

DEGENERATIONS OF THE VERONESE AND APPLICATIONS

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INTRODUCTION

The classical *Hermite interpolation* technique approximates a given differentiable function defined in an open subset of \mathbb{R}^r with a polynomial of a given degree which takes at an assigned set points p_1, \dots, p_n , usually assumed to be sufficiently general, the same values as the function and of its derivatives up to order m_1, \dots, m_n respectively.

A geometric counterpart of Hermite interpolation is the following. Fix general points p_1, \dots, p_n in the complex projective space \mathbb{P}^r , and multiplicities m_1, \dots, m_n . We will denote by $\mathcal{L} = \mathcal{L}_{r,d}(m_1, \dots, m_n)$ the linear system of hypersurfaces of degree d in \mathbb{P}^r having multiplicity at least m_i at p_i for each $i = 1, \dots, n$, and we will employ exponential notation $\mathcal{L}_{r,d}(m_1^{e_1}, \dots, m_h^{e_h})$ for repeated multiplicities. A basic question is to determine the dimension of the system \mathcal{L} . In the 1-dimensional case $r = 1$ this is easy. Ruffini's theorem says that

$$\dim(\mathcal{L}) = \max\{-1, d - \sum_i m_i\}.$$

As soon as $r \geq 2$ this problem becomes very complicated and so far it is still unsolved in this generality. In the present paper will mainly deal with the planar

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case $r = 2$, and we will write $\mathcal{L}_d(m_1, \dots, m_n)$ rather than $\mathcal{L}_{2,d}(m_1, \dots, m_n)$ if there is no ambiguity. Then, the *virtual dimension* of \mathcal{L} is

$$v(\mathcal{L}) = d(d+3)/2 - \sum_i m_i(m_i+1)/2$$

and the *expected dimension* is

$$e(\mathcal{L}) = \max\{-1, v\}.$$

One has $\dim(\mathcal{L}) \geq e(\mathcal{L})$ and the system \mathcal{L} is called *special* if $\dim(\mathcal{L}) > e(\mathcal{L})$. A special system is not empty.

Consider the blow-up X of the plane at the points p_1, \dots, p_n . We abuse notation and denote by \mathcal{L} also the proper transform of \mathcal{L} to X . Suppose \mathcal{L} not empty, and assume that there is a (-1) -curve C on X such that $C \cdot \mathcal{L} \leq -2$. This forces C to be a multiple fixed curve in the system and it is easy to see that \mathcal{L} is special in this case. We will then say that \mathcal{L} is (-1) -*special*. The Harbourne-Hirschowitz Conjecture (see also Gimigliano [25]) says that a system is special if and only if it is (-1) -special (see [14],[26],[27], etc.) Related to this conjecture, but weaker, is Nagata's Conjecture: if $n > 9$ and $d^2 < nm^2$ then $\mathcal{L}_d(m^n)$ is empty (see [29]).

There has been a substantial amount of partial progress on these conjectures; let us briefly recall some of the results. The Harbourne-Hirschowitz Conjecture is true for $n \leq 9$ (Castelnuovo, 1891 [6]; Nagata, 1960 [32]; Gimigliano [25]; Harbourne 1986 [26]). The Harbourne-Hirschowitz Conjecture is true for $m_i \leq 7$ (S. Yang, 2004 [41]; results have been announced for $m_i \leq 11$ by M. Dumnicki and W. Jarnicki, 2005 [19].) The Harbourne-Hirschowitz Conjecture is true for $\mathcal{L}_d(m^n)$ if $m \leq 20$ (see [9]; results have been announced for $m \leq 42$ by M. Dumnicki [20]). Nagata proved in [29] that his conjecture is true for $n = k^2$ points and he deduced from this a counterexample to the fourteenth problem of Hilbert. More specifically, the Harbourne-Hirschowitz Conjecture is true for $n = k^2$ points (see Evain 2005 [22]; also see [15] and [35]).

The results of Ciliberto and Miranda mentioned above (see [12], [13], [14], [15]; see also [9]) have systematically exploited a degeneration of the plane to the union of two surfaces, one of which is a plane and the other a ruled surface. In this work we describe some toric degenerations of the Veronese (see §§2–3) which we find useful in similar applications to interpolation results. In particular, using such degenerations, we give in §§4 and 6 a new proof of the following:

Theorem 0.1. *The linear systems*

$$\mathcal{L}_d(2^n) \text{ for } d \geq 5; \quad \mathcal{L}_{km}(m^{k^2}), \quad \mathcal{L}_{km+1}(m^{k^2})$$

all have the expected dimension.

These are all known results. In particular the first assertion is the planar instance of a famous theorem of Alexander-Hirschowitz (see [1]). However our purpose in this article is to introduce the technique of degenerations of the Veronese, and to illustrate its efficacy. This technique has interesting relations with tropical geometry (see [18]). It can be usefully applied to higher dimensional projective spaces and, more generally, to interpolation problems on toric varieties. In fact it has been recently applied to give a new combinatorial proof of Alexander-Hirschowitz Theorem in \mathbb{P}^3 (see [4]). The relation of double points interpolation problems with the study of secant varieties to the Veronese varieties is well known (see §1 and [8]).

Section 5 contains some speculations on applications of degeneration techniques to the problem of computing the degree of secant varieties of the Veronese surfaces.

1. DOUBLE POINTS, THE VERONESE, SECANT VARIETIES

Consider $\mathcal{L}_d(2^n)$, i.e. the linear system of plane curves of degree d with n general double points. Recall the *Veronese embedding*

$$v_d : \mathbb{P}^2 \longrightarrow \mathbb{P}^{d(d+3)/2}$$

with image V_d , the *Veronese surface* of degree d^2 . A plane curve of degree d corresponds via v_d to a hyperplane section of V_d ; and such a plane curve has a double point at p if and only if the corresponding hyperplane is tangent to V_d at $v_d(p)$. Therefore the linear system $\mathcal{L}_d(2^n)$ corresponds to the linear system \mathcal{H} of hyperplanes in $\mathbb{P}^{d(d+3)/2}$ which are tangent to V_d at n fixed (but general) points. The famous Terracini's Lemma (see [8]) relates this linear system to the tangent space to the a secant variety of V_d : the base locus of \mathcal{H} is the general tangent space to $\text{Sec}_{n-1}(V_d)$, the $(n-1)$ -*secant variety* to V_d , i.e. the variety described by all linear spaces of dimension $n-1$ which are n -secant to V_d . One thus concludes that $\mathcal{L}_d(2^n)$ is special if and only if $\text{Sec}_{n-1}(V_d)$ has smaller dimension than expected, namely V_d is $(n-1)$ -*defective*.

Next we will use toric degenerations of the Veronese surfaces to study interpolation problems and secant varieties.

2. TORIC DEGENERATIONS

In this section we quickly recall a few basic facts about toric degenerations of projective toric varieties that will be useful later. The interested reader is referred to [28], [33] for more information on the subject and to [24] for relations with tropical geometry.

Recall that the datum of a pair (X, \mathcal{L}) , where X is a projective, n -dimensional toric variety and \mathcal{L} is a base point free, ample line bundle on X , is equivalent to the datum of an n dimensional integral compact convex polytope P in \mathbb{R}^n , which is determined up to translation. Thus we may assume all points of P have non-negative coordinates (see [23], p. 72). If $m_i = (m_{i1}, \dots, m_{in})$, $0 \leq i \leq r$, are the integral points of P , we can consider the monomial map

$$\phi_P : x \in (\mathbb{C}^*)^n \rightarrow [x^{m_0} : \dots : x^{m_r}] \in \mathbb{P}^r$$

where $x = (x_1, \dots, x_n)$ and

$$x^{m_i} := x_1^{m_{i1}} \dots x_n^{m_{in}}.$$

The closure of the image of ϕ_P is the image X_P of X via the morphism $\phi_{\mathcal{L}}$ determined by the line bundle \mathcal{L} .

For example, if P is the triangle

$$\Delta_d := \{(x, y) : x \geq 0, y \geq 0, x + y \leq d\}$$

then X_{Δ_d} is the Veronese surface V_d .

If P is the rectangle

$$R_{a,b} := \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$$

with a, b positive integers, then $X_{R_{a,b}}$ is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ embedded in \mathbb{P}^{ab+a+b} via the complete linear system $\mathcal{L}_{(a,b)}$ of curves of bidegree (a, b) .

