''}(F_{\gamma''}).$$

Since,

$$\deg(\mathcal{N}_{\gamma''|X''_1}) = \deg(\mathcal{N}_{\gamma|X_i}) - 2, \ i = 1, 2, \qquad \deg F_{\gamma''} = f_3(\gamma) + 2,$$

then,

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - 1 = d_\gamma + 1 > d_\gamma. \tag{7.21}$$

If the point p_1 is not an ordinary double point, we again blow-up p_1 as above. Now the exceptional divisor E'' of $\mathcal{X}''$ is a singular quadric in $\mathbb{P}^3$, which can only be either a quadric cone or it has to consist of two distinct planes E_1'', E_2''. Note that if p_1 lies on a double line of E' (i.e. p_1 is in the intersection of two irreducible components of E'), then only the latter case occurs since E'' has to contain a curve C_{E_4}.

Let $p_2 = E'' \cap \gamma''$. In the former case, if p_2 is not the vertex of the quadric cone, then the total space $\mathcal{X}''$ is smooth at p_2 and we can apply Corollary 7.17 and we get (7.21) as before.

If p_2 is the vertex of the quadric cone, then p_2 is a double point of $\mathcal{X}''$ and we can go on blowing-up $\mathcal{X}''$ at p_2. This blow-up procedure stops after finitely many, say h, steps and one sees that formula (7.21) has to be replaced by

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - 1 = d_\gamma + h > d_\gamma. \tag{7.22}$$

In the latter case, i.e. if E'' consists of two planes E_1'' and E_2'', let λ be the intersection line of E_1'' and E_2''. If p_2 does not belong to λ (see Figure 21), then p_2 is a smooth point of the total space $\mathcal{X}''$, therefore we can apply Corollary 7.17 and we get again formula (7.21).

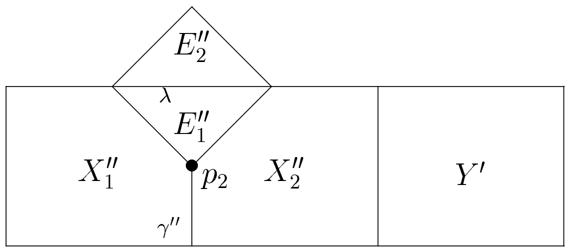


FIGURE 21. E'' splits in two planes E_1'', E_2'' and $p_2 \notin E_1'' \cap E_2''$

If p_2 lies on λ, then p_2 is a double point for the total space $\mathcal{X}''$ (see Figure 22). We can thus iterate the above procedure until the process terminates after finitely many, say h, steps by getting rid of the singularities which are infinitely near to p along γ. At the end, one gets again formula (7.22). $\square$

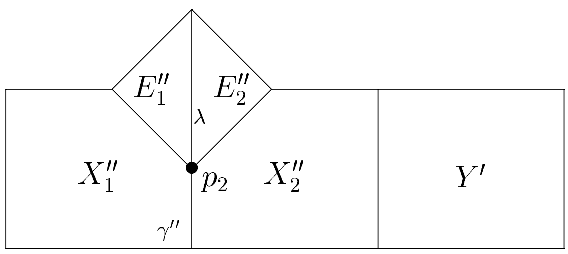


FIGURE 22. E''' splits in two planes E_1''', E_2''' and $p_3 \in E_1''' \cap E_2'''$

Remark 7.23. We observe that the proof of Theorem 7.2 proves a stronger result than what we stated in (7.3). Indeed, the idea of the proof is that we blow-up the total space $\mathcal{X}$ at each Zappatic singularity p in a sequence of singular points p, p_1, p_2, $\dots$, p_{h_p}, each infinitely near one to the other along γ. Note that p_i, $i = 1, \dots, h_p$, is a double point for the total space.

The above proof shows that the first inequality in (7.3) is an equality if and only if each Zappatic singularity of $\mathcal{X}$ has no infinitely near singular point. Moreover (7.22) implies that

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n \geqslant 4}(\rho_n(\gamma) + f_n(\gamma)) = d_\gamma + \sum_{p \in \gamma} h_p.$$

In other words, as natural, every infinitely near double point along γ counts as a double point of the original total space along γ.

8. On some results of Zappa

In [44, 45, 46, 47, 48, 49, 50], Zappa considered degenerations of projective surfaces to a planar Zappatic surface with only R_3-, S_4- and E_3-points. One of the results of Zappa's analysis is that the invariants of a surface admitting a good planar Zappatic degeneration with mild singularities are severely restricted. In fact, translated in modern terms, his main result in [49] can be read as follows:

Theorem 8.1 (Zappa). *Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration, where the central fibre $\mathcal{X}_0 = X$ has at most R_3- and E_3-points. Then, for $t \neq 0$, one has*

$$K^2 := K^2_{\mathcal{X}_t} \leqslant 8\chi + 1 - g, \tag{8.2}$$

where $\chi = \chi(\mathcal{O}_{\mathcal{X}_t})$ and g is the sectional genus of $\mathcal{X}_t$.

