Interpolation on Curvilinear Schemes

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A curvilinear subscheme of $\mathbb{P}^r$ is a subscheme of finite length ℓ of the form $\mathcal{O}_{C,p}/m_p^\ell$ for some smooth point p on a reduced curve C. Such a scheme S imposes linear conditions on the space V of hypersurfaces of degree d, by requiring the hypersurface H to contain it; this is equivalent to the intersection number of H with C at the point p being at least ℓ. V has dimension $\binom{d+r}{r}$, and for a hypersurface of high enough degree to contain a curvilinear subscheme of length ℓ is a priori ℓ linear conditions.

In this note we present an elementary proof of the following.

Theorem: *A general collection of curvilinear subschemes $S_1, \ldots, S_k$ of $\mathbb{P}^r$ of lengths $\ell_1, \ldots, \ell_k$ respectively impose independent conditions on hypersurfaces of degree d whenever $\sum_i \ell_i \leq \binom{d+r}{r}$.*

Proof: It suffices to prove the statement when $\ell = \sum_i \ell_i = \binom{d+r}{r}$; in this case we must prove that the system of hypersurfaces containing all the S_i's is empty. By semicontinuity, it suffices to find a single such collection of subschemes. In fact we will find a single curvilinear subscheme which does the job.

Fix a smooth curve C which does not lie on any hypersurface of degree d; then the linear series on C cut out by the hypersurfaces of degree d is a $g_\Delta^{\ell-1}$ (where $\Delta = d \cdot \deg(C)$). If the theorem is false, then for general $p \in C$, the subscheme $\ell \cdot p$ is a member of this linear series; that is, p is an inflection point for the $g_\Delta^{\ell-1}$. This is a contradiction, since a linear series on a smooth curve has only finitely many inflection points. Q.E.D.

The above argument generalizes a suggestion made to the authors by J. Harris in the planar case; this planar case was also handled in [CG]. The reader may consult [E] and [AH] for a similar result where the subschemes happen to lie on straight lines.

[AH] J. Alexander and A. Hirschowitz: *Interpolation on Jets.* To appear in J. of Algebra (1997).

[CG] M.V.Catalisano, A.Gimigliano: *On curvilinear subschemes of* $\mathbb{P}^2$, J. of Pure and Applied Algebra 93 (1994), 1-14.

[E] A. Eastwood: *Interpolation à N variables.* Journal of Algebra, vol. 139 (1991), 273-310.

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