If P is the trapezoid

$$T_{a,b} := \{(x, y) : x \geq 0, 0 \leq y \leq b, x + y \leq a\}$$

with $a > b$ positive integers, then $X_{T_{a,b}}$ is \mathbb{F}_1 , i.e. the plane blown up at a point p , embedded in \mathbb{P}^r , $r = ab + a - b(b + 1)/2$, via the proper transform of the linear system of curves of degree a with a point of multiplicity $a - b$ in p .

Now, consider a *subdivision* D of P into convex subpolytopes. This is a finite family of n dimensional convex polytopes whose union is P and such that any two of them intersect only along a face (which may be empty). Such a subdivision is called *regular* if there is a piecewise linear, positive function F defined on P such that:

- (i) the polytopes of D are the orthogonal projections on the hyperplane $z = 0$ of \mathbb{R}^{n+1} of the n -dimensional faces of the *graph polytope*

$$G(F) := \{(x, z) \in P \times \mathbb{R} : 0 \leq z \leq F(x)\}$$

which are neither vertical, nor equal to P ;

- (ii) the function F is *strictly convex*, i.e., the hyperplanes determined by each of the faces of $G(F)$ intersect $G(F)$ only along that face.

Once one has a regular subdivision D as above, one can construct a flat, projective degeneration of X_P parametrized by the affine line \mathbb{C} , to a reducible variety X_0 which is the union of the toric varieties X_Q , with Q in D . The intersection of the components X_Q of X_0 is dictated by the incidence relations of the corresponding polytopes: if Q and Q' have a common face R , then X_Q intersects $X_{Q'}$ along the toric subvariety of both determined by the face R .

The degeneration can be described as follows. Consider the morphism

$$\phi_D : (x, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \rightarrow [t^{F(m_0)}x^{m_0} : \dots : t^{F(m_r)}x^{m_r}] \in \mathbb{P}^r.$$

The closure of the image of $(\mathbb{C}^*)^n \times \{t\}$, $t \neq 0$, is a variety X_t which is projectively equivalent to X_P . The flat limit of X_t when t tends to 0 is the variety X_0 . To see that X_0 is the union of the varieties X_Q , with $Q \in D$, one argues as follows. Suppose that $F|_Q$ is the linear function $a_1x_1 + \dots + a_nx_n + b$. First act with the torus in the following way:

$$(x_1, \dots, x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \rightarrow (t^{-a_1}x_1, \dots, t^{-a_n}x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^*.$$

Then compose with ϕ_D , getting

$$(x_1, \dots, x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \rightarrow [\dots : t^{F(m_i)}t^{-a_1m_{i1}-\dots-a_nm_{in}}x^{m_i} : \dots] \in \mathbb{P}^r.$$

Note that the point

$$[\dots : t^{F(m_i)}t^{-a_1m_{i1}-\dots-a_nm_{in}}x^{m_i} : \dots]$$

in \mathbb{P}^r equals

$$t^{-b}[\dots : t^{F(m_i)}t^{-a_1m_{i1}-\dots-a_nm_{in}}x^{m_i} : \dots]$$

i.e. the point

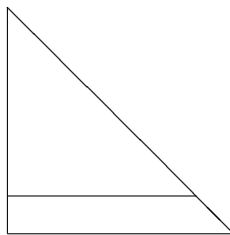
$$[\dots : t^{F(m_i)-F|_Q(m_i)} x^{m_i} : \dots].$$

Then by the very definition of a regular subdivision, by letting $t \rightarrow 0$ in the above expression, we see that X_Q sits in the flat limit X_0 of X_t .

3. DEGENERATIONS OF THE VERONESE

Recall that a *rational normal scroll* $S(a, b)$, with $0 < a \leq b$, is a smooth scroll surface of degree $d = a + b$ in \mathbb{P}^{d+1} , which is described by the lines joining corresponding points of two rational normal curves of degrees a and b lying in two linearly independent subspaces of dimensions a and b respectively. As an abstract surface, $S(a, b)$ is isomorphic to the *Hirzebruch surface* \mathbb{F}_{b-a} .

First of all, the Veronese V_d degenerates to the union of V_{d-1} and a scroll $S(d-1, d)$:



This is a toric degeneration: the subdivision of the triangle Δ_d is clearly regular in this case. However, this degeneration can also be directly described as follows. In a trivial family \mathcal{X} of \mathbb{P}^2 's parametrized by a disk Δ , blow up the central fibre along a line R , thus getting a new family \mathcal{X}' . The general fibre of \mathcal{X}' is still a plane, whereas the central fibre consists of the old plane \mathbb{P} and of the exceptional divisor \mathbb{F} , which is a \mathbb{F}_1 , meeting \mathbb{P} along the line R . Note that R is also the (-1) -curve on \mathbb{F} , meeting the ruling F in one point.

Consider the line bundle which is the pull-back to \mathcal{X} of $\mathcal{O}_{\mathbb{P}^2}(d)$. Call $\mathcal{O}(d)$ the pull-back of this line bundle to \mathcal{X}' and consider the bundle $\mathcal{O}(d) \otimes \mathcal{O}(-\mathbb{F})$. The restriction of it to the general fibre is still $\mathcal{O}_{\mathbb{P}^2}(d)$, whereas its restriction to \mathbb{P} is $\mathcal{O}_{\mathbb{P}^2}(d-1)$ and to \mathbb{F} is $\mathcal{O}_{\mathbb{F}}(dF + R)$, which embeds \mathbb{F} as $S(d-1, d)$. The projective degeneration in question can be obtained by using the sections of $\mathcal{O}(d) \otimes \mathcal{O}(-\mathbb{F})$.

Similarly, $S(d-1, d)$ degenerates to a quadric and a scroll $S(d-2, d-1)$:



Again this is a toric degeneration. On the other hand one can easily figure out how to imitate the previous construction in order to directly perform this degeneration: in the central fibre of a trivial family of \mathbb{F}_1 's one has to blow up a ruling, thus creating an exceptional divisor which is a \mathbb{F}_0 . We leave to the reader the task of computing which *twisting operation* of line bundles has to be made in order to finish the job.

The first construction can be iterated, and we thus see that V_d degenerates to a union of a plane V_1 and a sequence of scrolls $S(1, 2), S(2, 3), \dots, S(d-1, d)$ (see Figure 1). Also the second construction can be iterated, and we have that $S(a-1, a)$ degenerates to $a-1$ quadrics and a plane (see Figure 2).

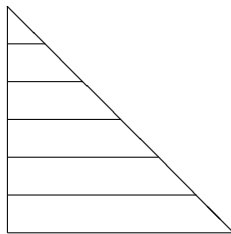


FIGURE 1.

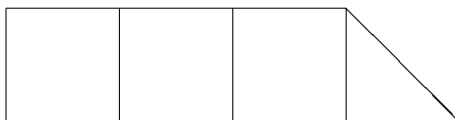
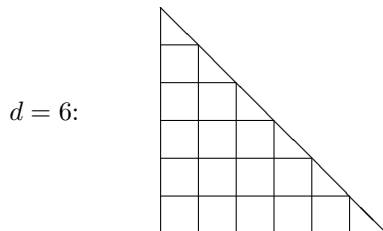


FIGURE 2.

It is less obvious that these degenerations can be combined in order to give rise to a degeneration of V_d to a union of d planes and $\binom{d}{2}$ quadrics, which we illustrate below for $d = 6$:



This is again a toric degeneration: one has to prove that the subdivision of Δ_d in d triangles and $\binom{d}{2}$ squares is regular, which is not difficult to see, especially if one translates the problem into the existence of the corresponding tropical curve (see [24]). Another way of proceeding is noticing that the *horizontal segments* in Figure 1 correspond to rational normal curves of degrees $1, 2, \dots, d-1, d$ lying in linearly independent subspaces. The rulings of the scroll determine correspondences between these curves. The degeneration can be performed by letting each of these curves simultaneously degenerate to chains of lines (meanwhile also the correspondences between the curves degenerate). The existence of the degeneration can be also directly, but more painfully, verified by performing blow-up, blow-down and twist operations as illustrated above. Since we are now interested in drawing geometrical consequences from the existence of the degeneration, we skip these technical details and we leave them to the interested reader.

Summing up, the *vertices* of the last configuration of planes and quadrics are independent and therefore can be taken as the coordinate points of the ambient $\mathbb{P}^{d(d+3)/2}$. Each double curve in the above degeneration, along which two adjacent surfaces meet, is a line in the ambient space. We will refer to this degeneration as the *quadrics degeneration* of the Veronese.