Theorem 8.1 has the following interesting consequence:

Corollary 8.3 (Zappa). *If $\mathcal{X}$ is a good, planar Zappatic degeneration of a scroll $\mathcal{X}_t$ of sectional genus $g \geqslant 2$ to $\mathcal{X}_0 = X$, then X has worse singularities than R_3- and E_3-points.*

Proof. For a scroll of genus g one has $8\chi + 1 - g - K^2 = 1 - g$. □

Actually Zappa conjectured that for most of the surfaces the inequality $K^2 \leqslant 8\chi + 1$ should hold and even proposed a plausibility argument for this. As well-known, the correct bound for all the surfaces is $K^2 \leqslant 9\chi$, proved by Miyaoka and Yau (see [31, 43]) several decades after Zappa.

We will see in a moment that Theorem 8.1 can be proved as consequence of the computation of K^2 (see Theorem 6.1) and the Multiple Point Formula (see Theorem 7.2).

Actually, Theorems 6.1 and 7.2 can be used to prove a stronger result than Theorem 8.1; indeed:

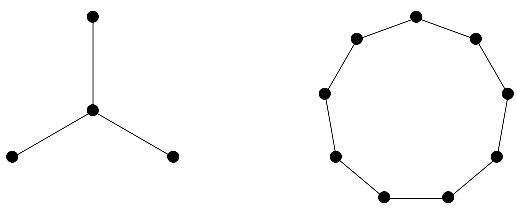
Theorem 8.4. *Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration, where the central fibre $\mathcal{X}_0 = X$ has at most R_3-, E_3-, E_4- and E_5-points. Then*

$$K^2 \leqslant 8\chi + 1 - g. \tag{8.5}$$

Moreover, the equality holds in (8.5) if and only if $\mathcal{X}_t$ is either the Veronese surface in $\mathbb{P}^5$ degenerating to 4 planes with associated graph S_4 (i.e. with three R_3-points, see Figure 23.a), or an elliptic scroll of degree $n \geqslant 5$ in $\mathbb{P}^{n-1}$ degenerating to n planes with associated graph a cycle E_n (see Figure 23.b).

Furthermore, if $\mathcal{X}_t$ is a surface of general type, then

$$K^2 < 8\chi - g. \tag{8.6}$$



(a) (b)

Figure 23

Proof. Notice that if X has at most R_3-, E_3-, E_4- and E_5-points, then formulas (6.3) and (6.5) give $K^2 = 9v - 10e + 6f_3 + 8f_4 + 10f_5 + r_3$. Thus, by (3.13) and (3.15), one gets

$$8\chi + 1 - g - K^2 = 8v - 8e + 8f_3 + 8f_4 + 8f_5 + 1 - (e - v + 1) - K^2 = e - r_3 + 2f_3 - 2f_5 =$$

$$= \frac{1}{2}(2e - 2r_3 + 3f_3 - 4f_4 - 5f_5) + \frac{1}{2}f_3 + 2f_4 + \frac{1}{2}f_5 \overset{(*)}{\geqslant} \frac{1}{2}f_3 + 2f_4 + \frac{1}{2}f_5 \geqslant 0$$

where the inequality $(*)$ follows from (7.6). This proves formula (8.5) (and Theorem 8.1).

If $K^2 = 8\chi + 1 - g$, then $(*)$ is an equality, hence $f_3 = f_4 = f_5 = 0$ and $e = r_3$. Therefore, by formula (3.17), we get

$$\sum_i w_i(w_i - 1) = 2r_3 = 2e, \tag{8.7}$$

where w_i denotes the valence of the vertex v_i in the graph G_X. By definition of valence, the right-hand-side of (8.7) equals $\sum_i w_i$. Therefore, we get

$$\sum_i w_i(w_i - 2) = 0. \tag{8.8}$$

If $w_i \geqslant 2$, for each $1 \leqslant i \leqslant v$, one easily shows that only the cycle as in Figure 23 (b) is possible. This gives

$$\chi = 0,\ K^2 = 0,\ g = 1,$$

which implies that $\mathfrak{X}_t$ is an elliptic scroll.