Finally, each quadric can independently degenerate in its own space \mathbb{P}^3 , to a union of two planes, in two ways:



Such a degeneration, which again can be understood from a toric, or tropical, viewpoint, can be directly performed by moving the quadric in a pencil in its embedding \mathbb{P}^3 and leaving the corresponding quadrilateral of double lines fixed.

If we degenerate each quadric to two planes, we obtain degenerations of V_d to d^2 planes; we will refer to these as to *planar degenerations* of the Veronese. Again, each double curve is a line in the ambient projective space, and the union of the planes spans this space. Each plane contains exactly three of the coordinate points of the projective space.

Since the vertices of these configurations are independent in the ambient projective space, any subset of n of the planes which are pairwise disjoint will span a maximal dimensional space, of dimension $3n - 1$. Any such subset of a given planar degeneration D of V_d will be called a *skew n -set* of planes of D , and we will denote by $\nu_n(D)$, or simply by ν_n the number of such skew n -sets.

For the degenerations considered in this section and applications to secant varieties as in §5, see also [38].

4. DOUBLE POINT INTERPOLATION PROBLEMS

Consider a double point interpolation system $\mathcal{L}_d(2^n)$, the linear system of plane curves of degree d with n general double points. As noted above, this system corresponds to the system of hyperplanes in $\mathbb{P}^{d(d+3)/2}$ which are tangent to the Veronese surface at n general points. A famous theorem of Alexander–Hirschowitz (see [1], see also [27] and [2]) implies that $\mathcal{L}_d(2^n)$ is non-special, unless $d = 2, n = 2$ and $d = 4, n = 5$. The standard proof is based on an infinitesimal deformation analysis of singular hypersurfaces, implicit in Terracini’s Lemma: it implies that speciality of $\mathcal{L}_{r,d}(2^n)$ forces the general element of this system to have non isolated singularities at the double base points p_1, \dots, p_n (see [2], [10], [7]). Employing a planar degeneration of the Veronese, we are able to reduce this result to a purely combinatorial property of the resulting configuration of planes.

Lemma 4.1. *Suppose that there exists a planar degeneration D of V_d , and a skew n -set S of planes of D . Then the linear system $\mathcal{L}_d(2^n)$ has the expected dimension $d(d+3)/2 - 3n$.*

In particular, if there is a skew n -set of planes of D whose planes contain all of the $(d+1)(d+2)/2$ coordinate points of the configuration, then $3n = (d+1)(d+2)/2$ and the linear system $\mathcal{L}_d(2^n)$ is empty.

Proof. Consider the degeneration D , and let the general points p_1, \dots, p_n on V_d degenerate in such a way that each point goes to a general point of the planes in the subset S . The limit of the system of hyperplanes tangent to V_d at the points p_1, \dots, p_n is the system of hyperplanes tangent to the configuration D at each limit point; but a hyperplane which is tangent to a plane at a point must contain that plane. Therefore the limiting system of hyperplanes is the system

that contains the subset S of n planes in the configuration, which is the system of hyperplanes containing the span of S . Since S consists of pairwise disjoint planes, it has maximal dimensional span, of dimension $3n - 1$; and therefore this limiting system of hyperplanes has codimension equal to $3n$.

By semicontinuity, we conclude that the system $\mathcal{L}_d(2^n)$ has codimension at least $3n$ in \mathcal{L}_d ; but this is also the maximum possible codimension, since we are imposing $3n$ linear conditions on the plane curves.

In particular, if one can find a skew n -set S of planes in D that contain all of the coordinate points of the configuration, then S will span the ambient space. Hence there can be no hyperplane that contains all of the planes of S , and we conclude, using the same argument as above, that the corresponding linear system must be empty. \square

As a first application, we have:

Lemma 4.2. $\mathcal{L}_5(2^7)$ is empty, and $\dim(\mathcal{L}_6(2^9)) = 0$, i.e. these systems have the expected dimensions.

Proof. We illustrate below a skew 7-subset (respectively 9-subset) for a planar degeneration D_5 of V_5 (respectively D_6 of V_6):

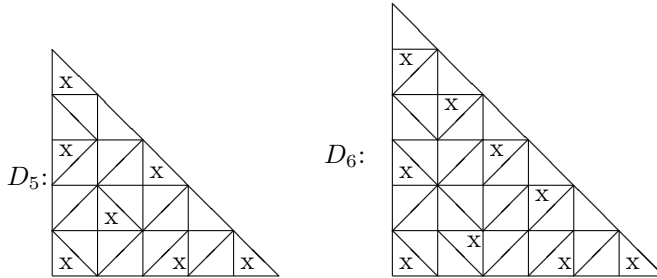
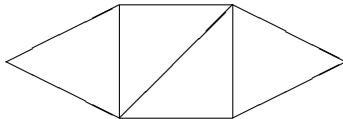


FIGURE 3.

Note that in the $d = 5$ case, the planes indicated by an 'x' form both a spanning 7-subset and a skew 7-subset of the indicated total planar degeneration. In the $d = 6$ example, the only vertex not covered by the 9-subset is the one at the upper left. \square

Remark 4.3. In the above lemmas, the existence of a certain degeneration of the Veronese was used to prove that certain linear systems have the expected dimension. This argument can be reversed, if one knows that linear systems do not have the expected dimension. For example, the following union of four planes in \mathbb{P}^5 is *not* a degeneration of the Veronese V_2 :



If it were, then the pair of planes on the two ends forms a spanning 2-subset. Therefore one could degenerate two points of V_2 so that one point goes on each

end, and this would prove by the above Lemma that $\mathcal{L}_2(2^2)$ is empty, which it is not (for another, perhaps more conceptual explanation, see [5]).

It is on the other hand clear that the above configuration of planes is a degeneration of the scroll $S(2, 2)$.

As announced, the lemmas above enable us to reduce the problem of determining the dimension of $\mathcal{L}_d(2^n)$ to a tractable combinatorial one.

Theorem 4.4. *The linear system $\mathcal{L}_d(2^n)$ has the expected dimension whenever $d \geq 5$.*

Proof. The proof will be by induction on the degree d . Fix $n_0 = \lfloor (d+1)(d+2)/6 \rfloor$; with this number of points, we see that the virtual dimension of $\mathcal{L}_d(2^{n_0})$ is

$$v = d(d+3)/2 - 3n_0 = \begin{cases} -1 & \text{if } d \equiv 1, 2 \pmod{3} \\ 0 & \text{if } d \equiv 0 \pmod{3}. \end{cases}$$

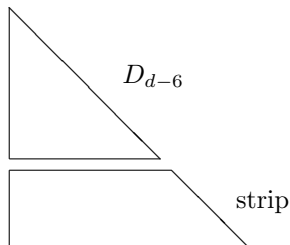
Suppose that the theorem is true for this $n = n_0$. Since the virtual dimension of $\mathcal{L}_d(2^{n_0})$ is at least -1 , we conclude that the $3n_0$ conditions imposed by the n_0 double points are independent. Hence any fewer number of points will also impose independent conditions, and so $\mathcal{L}_d(2^k)$ will have the expected dimension for any $k < n_0$.

If $k > n_0$, then we must show that the system $\mathcal{L}_d(2^k)$ is empty; this is obvious if d is not divisible by three, since by the computation above we have that $\mathcal{L}_d(2^{n_0})$ is already empty. If $3|d$, then we have that $\mathcal{L}_d(2^{n_0})$ has dimension zero, and so consists of a unique divisor; imposing any additional double points off this divisor will make the system empty.

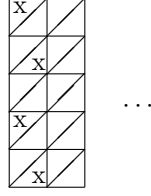
Therefore it will be sufficient to prove the theorem for $\mathcal{L}_d(2^{n_0})$; and this we will do by induction on d , employing Lemma (4.1). What we will show is that there is a skew n_0 -subset S of planes for a certain planar degeneration D of V_d , if there is one for V_{d-6} .

To get the induction started, we must illustrate such degenerations and subsets for $5 \leq d \leq 10$; the $d = 5$ and $d = 6$ cases have been done above in Lemma (4.2). We leave to the reader the task of finding these for $7 \leq d \leq 10$, by drawing similar pictures to the ones in Lemma (4.2); it is not difficult at all, there are many ways to do each one, and it is an amusing exercise (see also [4]).

By induction, we assume that a total planar degeneration D_{d-6} and a maximal skew subset S' of it are available. The required total planar degeneration D_d and the skew n_0 -subset S of it will be formed by grafting an appropriate strip onto D_{d-6} , as the following diagram indicates:

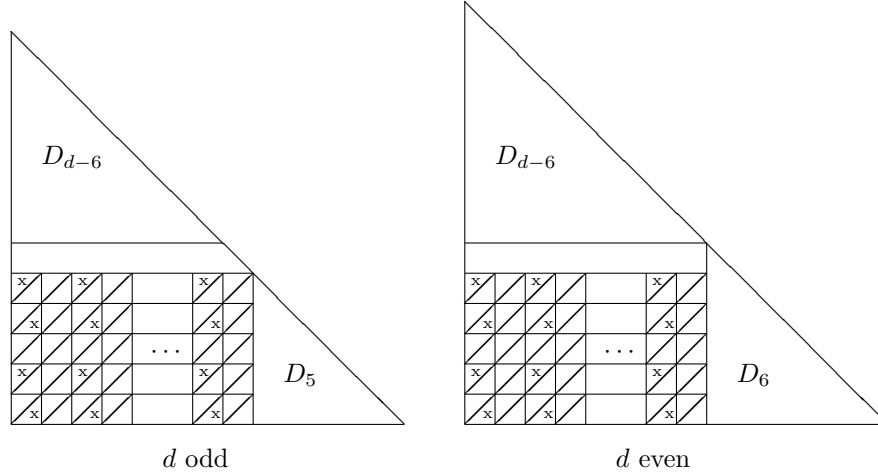


The strip is exactly 6 vertices high, and the configuration on the strip consists of $\lfloor (d+1)/2 \rfloor - 3$ copies of the rectangle



organized side by side; then on the far right of the strip, one has either the triangle with 6 vertices on a side, or a 5×1 rectangle adjacent to such a triangle, depending on the parity of d . These two pieces may be constructed using the configurations D_5 and D_6 presented in Lemma (4.2): the D_5 triangle configuration is used as is; the D_6 configuration is used after deleting the top vertex, creating the 5×1 rectangle adjacent to the triangle.

To be specific, we have the configurations



which show that if we have a solution for degree $d - 6$, then we can construct one for d ; this finishes the proof. \square

The above proof is similar to a recent one by J. Draisma in [18], which uses tropical geometry rather than degenerations. These ideas have been exploited by S. Brannetti in [4] in order to give a similar combinatorial proof of the Alexander–Hirschowitz theorem in \mathbb{P}^3 : the system $\mathcal{L}_{3,d}(2^n)$ has the expected dimension as soon as $d \geq 5$ (see also [39] and [36] for the classical approach of Terracini to this result). A more general result in the planar case has been proved by J. Roé in [34]. It would be very nice to have a full proof of the Alexander–Hirschowitz theorem using this technique.

5. DEGREE OF SECANT VARIETIES

In this section we want to indicate an enumerative application of planar degenerations of the Veronese surfaces.

The general question we want to address is: given a projective, irreducible variety X of dimension n in \mathbb{P}^r , such that $\text{Sec}_k(X) \neq \mathbb{P}^r$, what is the degree $\mu_k(X)$ of $\text{Sec}_k(X)$?

A similar question can be asked even if $\text{Sec}_k(X) = \mathbb{P}^r$ and $r = nk + n + k$. In this case given a general point $p \in \mathbb{P}^r$ there is a finite number $\mu_k(X)$ of $(k+1)$ -secant \mathbb{P}^k to X passing through p : what is this number?

Recent contributions to these problems can be found in [16].

If X is smooth, $\mu_1(X)$ coincides with the number of *apparent double points* of a general projection of X to \mathbb{P}^{2n} (see [11], [16]), and one has the so-called *double point formula*

$$2\mu_1(X) = d^2 - c_n(N_f)$$

where d is the degree of X and N_f is the normal bundle to the projection morphism $f : X \rightarrow \mathbb{P}^{2n}$, defined by the exact sequence of vector bundles on X

$$0 \rightarrow T_X \rightarrow f^*(T_{\mathbb{P}^{2n}}) \rightarrow N_f \rightarrow 0;$$

hence $c_n(N)$ can be computed in terms of projective and birational invariants of X .

In the curve case, this is nothing but *Hurwitz formula*, i.e.

$$\mu_1(X) = \binom{d-1}{2} - g$$

where d is the degree and g the genus of X . In the surface case one has

$$(5.1) \quad \mu_1(X) = \frac{d(d-5)}{2} + 6\chi(\mathcal{O}_X) - 5(g-1) - K_X^2$$

where d is the degree and g the sectional genus of X .

Similar, but more complicated, formulas are available for the degree of $\mu_k(X)$ in the case of curves (see [3], chapter VIII) and in the case of surfaces for $k \leq 5$ (see [30], [40], [31]). The question of computing $\mu_k(X)$ is open in general, even in the surface case and for Veronese surfaces. We devote this section to some speculations showing how planar degenerations of Veronese surfaces may be used to attack this problem.

Consider the Veronese surface V_d for $d = 3$ or $d \geq 5$ and set $k_0 = n_0 - 1$, where n_0 was defined at the beginning of the proof of Theorem 4.4. We wish to compute the numbers $\mu_{d,k} := \mu_k(V_d)$, which make sense for $1 \leq k \leq k_0$. Note that Theorem 4.4 implies that all secant varieties of V_d have the expected dimension. In particular, $\text{Sec}_{k_0}(V_d)$ is a hypersurface if $d \equiv 0$ modulo 3, whereas it fills up the ambient space if $d \equiv 1, 2$ modulo 3.

Proposition 5.2. *Let D be a planar degeneration of V_d , for $d = 3$ or $d \geq 5$. Then*

$$(5.3) \quad \mu_{d,k} \geq \nu_{k+1}(D).$$

Proof. First, let us look at the cases $d \equiv 0$ modulo 3 and $d \equiv 1, 2$ modulo 3 and $k < k_0$, in which $\text{Sec}_k(V_d)$ is a proper subvariety of the ambient space. The proof of Theorem 4.4 implies that for any skew $(k+1)$ -subset S of planes of D , the span of these planes sits in the flat limit of $\text{Sec}_k(V_d)$. This proves the assertion in this case.

Similarly, if $d \equiv 1, 2$ modulo 3 and $k = k_0$, given a planar degeneration D of V_d , any skew n_0 -subset S of planes of D span the whole space, and each such subset contributes to the number μ_{d,k_0} . \square

We are inclined to believe that, for most degenerations D , equality should hold in (5.3) in case $k = k_0$. Would this be true, the computation of μ_{d,k_0} would turn in a combinatorial problem. The only evidence supporting our belief is given by following examples.

Example 5.4. If $d = 3$, then $n_0 = 3$ and $\mu_{3,2} = 4$. In fact $\text{Sec}_2(V_3)$ is the hypersurface in \mathbb{P}^9 consisting of points corresponding to cubic curves which, after a change of variables, can be written in the form $x_0^3 + x_1^3 + x_2^3 = 0$. These are the so-called *equianharmonic cubics*, i.e. the ones with J -invariant equal to 0. It is a classical result of invariant theory due to Aronhold that the degree of this hypersurface is 4 (see [21], p. 194). Note that this also follows by Le Barz formula in [30] (see also [16]).

Consider the planar degeneration D_3 of V_3 corresponding to the picture:

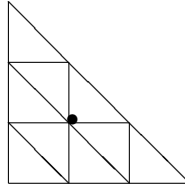
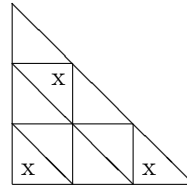
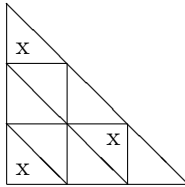
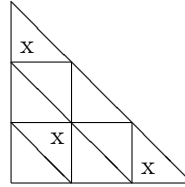
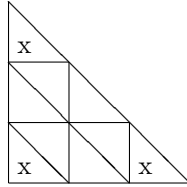


FIGURE 4.

There are four skew 3-sets of planes in this picture, given by:



Thus we see that $\mu_{3,2} = \nu_3(D_3) = 4$.

Example 5.5. If $d = 5$, then $n_0 = 7$ and $\text{Sec}_6(V_5)$ fills up the whole space \mathbb{P}^{20} . It is a result by Shepherd–Barron [37], that $\mu_{5,6} = 1$ (see also [16]).