Easy combinatorial computations show that, if there is a vertex with valence $w_i \neq 2$, then there is exactly one vertex with valence 3 and three vertex of valence 1. Such a graph, with v vertices, is associated to a planar Zappatic surface of degree v in $\mathbb{P}^{v+1}$ with

$$\chi = 0,\ p_g = 0,\ g = 0.$$

Thus, by hypothesis, $K^2 = 9$ and, by properties of projective surfaces, the only possibility is that $v = 4$, G_X is as in Figure 23 (a) and $\mathfrak{X}_t$ is the Veronese surface in $\mathbb{P}^5$.

Suppose now that $\mathfrak{X}_t$ is of general type. Then $\chi \geqslant 1$ and $v = \deg(\mathfrak{X}_t) < 2g - 2$. Formulas (3.13) and (3.15) imply that $\chi = f - g + 1 \geqslant 1$, thus $f \geqslant g > v/2 + 1$. Clearly $v \geqslant 4$, hence $f \geqslant 3$. Proceeding as at the beginning of the proof, we have that:

$$8\chi - g - K^2 \geqslant \frac{1}{2}f_3 + 2f_4 + \frac{1}{2}f_5 - 1 \geqslant \frac{1}{2}f - 1 > 0,$$

or equivalently $K^2 < 8\chi - g$. □

Remark 8.9. By following the same argument of the proof of Theorem 8.4, one can list all the graphs and the corresponding smooth projective surfaces in the degeneration, for which $K^2 = 8\chi - g$. For example, one can find $\mathfrak{X}_t$ as a rational normal scroll of degree n in $\mathbb{P}^{n+1}$ degenerating to n planes with associated graph a chain R_n. On the other hand, one can also have a del Pezzo surface of degree 7 in $\mathbb{P}^7$.

Let us state some applications of Theorem 8.4.

Corollary 8.10. *If $\mathfrak{X}$ is a good, planar Zappatic degeneration of a scroll $\mathfrak{X}_t$ of sectional genus $g \geqslant 2$ to $\mathfrak{X}_0 = X$, then X has worse singularities than R_3-, E_3-, E_4- and E_5-points.*

Corollary 8.11. *If $\mathfrak{X}$ is a good, planar Zappatic degeneration of a del Pezzo surface $\mathfrak{X}_t$ of degree 8 in $\mathbb{P}^8$ to $\mathfrak{X}_0 = X$, then X has worse singularities than R_3-, E_3-, E_4- and E_5-points.*

Proof. Just note that $K^2 = 8$ and $\chi = g = 1$, thus $\mathfrak{X}_t$ satisfies the equality in (8.5). □

Corollary 8.12. *If $\mathfrak{X}$ is a good, planar Zappatic degeneration of a minimal surface of general type $\mathfrak{X}_t$ to $\mathfrak{X}_0 = X$ with at most R_3-, E_3-, E_4- and E_5-points, then*

$$g \leqslant 6\chi + 5.$$

Proof. It directly follows from (8.6) and Noether's inequality, i.e. $K^2 \geqslant 2\chi - 6$. □

Corollary 8.13. *If $\mathfrak{X}$ is a good planar Zappatic degeneration of a m-canonical surface of general type $\mathfrak{X}_t$ to $\mathfrak{X}_0 = X$ with at most R_3-, E_3-, E_4- and E_5-points, then*

(i) *$m \leqslant 6$;*
(ii) *if $m = 5, 6$, then $\chi = 3$, $K^2 = 1$;*
(iii) *if $m = 4$, then $\chi \leqslant 4$, $8\chi \geqslant 11K^2 + 2$;*
(iv) *if $m = 3$, then $\chi \leqslant 6$, $8\chi \geqslant 7K^2 + 2$;*
(v) *if $m = 2$, then $K^2 \leqslant 2\chi - 1$;*
(vi) *if $m = 1$, then $K^2 \leqslant 4\chi - 1$.*

Proof. Take $\mathfrak{X}_t = S$ to be m-canonical. First of all, by Corollary 8.12, we immediately get (i). Then, by formula (8.6), we get

$$8\chi - 2 \geqslant \frac{(m^2 + m + 2)}{2} K^2.$$

Thus, if m equals either 1 or 2, we find statements (v) and (vi).

Since S is of general type, by Noether's inequality we get

$$8\chi - 2 \geqslant (2\chi - 6)\frac{(m^2 + m + 2)}{2}.$$

This gives, for $m \geqslant 3$,

$$\chi \leqslant 3 + \frac{22}{(m^2 + m - 6)}$$

which, together with the above inequality, gives the other cases of the statement. □

It would be interesting to see whether the numerical cases listed in the above corollary can actually occur.