Consider the planar degeneration D_5 of V_5 shown in Figure 3. It is not difficult to see that there is a unique skew 7–set of planes in D_5 , i.e. the one indicated in Figure 3. Again we find $\mu_{5,6} = \nu_{5,7} = 1$.

Let us turn now to the other extreme case, i.e. to $\mu_{d,1}$. For $d \geq 5$ and $d = 3$, the variety $\text{Sec}(V_d)$ has dimension 5 and, by (5.1), its degree is

$$(5.6) \quad \mu_{d,1} = \frac{d^4 - 10d^2 + 15d}{2} - 3.$$

Is there some planar degeneration of V_d for which equality holds in (5.3) for $k = 1$?

Surprisingly enough, the answer is no, in general. For instance, $\mu_{3,1} = 15$. However, if we look at the planar degeneration D_3 of V_3 described in Figure 4, one sees only 12 pairs of disjoint planes. Therefore the conclusion is that the union of the twelve \mathbb{P}^5 's generated by these skew 2–sets is only a part of the flat limit of the secant variety of V_3 . This seems to contrast with our belief concerning the equality in (5.3) for $k = k_0$. However, as we shall see in a moment, the phenomenon which shows up in the computation of $\mu_{d,1}$, does not occur in the computation of μ_{d,k_0} .

The reason for the discrepancy between (5.6) and the number of skew 2–sets of planes in a degeneration can be understood as follows. As we said, if X a smooth surface, $\mu_1(X)$ computes the number of double points of a general projection X' of X to \mathbb{P}^4 . These double points are the only non Cohen–Macaulay points of X' ; thus $\mu_1(X)$ measures the length of the scheme of non Cohen–Macaulay points of X' , if this has finite length.

Now let us go back to V_d and more specifically to V_3 . When we project down to \mathbb{P}^4 the union of planes depicted in Figure 4, we find a reducible surface D which has 12 nodes corresponding to the pairs of planes which are disjoint in D_3 , but whose projections meet in \mathbb{P}^4 . There is however one further non Cohen–Macaulay point, namely the *central* sextuple point \bullet . This used to be Cohen–Macaulay in \mathbb{P}^9 (its original embedding dimension is 5), but it is no longer Cohen–Macaulay in \mathbb{P}^4 . We claim its *Cohen–Macauliness defect* is 3, i.e. *it counts for 3 more nodes of X'* . This restores the number 15 for $\mu_{3,1}$.

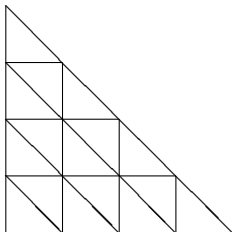
To give evidence to our claim, consider the following picture:



The left hand sides shows a polytope P with horizontal and vertical sides of length 1. The corresponding projective toric variety X_P is a del Pezzo surface of degree 6 in \mathbb{P}^6 , i.e. the plane blown up at three distinct points anticanonically embedded in \mathbb{P}^6 . The degree of the secant variety to X_P is 3 by (5.1). The right hand side shows a

regular subdivision of P , corresponding to a degeneration of X_P to 6 planes, which form a cone from the central vertex \bullet over a cycle of six lines spanning a \mathbb{P}^5 . Since there are no pairs of disjoint planes here, we see that the singularity at \bullet has to count with multiplicity 3 in the computation of the degree of the secant variety.

Similarly, $\mu_{4,1} = 75$. On the other hand in the planar degeneration of V_4 corresponding to the picture



we see only 66 pairs of disjoint planes. However the three sextuple points contribute each by three in this computation restoring the right degree $75 = 66 + 9$.

Similar computations should be available for $\mu_{d,k}$, with $1 \leq k \leq k_0$. However the situation may turn out to be more complicated, and in general, the question is: how do the singularities of the configuration influence the combinatorial computation of $\mu_{d,k}$?

Remark 5.7. The considerations we made in the present section can be more generally applied to toric varieties. As an example, consider $S(2,2)$. This is well known to be a OADP- variety, i.e. there is a unique secant line to $S(2,2)$ passing through the general point of \mathbb{P}^5 (see [11]).

To see this combinatorially, look at the planar degeneration of $S(2,2)$ showed in Remark 4.3 and note that there is here a unique skew 2-set of planes, given by the two end planes of the configuration.

The problem of studying the secant variety to a toric variety has been considered in [17].

6. POINTS OF HIGHER MULTIPLICITY

In this section we go back to interpolation and we want to use the planar degenerations of the Veronese surfaces in order to study multiple points interpolation problems. We recall a basic fact.

Lemma 6.1. *Let X be a variety, M a line bundle on X , and D an effective Cartier divisor on X . Set $M(D) = M \otimes \mathcal{O}_X(D)$. Suppose that $H^0(X, M) = H^1(X, M) = 0$. Then the restriction map from $H^0(X, M(D))$ to $H^0(X, M(D)|_D)$ is an isomorphism.*

This lemma applies in the following two cases:

Lemma 6.2.

- (a) *Let X be the blow up of \mathbb{P}^2 at a point, with exceptional divisor E and line class H . Let $M = \mathcal{O}_X((m-1)H - mE)$. Then $H^0(M) = H^1(M) = 0$.*
- (b) *Let X be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two general points, with exceptional divisors E_1 and E_2 , and denote by V the vertical fiber class and by H the horizontal fiber class. Let $M = \mathcal{O}_X((m-1)H + mV - mE_1 - mE_2)$ (or, symmetrically, $M = \mathcal{O}_X(mH + (m-1)V - mE_1 - mE_2)$). Then $H^0(M) = H^1(M) = 0$.*

Proof. In both cases, the systems are empty, so that $H^0 = 0$, and by using Serre Duality, H^2 is also zero. Riemann-Roch gives $\chi = 0$ as well, which implies $H^1 = 0$. \square

We can apply the previous Lemmas to different divisors D . It is useful for our applications, given the degeneration constructions we have introduced, that the divisors D be subdivisors of the double curves of the planes or quadrics in the degeneration.

In the planar case, this means that we will be applying the lemmas for a divisor D consisting of a subdivisor of a triangle of lines $L_1 + L_2 + L_3$. We have the following list of $M(D)$'s in this case.

Lemma 6.3. *Let X be the blow up of \mathbb{P}^2 at a point, with exceptional divisor E and line class H . Let L_1, L_2 , and L_3 be a triangle T of lines not passing through the point. Let $M = \mathcal{O}_X((m-1)H - mE)$. Then the restriction maps*

$$\begin{aligned} (1) : H^0(X, \mathcal{O}_X(mH - mE)) &\rightarrow H^0(L_i, \mathcal{O}_X(mH)|_{L_i}) \\ (2) : H^0(X, \mathcal{O}_X((m+1)H - mE)) &\rightarrow H^0(L_i + L_j, \mathcal{O}_X((m+1)H - mE)|_{L_i + L_j}) \\ (3) : H^0(X, \mathcal{O}_X((m+2)H - mE)) &\rightarrow H^0(T, \mathcal{O}_X((m+2)H - mE)|_T) \end{aligned}$$

are isomorphisms.

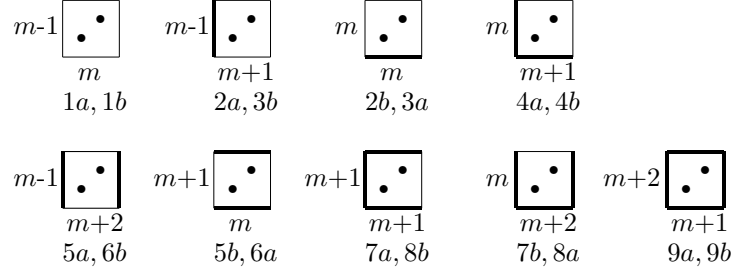
We note that these three spaces have dimensions $m+1$, $2m+3$, and $3m+6$, respectively.

In the quadric case, we present the information in a table. Let X be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two general points, with exceptional divisors E_1 and E_2 , and denote by V the vertical fiber class and by H the horizontal fiber class. We fix two vertical fibers V_1 and V_2 , and two horizontal fibers H_1 and H_2 , not passing through the two points; our divisor D will be a subdivisor of $H_1 + H_2 + V_1 + V_2$. By Lemma 6.2, there are two possibilities for the line bundle M for each divisor D ; these are presented in the last two columns of the table.

Lemma 6.4. *Using the above notation, the restriction map from $H^0(X, M(D))$ to $H^0(D, M(D)|_D)$ is an isomorphism, for all D and $M(D)$ in the following table:*

<i>Divisor D</i>	<i>$M(D)$-a</i>	<i>$M(D)$-b</i>
1. 0	$mV + (m-1)H$	$(m-1)V + mH$
2. V_1 or V_2	$(m+1)V + (m-1)H$	$mV + mH$
3. H_1 or H_2	$mV + mH$	$(m-1)V + (m+1)H$
4. $V_i + H_j$	$(m+1)V + mH$	$mV + (m+1)H$
5. $V_1 + V_2$	$(m+2)V + (m-1)H$	$(m+1)V + mH$
6. $H_1 + H_2$	$mV + (m+1)H$	$(m-1)V + (m+2)H$
7. $H_1 + H_2 + V_i$	$(m+1)V + (m+1)H$	$mV + (m+2)H$
8. $V_1 + V_2 + H_j$	$(m+2)V + mH$	$(m+1)V + (m+1)H$
9. $V_1 + V_2 + H_1 + H_2$	$(m+2)V + (m+1)H$	$(m+1)V + (m+2)H$

(We abused notation and denoted the $M(D)$'s using the divisor classes only.)

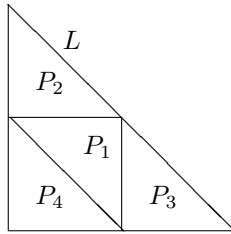


We will apply these lemmas by constructing a degeneration of the d -fold Veronese V_d to a union of planes and quadrics as described above. We will degenerate the bundle $\mathcal{O}(d)$ to a bundle on the degenerate configuration which will have certain degrees on the planes and bidegrees on the quadrics. The general multiple points will degenerate either to one point on a plane or to two general points on a quadric. Our goal is to show that we may determine the dimension of the limit system by analyzing, via the above lemmas, the dimensions on each surface in the degeneration and then putting the information together.

This is the general form of the proof outlined above for double points, in which we had a degeneration only to planes. However for the combinatorial analysis there, it was useful to focus on the three coordinate points in each plane, rather than the divisor of double curves. For higher multiplicities, it is necessary to relate the linear systems on the surfaces and on the double curves. For this reason the above lemmas are the most appropriate tools.

An example will illustrate the argument.

Consider the linear system of curves of degree $2m$ with 4 points of multiplicity m . It is well known that this is a non-special system of dimension m , composed with the pencil of conics through the four points; each element of the system consists of m conics in the pencil. This is the expected dimension of the system $\mathcal{L}_{2m}(m^4)$: $v = 2m(2m + 3)/2 - 4m(m + 1)/2 = m$. We want exploit the degeneration of the Veronese surface V_2 to four planes to give a combinatorial proof of non-speciality of $\mathcal{L}_{2m}(m^4)$. The degeneration to four planes P_1, \dots, P_4 , may be illustrated as below:



Lemma 6.5. *The system $\mathcal{L}_{2m}(m^4)$ has the expected dimension m .*

Proof. Since this is a degeneration of V_2 , the bundle $\mathcal{O}(m)$ (which has degree m on each plane) corresponds to the bundle $\mathcal{O}(2m)$ on \mathbb{P}^2 . We degenerate the four points to one in each plane.

We note that on each plane, we have curves of degree m with one point of multiplicity m . Therefore we are in the situation of Lemma 6.3, and we see that the linear system on any one of the planes is determined by the linear system

restricted to any one of the three lines on that plane indicated in the figure; indeed, the map on global sections is an isomorphism.

Denote by L_j the line of intersection between the interior plane P_1 and one of the other three planes P_j .

To determine the space of sections on the entire configuration of four planes, we choose any one of the other non-double lines, say the line L indicated in the figure, on the plane P_2 . The restriction of the system to L has degree m , and the vector space of sections has dimension $m + 1$.

By Lemma 6.3, any section restricted to L determines a section on the plane P_2 . This then determines the section on the double line L_2 , via the corresponding restriction isomorphism. Again by Lemma 6.3, the section on L_2 determines the section on the interior plane P_1 . Similarly, this determines sections on the two other double lines L_3 and L_4 . Finally the Lemma now implies that the sections on the other two planes are determined as well.

All of these determinations are made via isomorphisms, and the result is that the space of sections of $\mathcal{O}(m)$ on the configuration of four planes, with the four multiple points, is isomorphic to the space of sections on the restriction to the line L . We conclude that the limiting bundle has $h^0 = m + 1$ as desired. \square

Another phenomenon occurs when a configuration of surfaces meets in a cycle, with an interior vertex. To fix notation, suppose that we have n surfaces S_1, \dots, S_n , meeting in a cycle with a common vertex. i.e., there is a smooth rational curve $L_i = S_i \cap S_{i+1}$ (with indices taken modulo n) and a point v common to all the curves L_i . Therefore on the surface S_i we have the two curves L_i and L_{i-1} , meeting at v .

We assume that we have a linear system M_i on each surface S_i , which restricts to linear systems on both L_i and L_{i-1} . We assume that these are compatible with the configuration, in the sense that the two linear systems on each L_i (one restricted from M_i on S_i and one restricted from M_{i+1} on S_{i+1}) are the same, corresponding to a vector space V_i of sections. We assume that not every section in M_i vanishes at the interior vertex v .

We further assume that these restriction maps from the vector space of sections corresponding to M_i to both L_{i-1} and L_i are isomorphisms. The restriction isomorphisms give a map $g_i : V_{i-1} \rightarrow V_i$, as the composition of the inverse of the restriction from S_i to L_{i-1} , followed by the restriction to L_i . Since these restrictions are assumed to be isomorphisms, each g_i is an isomorphism. The composition of these in sequence around the cycle of surfaces gives a map $F : V_1 \rightarrow V_1$.

We have the following result.

Lemma 6.6. *In the above settings, if the maps g_i are general enough, then there is a unique non-zero section of V_1 (up to scalar) which does not vanish at v and is fixed by F .*

Proof. The generality of the maps allows us to assume that the composition F , which is an automorphism of V_1 , can be diagonalized. Consider the subspace $W \subset V_1$ of sections vanishing at the interior vertex v . Note that W has codimension one in V_1 , by the assumption that not every section in M_1 vanishes at v . Furthermore, W is F -invariant: indeed, a section vanishing at v is mapped to a section vanishing at v by all restriction maps, hence by all maps g_i , and hence by F .

Therefore there is exactly one eigenspace $W_1 \subset V_1$ such that $V_1 = W \oplus W_1$.

We finish the proof by showing that the eigenvalue λ for the one-dimensional eigenspace W_1 is $\lambda = 1$. Choose a section $s \in W_1$; note that $s(v) \neq 0$. We also have $F(s)(v) = s(v)$; that corresponding sections have the same value at v is true for all restrictions, hence for all g_i , and hence for F . We have $F(s) = \lambda s$; evaluating at v , and noting that $s(v) \neq 0$, we see that $\lambda = 1$. \square

We apply this to the following case.

Lemma 6.7. *The system $\mathcal{L}_{3m}(m^9)$ has the expected dimension 0.*

Proof. We make the total planar degeneration D_3 of V_3 to nine planes showed in Figure 4. The line bundle $\mathcal{O}(m)$ which has degree m on each of the nine planes of the degeneration of V_3 is a degeneration of the bundle $\mathcal{O}(3m)$ on \mathbb{P}^2 . Hence on each plane we have curves of degree m with a general point of multiplicity m that agree on the double lines; moreover there is exactly one interior vertex \bullet .

In this configuration we see six planes adjacent to the interior vertex \bullet and linear systems that satisfy the hypothesis of the Lemma 6.6, each being determined by the restriction to a line. Indeed, they each satisfy the conditions of the first case in Lemma 6.3.

By Lemma 6.6, there is a unique non-zero section, up to scalar, on the six middle planes, which does not vanish on the interior vertex \bullet and agrees on all of the double curves around the vertex. This gives a unique section on each of the three double lines that meet the three corner planes in the configuration. Again by the first case in Lemma 6.3, these sections lift uniquely to sections of $\mathcal{O}(m)$ on the three corner planes.

On the other hand there is no non-zero section vanishing on the interior vertex \bullet . Any such a section s , in fact, would vanish also on each of the 9 points of multiplicity m located in each of the planes of D_3 . In particular the divisor of s would consist, around v of six general lines passing through v , one in each of the six planes surrounding v . This however is clearly not a Cartier divisor on the reducible surface D_3 , which gives a contradiction.