We remark that Corollary 8.10 implies in particular that one cannot hope to degenerate all surfaces to unions of planes with only global normal crossings, namely double lines and E_3-points; indeed, one needs at least E_n-points, for $n \geqslant 6$, or R_m-, S_m-points, for $m \geqslant 4$.

From this point of view, another important result of Zappa is the following:

Theorem 8.14 (Zappa). *For every $g \geqslant 2$ there are families of scrolls of sectional genus g with general moduli having a planar Zappatic degeneration with at most R_3-, S_4- and E_3-points.*

One of the key steps in Zappa's argument for the proof of Theorem 8.14 is the following nice result:

Proposition 8.15 (Zappa). *Let $C \subset \mathbb{P}^2$ be a general element of the Severi variety $V_{d,g}$ of irreducible curves of degree d and geometric genus g, with $d \geqslant 2g + 2$. Then C is the plane section of a scroll $S \subset \mathbb{P}^3$ which is not a cone.*

It is a natural question to ask which Zappatic singularities are needed in order to Zappatically degenerate as many smooth, projective surfaces as possible. Note that there are some examples (cf. [7]) of smooth projective surfaces S which certainly cannot be degenerated to Zappatic surfaces with E_n-, R_n-, or S_n-points, unless n is large enough.

However, given such a S, the next result — i.e. Proposition 8.16 — suggests that there might be a birational model of S which can be Zappatically degenerated to a surface with only R_3- and E_n-points, for $n \leqslant 6$.

Proposition 8.16. *Let $\mathcal{X} \to \Delta$ be a good planar Zappatic degeneration and assume that the central fibre X has at most R_3- and E_m-points, for $m \leqslant 6$. Then*

$$K^2 \leqslant 9\chi.$$

Proof. The bounds for K^2 in Theorem 6.1 give $9\chi - K^2 = 9v - 9e + \sum_{m=3}^{6} 9f_m - K^2$. Therefore, we get:

$$2(9\chi - K^2) \geqslant 2e + 6f_3 + 2f_4 - 2f_5 - 6f_6 - 2r_3 \tag{8.17}$$

If we plug (7.6) in (8.17), we get

$$2(9\chi - K^2) \geqslant (2e + 3f_3 - 4f_4 - 5f_5 - 6f_6 - 2r_3) + (3f_3 + 6f_4 + 3f_5),$$

where both summands on the right-hand-side are non-negative. □

In other words, Proposition 8.16 states that the Miyaoka-Yau inequality holds for a smooth projective surface S which can degenerate to a good planar Zappatic surface with at most R_3- and E_n-points, $3 \leqslant n \leqslant 6$.

Another interesting application of the Multiple Point Formula is given by the following remark.

Remark 8.18. Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration. Denote by δ the *class* of the general fibre $\mathcal{X}_t$ of $\mathcal{X}$, $t \neq 0$. By definition, δ is the degree of the dual variety of $\mathcal{X}_t$, $t \neq 0$. From Zeuthen-Segre (cf. [14] and [24]) and Noether's formula (cf. [20], page 600), it follows that:

$$\delta = \chi_{\mathrm{top}} + \deg(\mathcal{X}_t) + 4(g-1) = (9\chi - K^2) + 3f + e. \tag{8.19}$$

Therefore, (7.6) implies that:

$$\delta \geqslant 3f_3 + r_3 + \sum_{n\geqslant 4}(12-n)f_n + \sum_{n \geqslant 4}(n-1)\rho_n - k.$$

In particular, if X is assumed to have at most R_3- and E_3-points, then (8.19) becomes

$$\delta = (2e + 3f_3 - 2r_3) + (3f_3 + r_3),$$

where the first summand in the right-hand side is non-negative by the Multiple Point Formula; therefore, one gets

$$\delta \geqslant 3f_3 + r_3.$$

Zappa's original approach, indeed, was to compute δ and then to deduce formula (8.2) and Theorem 8.1 from this (cf. [44]).

In [7], we collect several examples of degenerations of smooth surfaces to planar Zappatic surfaces, namely:

(i) rational and ruled surfaces as well as abelian surfaces given by product of curves;
(ii) del Pezzo surfaces, rational normal scrolls and Veronese surface, by using some results from [33], [37], [38];
(iii) $K3$ surfaces, as in [9] and in [10];
(iv) complete intersections, giving a generalization of the approach of CM surfaces in $\mathbb{P}^4$ as in [17].

We also discuss some examples of non-smoothable Zappatic surfaces and we pose open questions on the existence of degenerations to planar Zappatic surfaces for other classes of surfaces like, e.g., *Enriques' surfaces*. For more details, the reader is referred to [7].

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