Hence the dimension of the linear system is zero, which is the expected dimension. \square

Note that the unique curve in the limit of $\mathcal{L}_{3m}(m^9)$ consists of the unique cubic curve in $\mathcal{L}_3(1^9)$ counted with multiplicity m .

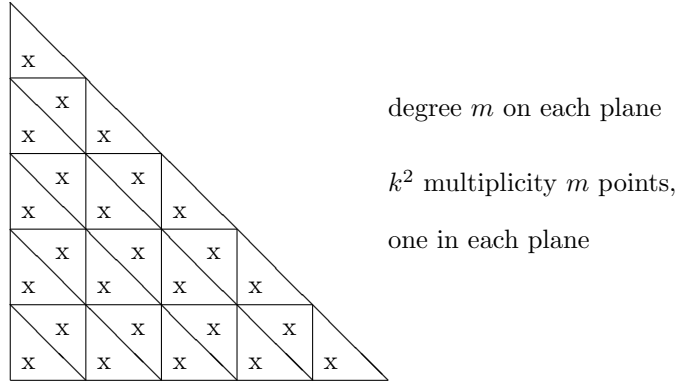
Remark 6.8. Lemma 6.6 is useful when there are cyclical configurations of surfaces that overlap. In this case, in each of the cycles of surfaces, there is a unique section up to scalar satisfying the matching conditions. However these two sections will not agree on the overlap. Hence we conclude that any section satisfying the matching conditions must be zero.

We have seen in Lemmas 6.3, 6.5 and 6.7 that the linear system $\mathcal{L}_{km}(m^{k^2})$ has the expected dimension for $k = 1, 2, 3$. We can now prove the more general statement, which is slightly better than Nagata's conjecture in this case, but it is weaker than Harbourne-Hirschowitz.

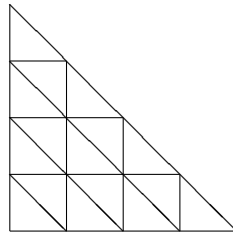
Theorem 6.9. *The system $\mathcal{L}_{km}(m^{k^2})$ has the expected dimension, in particular it is empty for $k \geq 4$.*

Proof. As above, we consider the system associated to the line bundle $\mathcal{O}(m)$ on the k -fold Veronese V_k . Degenerating, we form a total planar degeneration to k^2

planes, and on each plane we have the linear system of curves of degree m . We degenerate the k^2 points by putting one in general position on each plane of the degeneration; for example, a $k = 5$ example is illustrated below.



The cases $k = 1, 2$ and 3 follow by the lemmas 4.3, 4.5 and 4.7 respectively. For $k \geq 4$, the system is expected to be empty. For $k = 4$, we have the following configuration:



By Lemma 6.6, there is a unique divisor satisfying the matching conditions on the six planes adjacent to each of the three interior vertices. However for any two of these interior vertices, there are adjacent planes in common. The divisors will not agree on these common adjacent planes. Hence the system is empty as expected.

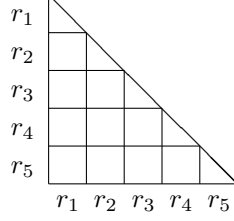
Finally for $k > 4$, if we form the same type of configuration, by induction, the top $(k - 1)^2$ planes already cannot support a divisor. The system will thus be empty. \square

7. LINE BUNDLES ON QUADRICS DEGENERATIONS OF THE VERONESE

One can acquire some more flexibility in the limiting line bundle on the configuration by using the quadrics degeneration of the Veronese that we introduced in §3. Recall that this is the triangular configuration of $\binom{d}{2}$ quadrics, meeting along lines, with d planes on the 'hypotenuse' of the configuration. Let us coordinatize the configuration, and index the surfaces in the configuration as T_{ij} , with $i \geq 1$, $j \geq 1$, and $i + j \leq d + 1$; the quadrics are the surfaces with $i + j \leq d$, and the planes are the surfaces $T_{i,d+1-i}$. We have that T_{ij} meets $T_{k\ell}$ along a line if and only if either $i = k$ and $|j - \ell| = 1$ or $j = \ell$ and $|i - k| = 1$.

We can form a line bundle on this partial quadrics degeneration by putting a line bundle on each surface such that on each double curve the restriction of the two bundles agree. This can be done by choosing d integers r_1, r_2, \dots, r_d , and for $i + j \leq d$ putting the bundle of bidegree (r_i, r_{d+1-j}) on the quadric T_{ij} ; on the plane

$T_{i,d+1-i}$ one puts the bundle of degree r_i . This can be conveniently visualized with the following picture referring to the case $d = 5$:



One can prove that this line bundle is the limit of the line bundle $\mathcal{O}_{\mathbb{P}^2}(r)$, with $r = r_1 + r_2 + \dots + r_d$. This can be seen by using the theory of toric degenerations (see §2) but also by performing a series of blow-up, blow-down and twist operations as indicated in §3. We leave the details to the reader.

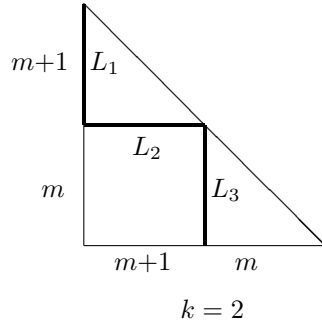
We want to illustrate the technique, invoking Lemmas 6.3 and 6.4, by proving the following theorem:

Theorem 7.1. *The system $\mathcal{L}_{km+1}(m^{k^2})$ has the expected dimension. In particular, it is empty for $k \geq 6$.*

Proof. First note that the expected dimension is $v = mk(5 - k)/2 + 2$. We will degenerate V_k to a union of k planes and $\binom{k}{2}$ quadrics placing one point on each plane and two points on each quadric and the bundle degeneration is as indicated above: we will see for each value of k which values of r_1, \dots, r_k is convenient to take. In particular, for $k \leq 4$ we will take $r_1 = m + 1, r_2 = \dots = r_k = m$.

The $k = 1$ case is proved by noting that the linear system $\mathcal{L}_{m+1}(m)$ has the expected dimension $2m + 2$. This is easy to see in many ways, but let us use Lemma 6.3, to demonstrate the method. The second case of this lemma tells us that this linear system is isomorphic to the linear system restricted to two general lines of \mathbb{P}^2 . This is a linear system on two lines, each of degree $m + 1$, which meet at one point. The space of sections on each line has dimension $m + 2$, and the matching of the sections at the point of intersection gives one condition, so that the space of sections on the lines has dimension $2m + 3$. Hence the projective dimension of the linear system is $2m + 2$ as expected.

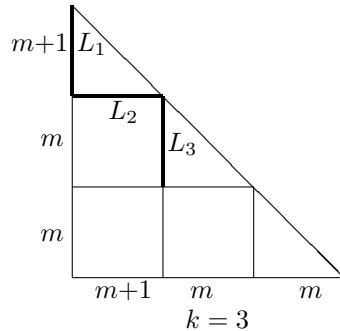
Case $k = 2$. Here the expected dimension is $v = 3m + 2$. We claim that in this case the linear system is determined by the restriction to the three lines L_1, L_2 , and L_3 outlined in the picture below.



Indeed, consider first the upper plane $T_{1,2}$. Applying case (2) of Lemma 6.3, one gets a unique divisor in the linear system $\mathcal{L}_{m+1}(m)$ that restricts to the given divisor on $L_1 + L_2$. For the lower quadric $T_{1,1}$, invoking Lemma 6.4 (fourth case) we conclude the existence of a unique divisor in $\mathcal{L}_{(m,m+1)}(m^2)$ that restricts to the given divisor on $L_2 + L_3$. Similarly for the lower plane $T_{2,1}$, using Lemma 6.3 again, we get a unique divisor restricting to the given divisor on L_3 .

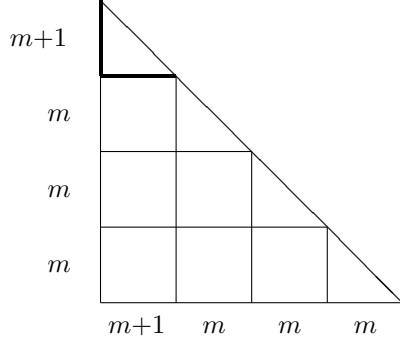
This proves the claim that the divisor on the union of the three surfaces is determined by the divisor on the union of the three lines. Hence the dimension of $\mathcal{L}_{2m+1}(m^4)$ is given by the number of conditions imposed by fixing linear systems on three lines that agree at two points of intersection. The three systems have degrees $m + 1$, $m + 1$, and m on L_1 , L_2 , and L_3 respectively, and so we have vector spaces on the three lines of dimension $m + 2$, $m + 2$, and $m + 1$; the two points of intersection give two conditions, and hence the projective dimension is $\dim(\mathcal{L}_{2m+1}(m^4)) = (m + 2) + (m + 2) + (m + 1) - 2 - 1 = 3m + 2 = v$ as expected.

Case $k = 3$. Here the expected dimension is again $v = 3m + 2$. We claim that the linear system is determined by the restriction to the three lines L_1 , L_2 , and L_3 as in the picture:



By the previous case is enough to prove that the divisor D in the upper three surfaces determines a unique one on each of the three surfaces from the lowest row. Indeed we get a unique divisor on the quadric $T_{2,1}$ using Lemma 6.4 (third case); then the restriction to the right line determines a unique divisor on the plane $T_{3,1}$ using Lemma 6.3 (second case), and the restriction to the left line together with the restriction of D determines the divisor on $T_{1,1}$ using Lemma 6.4 (fourth case).

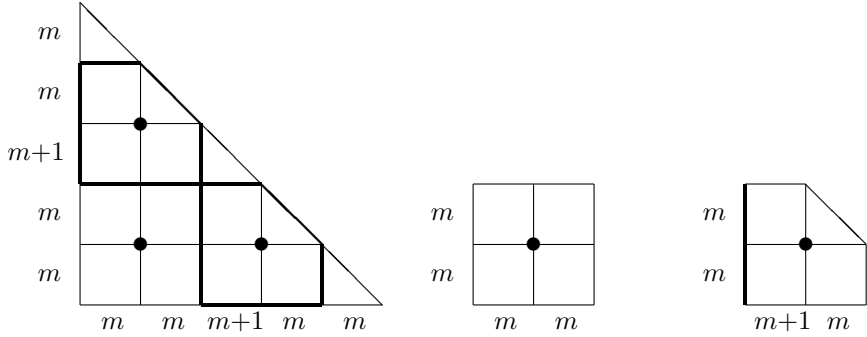
Case $k = 4$. The virtual dimension is $v = 2m + 2$. We want to determine a unique divisor that restricts to the fixed two lines in the configuration below.



$k = 4$

On the plane $T_{1,4}$ the divisor is fixed by Lemma 6.3. On the six surfaces $T_{i,j}$ with $i \geq 2$, we have a unique section, that agrees at the intersection point of $T_{1,4}$ and $T_{2,3}$, by Lemma 6.6 (see also the argument in the proof of Lemma 6.7). This then determines the sections on the lines of intersection between $T_{1,j}$ and $T_{2,j}$ for $j = 1, 2, 3$. At this point the divisor on $T_{1,3}$ is determined by Lemma 6.4; then the divisor on $T_{1,2}$ is determined as well; and finally the divisor on $T_{1,1}$ is determined.

Case $k = 5$. Here the virtual dimension is $v = 2$ and therefore we expect to find three points in the configuration such that the values of the section at these three points determine the section uniquely. We fix the degrees on the planes as indicated in the leftmost figure below, i.e. we take $r_1 = r_2 = r_4 = r_5 = m, r_3 = m + 1$:



$k = 5$

First consider the union of the four quadrics that form the lower left square $T_{1,1}$, $T_{1,2}$, $T_{2,1}$ and $T_{2,2}$; these are indicated in the middle picture of the figure. On each quadric we have the linear system of curves of bidegree (m, m) with two points of multiplicity m , each of which is determined by the restriction to a line. We are therefore in the hypothesis of Lemma 6.6, and so we conclude that the space of sections is one dimensional being determined by the value at the interior point. Note that this section determines the section on the boundaries.

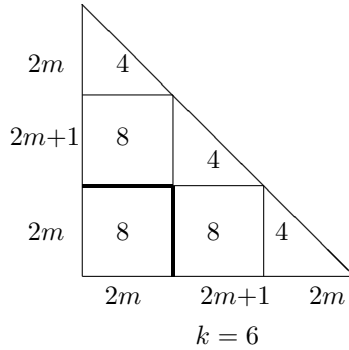
Next we analyse the union of the 4 surfaces $T_{3,1}$, $T_{3,2}$, $T_{4,1}$ and $T_{4,2}$. By the above, we see that the section is determined on the two leftmost vertical lines. This is indicated by the rightmost picture in the figure. We are again in the hypothesis

of Lemma 6.6 with these four surfaces; hence the section is uniquely determined by its value at the interior point.

Similarly, the section on the four surfaces $T_{1,3}$, $T_{1,4}$, $T_{2,3}$ and $T_{2,4}$ (with the lower two horizontal lines determined) is determined by the value at the interior point.

Finally the sections on the planes $T_{1,5}$, $T_{5,1}$ and $T_{3,3}$ are now determined using Lemma 6.3.

Case $k = 6$. This is the first case where we must show that the system is empty. Here we depart from the above pattern a bit and consider the following degeneration of the plane into six surfaces, three re-embedded quadrics and three Veronese surfaces, with degrees indicated, that sum to $6m + 1$:



The number of points on each quadric is 8, while the number on each Veronese is 4; note that the total number is 36 as required.

Focus on the lower left quadric $T_{1,1}$, where we have the linear system of curves of bidegree $(2m, 2m)$ on a quadric, with 8 points of multiplicity m . This has a space of sections of dimension one, namely a unique divisor, the m -fold curve in the linear system of bidegree $(2, 2)$ through the 8 points. The restriction of this space of sections to both the right double curve and the top double curve has dimension one.

Now consider the quadric $T_{2,1}$ just to the right of this, and consider the restriction to the double curve on the left with $T_{1,1}$. This restriction of sections is onto (the sheaf is the sheaf of degree $2m$ on that vertical curve), and the kernel has dimension one (as a vector space), with the similar analysis as above. Therefore the space of sections here that could agree with an element of the dimension one space of sections coming from $T_{1,1}$ has dimension two, one coming from the restriction and one coming from the kernel.

This same analysis holds for the quadric $T_{1,2}$: there is a dimension two space of sections there that restrict to some element of the dimension one space of sections of the double curve where this quadric meets the lower left quadric.

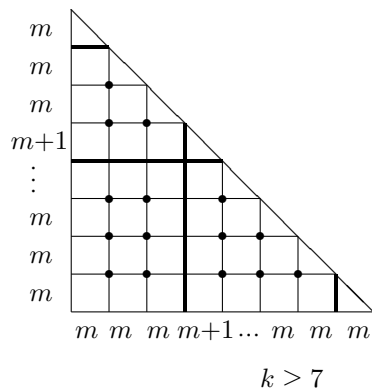
The space of sections now on these three quadrics has dimension three: 2 each on the two quadrics, but there is a condition that the sections agree at the point of intersection, which is the interior point of the configuration.

Now look at one of the corner planes, e.g., $T_{1,3}$. The system there is of degree $2m$, with four m -fold points. We know that this is the linear system composed with the pencil of conics through the four points. Therefore the restriction of this system to the double line is the system (of vector space dimension $m + 1$) of the intersections with the pencil of conics. If $m \geq 2$, no element of the 2-dimensional

space of sections on the adjoining quadric will match with any such element on the double line. (The ambient space has vector space dimension $2m+1$, and we have the restriction of a 2-dimensional space from the quadric and the $(m+1)$ -dimensional space from the plane, which will not intersect away from 0 if $m \geq 2$.)

We conclude that the section must be zero on that corner plane, also by symmetry on the other corner plane; then it must be zero as well on the quadrics, and finally on the center plane $T_{2,2}$.

Case $k \geq 7$. The virtual dimension is $v < 0$ and we must show the system is empty. We use a degeneration with $r_1 = r_2 = r_3 = m$, $r_4 = m+1$, and $r_i = m$ for $i \geq 5$:



The sections on the $3k-12$ lower left quadrics must be zero, using the overlapping interior vertices argument. Then on the eight surfaces just above these, we must also have zero sections; this applies as well to all of the surfaces to the right of these, except the final corner plane. This leaves only the two corner planes $T_{1,k}$ and $T_{k,1}$, and the final plane $T_{4,k-3}$; sections on these are now seen to be zero as well.

□

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